general agreement of the experimental data with the theory is fairly good for values of q above 400 MeV/c. However, the diffraction minimum predicted by the theory is not observed.

The failure of the data to show a diffraction minimum may not be due to a failure of the theory. It is possible that this effect is due to the enhancement of yet another unresolved level. Such an enhancement of a level would of course be more noticeable in a region where the cross section of the 19.5-MeV level is diminished.

At the same time that excitations in the region of 19 MeV were being studied, a search for higher energy levels was also conducted. A region of the inelastic electron spectrum corresponding to an excitation energy of up to 45 MeV was studied. No excitation of an excited state with an energy greater than the 19.5-MeV level was seen. In particular, the "giant resonance" and the level at 35.8 MeV predicted by Lewis and Walecka were not observed. However, because of the masking effects of the break-up continuum in the inelastic scattering spectrum, any state with a probability of excitation reduced by approximately a factor of 5 from that for the 19.5-MeV level would not have been observed.

CONCLUSION

We have studied the excitation of a level at about 19 MeV in C^{12}, and have presented two simple methods of obtaining cross sections. The agreement between the experimental results and the theory presented by Lewis and Walecka is good, except in the region of the Born-approximation minimum. Even there, however, the agreement may be better than it appears to be, if we allow the possibility that the peak observed at 19.5 MeV may really be two or more unresolved peaks, only some of which should be considered in the comparison with the theory. An experiment with improved resolution might be of great assistance in clarifying the many questions still remaining.

ACKNOWLEDGMENTS

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Information Obtainable from the Noncoplanar $(p,2p)$ Reaction

DAPHNE F. JACKSON
Department of Physics, University of Surrey, London, England*
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A description of the symmetric noncoplanar $(p,2p)$ reaction is given in terms of the distorted-wave impulse approximation and using a simple model of the three-body final state. It is shown that a study of noncoplanar scattering could give information on the population of substates in the final nucleus and on the way this is affected by distortion. Through such a study the advantages of the $(p,2p)$ reaction as a correlation experiment can be realized. It is also shown that coplanar and noncoplanar scattering are sensitive to deformation of the target nucleus and that a combined study of these processes could provide a method of investigating the spatial distribution of bound protons in nonspherical nuclei. A possible experimental arrangement for the noncoplanar scattering is discussed.

1. INTRODUCTION

A NEW treatment of the final state of the symmetric $(p,2p)$ reaction has recently been given[1] in which the two outgoing protons are described in terms of their relative motion and the motion of their center of mass in an optical potential which is twice that for a single proton. By means of the assumption that the proton-proton interaction is short range, a matrix element is obtained in distorted-wave impulse approximation which has exactly the same form as the matrix element for the pickup of a single nucleon, except that the distorted wave function for the deuteron in the pickup reaction is replaced by the distorted wave function for the center of mass of the two protons. Because of the presence of this wave function for the center of mass, this model is referred to as the "di-proton model," although it is not intended to imply that the two protons form a bound state or are correlated on leaving the nucleus. The three-body system is thus approximated by a two-body system with a consequent simplification in the calculation, and also, we believe, increased insight into the mechanism. It was pointed out in Ref. 1 that this model could be used for a simple

* Formerly Battersea College of Technology.
[1] D. F. Jackson, Nucl. Phys. **A90**, 209 (1967).

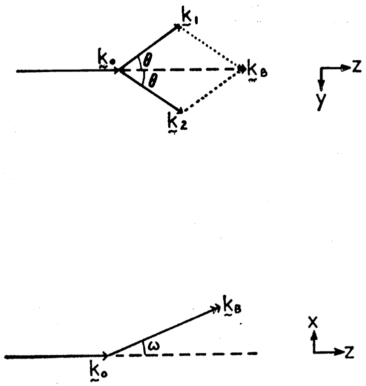

FIG. 1. The scattering system. (In the laboratory system, the scattering angle and the angle of noncoplanarity are represented by the symbols θ_L and ω_L.)

description of noncoplanar scattering. In this paper we give such a description, and discuss the information that can be obtained from noncoplanar scattering.

In the symmetric $(p,2p)$ reaction, the two outgoing protons have equal energies and make equal angles with the incident beam (see Fig. 1). In the center-of-mass system, the momenta of the incoming and outgoing protons are taken to be $\mathbf{k}_0$, $\mathbf{k}_1$, and $\mathbf{k}_2$, respectively, with $|\mathbf{k}_1|=|\mathbf{k}_2|$, so that the momentum of the center of mass of the two outgoing protons (the "di-proton") is $\mathbf{k}_B=\mathbf{k}_1+\mathbf{k}_2$. We take the momenta $\mathbf{k}_0$, k_B to define the xz plane and the angle between them to be the angle of noncoplanarity ω. The cross section for the symmetric noncoplanar $(p,2p)$ reaction is given in distorted-wave impulse approximation by[2]

$$\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE}=K\frac{d\sigma}{d\Omega_{pp}}\sum_m |g_l^m|^2, \tag{1}$$

where K represents kinematical factors, $d\sigma/d\Omega_{pp}$ is the cross section for proton-proton scattering evaluated at 90° in the c.m. system, and g_l^m is given in the di-proton model by[1]

$$g_l^m=\langle\chi_B^-(\mathbf{k}_B,\mathbf{r})|\psi_p(\mathbf{r})\chi_p^+(\mathbf{k}_0,a\mathbf{r})\rangle, \tag{2}$$

where χ_p^+, χ_B^- are distorted waves for the proton and the di-proton, $a=1-1/A$, and ψ_p is the overlap integral[3]

$$\psi_p(\mathbf{r})=\int\Phi_{A-1}^*(\xi)\Phi_A(\xi,\mathbf{r})d\xi, \tag{3}$$

which we shall approximate by a single-particle wave function

$$\psi_l^m(\mathbf{r})=R_{nl}(r)Y_l^m(\hat{r}). \tag{4}$$

In the plane-wave approximation the matrix element g_l^m reduces to

$$g_l^m=\int e^{i\mathbf{Q}\cdot\mathbf{r}}\psi_l^m(\mathbf{r})d\mathbf{r}, \tag{5}$$

where

$$\mathbf{Q}=a\mathbf{k}_0-\mathbf{k}_B \tag{6}$$

is the recoil momentum of the residual nucleus in the laboratory system. We take the z axis along the direction of the incident beam. For symmetric coplanar scattering $(\omega=0)$ the symmetry of the system with respect to rotation about this axis[4] leads to a reduction in the number of terms g_l^m. In the plane-wave approximation and the distorted-wave impulse approximation with the di-proton model, the terms with $m\neq0$ are identically zero. In an exact treatment of the distortion within the framework of the impulse approximation,[4] there are contributions from terms with all even values of m. In symmetric and nonsymmetric coplanar scattering, the rotation symmetry is lost and contributions from all values of m are to be expected.

The effect of noncoplanarity has been discussed before.[5,6] Jacob and Maris[5] used a diffraction model based on strong-absorption theories, but the predictions of their model do not appear to be in agreement with such experimental data[7] as exist at present. Sakamoto[6] used an approximate distorted-wave method and reached the conclusion that if the optical potential is real there are no contributions from $m\neq0$. In Sec. 2 we first give a simple description of symmetric noncoplanar scattering using plane waves and then discuss the effect of distortion. In Sec. 3 we discuss the information obtainable from such scattering.

2. THE MATRIX ELEMENT FOR NONCOPLANAR SCATTERING

In this section, we examine the behavior of the matrix element for noncoplanar scattering. We do this for plane waves first, to demonstrate simply the dependence on the angle of noncoplanarity ω, and then to examine the distorted-wave formulas.

A. Plane-Wave Formulas

In order to evaluate Eq. (5), it is convenient to resolve the recoil momentum $\mathbf{Q}$ into the components

$$Q_x=-k_B\sin\omega,\quad Q_z=ak_0-k_B\cos\omega.$$

We use oscillator functions of the form

$$R_{nl}(r)=A_{nl}r^l\,{}_1F_1\{-(n-1),l+\tfrac{3}{2};r^2/b^2\}e^{-r^2/b^2},$$

and substituting these into Eq. (5) we obtain the following formulas for the matrix element for the first few l values:

$$g_0^0=2\sqrt{2}\pi^{3/4}b^{3/2}e^{-\frac{1}{2}Q_x^2b^2}e^{-\frac{1}{2}Q_z^2b^2},$$

$$g_1^0=4\pi^{3/4}b^{3/2}e^{-\frac{1}{2}Q_x^2b^2}(Q_zb)e^{-\frac{1}{2}Q_z^2b^2},$$

$$g_1^\pm=2\sqrt{2}\pi^{3/4}b^{3/2}(Q_xb)e^{-\frac{1}{2}Q_x^2b^2}e^{-\frac{1}{2}Q_z^2b^2},$$

$$g_2^0=2\sqrt{3}\pi^{3/4}b^{3/2}(1-\tfrac{2}{3}Q_x^2b^2-\tfrac{2}{3}Q_z^2b^2)e^{-\frac{1}{2}Q_x^2b^2}e^{-\frac{1}{2}Q_z^2b^2},$$

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[2] Th. A. J. Maris, P. Hillmann, and H. Tyren, Nucl. Phys. **7**, 1 (1958); K. F. Riley, *ibid.* **13**, 407 (1959).

[3] T. Berggren, Nucl. Phys. **72**, 337 (1965).

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[5] G. Jacob and Th. A. J. Maris, Nucl. Phys. **20**, 40 (1960).

[6] Y. Sakamoto, Nucl. Phys. **46**, 293 (1963).

[7] G. Tibell, O. Sundberg, and U. Miklavzic, Phys. Letters **2**, 100 (1962).

etc., so that

$$\sum_m |g_l^m|^2 = f(Q^2b^2)e^{-Q^2b^2}.$$

For small ω, such that $Q_x^2 \ll 1$, it can easily be shown that the dependence on Q_x is given by $J_0(\sqrt{2}bQ_x)$ for $m=0$ and $J_1(\sqrt{2}bQ_x)$ for $m=\pm 1$. Thus we may conclude that terms with $m\neq 0$ do contribute as soon as $\omega > 0$. Hence even for the coplanar experiment the finite angular resolution will permit small contributions from terms with $m\neq 0$, although the magnitude of these contributions will be negligibly small (except when $Q_z=0$).

B. Distorted-Wave Formulas

In the di-proton model, the matrix element (2) is given by

$$g_l^m = \int \chi_B^{-*}(\mathbf{k}_B,\mathbf{r})\psi_l^m(\mathbf{r})\chi_p^+(\mathbf{k}_0,a\mathbf{r})d\mathbf{r},$$

and using a partial-wave expansion for the distorted waves, this becomes[1]

$$g_l^m = 4\pi \sum_{l_1 l_2} i^{l_1-l_2}(2l_1+1)[(2l+1)/(2l_2+1)]^{1/2}U(l_1l_2nl) \times (l_1 0 l m|l_2 m)(l_1 0 l 0|l_2 0)Y_{l_2}^m(\omega,0), \quad (7)$$

which for small ω can be written as

$$g_l^m \simeq (4\pi)^{1/2}\sum_{l_1 l_2} i^{l_1-l_2}(2l_1+1)(2l+1)^{1/2}U(l_1l_2nl) \times (l_1 0 l m|l_2 m)(l_1 0 l 0|l_2 0)J_m(\{l_2+\tfrac{1}{2}\}\omega), \quad (8)$$

where $U(l_1l_2nl)$ is the integral over the radial parts of the distorted waves and the single-particle wave function,

$$U(l_1l_2nl) = \int_0^\infty f_{l_1}(k_0ar)f_{l_2}(k_Br)R_{nl}(r)r^2dr.$$

Equation (8) reduces to a particularly simple form if one partial wave $l_1=l_0$ gives the dominant contribution to the matrix element. We then have

$$g_l^m \simeq (4\pi)^{1/2}\sum_{l_2=l_0+l}^{l_2=l_0-l} i^{l_0-l_2}(2l_0+1)(2l+1)^{1/2}U(l_0l_2nl) \times (l_0 0 l m|l_2 m)(l_0 0 l 0|l_2 0)J_m(\{l_2+\tfrac{1}{2}\}\omega).$$

For the special case of $l=0$, this reduces to

$$g_0^0 \simeq (4\pi)^{1/2}(2k_BR)U(l_0l_0n0)J_0(k_BR\omega),$$

where we have put $l_0+\frac{1}{2}=k_BR$ and have obtained the dependence on a zero-order Bessel function, as in the previous section. The assumption that a particular partial wave gives the dominant contribution to the matrix element is the condition for the validity of the diffraction model for direct reactions.[8] The diffraction model has been applied to the $(p,2p)$ reaction[1] and it too leads to the result that the dependence of g_l^m on ω is of the form $J_m(k_BR\sin\omega)$.

[8] K. A. Amos, I. E. McCarthy, and K. R. Greider, Nucl. Phys. **68**, 469 (1965).

3. INFORMATION OBTAINABLE FROM NONCOPLANAR SCATTERING

In this section we discuss the information which can be obtained from noncoplanar scattering. We use the result that, in impulse approximation, the cross section for the $(p,2p)$ reaction is expressed in terms of the cross section for proton-proton scattering at 90° in the center-of-mass system, and hence that the proton-proton scattering occurs only in singlet states. Thus our general conclusions are dependent only on the validity of impulse approximation. Particular calculations are carried out using the di-proton model to give predictions for the magnitudes of cross sections for noncoplanar scattering.

As we have seen, the effect of noncoplanarity is to introduce nonzero contributions from all possible substates for a given angular momentum l. In Sec. 2 A it was shown, using plane waves and oscillator functions, that $\sum_m |g_l^m|^2$ is a simple function of Q^2, but the true situation is likely to depart from this simple prediction owing to (i) angular localization caused by the distortion and (ii) departures from sphericity in the target nucleus.

The component ψ_l^0 of the overlap integral which contributes to g_l^0 can in principle be determined from coplanar scattering. This means that for a spherical nucleus the radial function $R_{nl}(r)$ is determined [at least that part of the radial function to which the $(p,2p)$ reaction is sensitive], and hence the other components of the overlap integral ψ_l^m are determined through Eq. (4). Information on $R_{nl}(r)$ is also given, of course, by other nuclear reactions. We assume, therefore, that the components of the overlap integral are known or can be determined. It then follows that a study of noncoplanar scattering in the $(p,2p)$ reaction on a spherical or nearly spherical nucleus will give further information on the effect of distortion and, in particular, on the way distortion affects the population of substates in the final nucleus. For example, if we consider knockout of a $p_{3/2}$ proton from C^{12}, leading to the ground state of B^{11}, or from O^{16}, leading to an excited state of N^{15} with $J_f=\frac{3}{2}^-$, then in the coplanar experiment the only substates to be populated are those with $M_f=\pm\frac{1}{2}$, whereas in the noncoplanar experiment all the substates are populated. This additional information should be expected from the $(p,2p)$ reaction since it is an angular-correlation experiment and, compared with reactions in which only one final particle is emitted, should reveal information concerning the population of nuclear substates, in the same way as the investigation of angular correlation in the $(p,p'\gamma)$ reaction[9] reveals more information than the (p,p') reaction. The formulas

[9] F. H. Schmidt, R. E. Brown, J. B. Gerhart, and W. A. Kolasinski, Nucl. Phys. **52**, 353 (1964), and references cited therein.

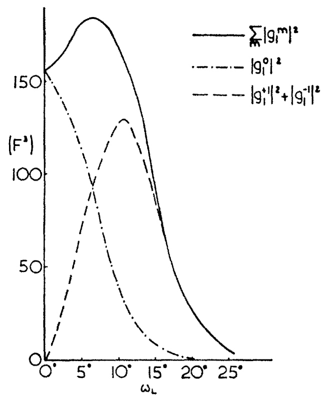

FIG. 2. The components of the nuclear matrix element for noncoplanar scattering calculated using plane waves. The dashed lines show the components for $m=0$ and $m=\pm1$ and the full line is the sum of these. The parameters are those of set B in Table I, the target is Li^6, and the incident energy is 185 MeV. The scattering angle θ_L is 35.5°.

for the $(p,2p)$ reaction are somewhat simpler than those for the $(p,p'\gamma)$ reaction, although in the former case there is at present the implicit complication due to our incomplete understanding of the structure of the overlap integral.[3,10]

From the theoretical point of view the simplest way to investigate such processes would be to compare coplanar scattering, for which Q_x is zero for all θ, with noncoplanar scattering carried out in such a way that Q_z is held at zero; this form of noncoplanar scattering is, however, not very practicable since it requires a simultaneous variation of ω and θ. Also, the finite energy and angular resolution of the experimental apparatus could have a confusing effect when both Q_x and Q_z are near zero. We therefore suggest that the best procedure is to fix the angles θ to give a value of Q_z corresponding to a maximum in the angular distribution for coplanar scattering. For $l\neq0$ there are two maxima and we choose the one corresponding to negative Q_z. As the angle of noncoplanarity ω is increased from zero, the value of $|Q_z|$ falls to zero, while Q_x rises from zero. The plane-wave formulas of Sec. 2 A then lead to the results shown in Fig. 2, while the distorted-wave formula given by Eq. (7) leads to the results shown in Fig. 3. The parameters for these figures are given in Table I. From Fig. 2 it can be seen that, as a result of the geometry we have chosen, the components of the matrix element with $m=0$ and those with $m=\pm1$ are important at different angles, giving rise to a sizeable cross section for noncoplanar angles up to 20°. (The cross sections corresponding to Figs. 2, 3, and 4 can be estimated in μb MeV^{-1} sr^{-2} by multiplying by a factor of 0.8. The correct multiplication factor is not a constant, but is nearly so.) By comparing the full-line curves in Figs. 2 and 3, it can be seen that distortion has the effect of filling in the dip at 0° and spreading out the angular distribution.

TABLE I. Parameters for Figs. 2, 3, and 4.

Curve	Deformation parameter ϵ	Length parameters (F) $a=\alpha^{-1/2}$	$b=\beta^{-1/2}$	(a^2b)
A	+0.4	1.67	2.08	5.8
B	0	1.80	1.80	5.8
C	−0.4	1.91	1.58	5.8

The evidence for the deformation of light and medium nuclei is increasing rapidly. Volkov[11] has made a detailed Hartree-Fock calculation of the equilibrium deformation in the ground state of nuclei in the $1p$ shell and finds prolate deformation favored for $A\leqslant8$, oblate deformation for $9\leqslant A\leqslant13$, while nuclei beyond $A=13$ are essentially spherical. Recent studies[3,10] of the properties of the overlap integral defined in Eq. (3) have shown that the overlap integral is not the same as the radial wave function in a self-consistent Hartree-Fock potential. It is, however, customary in calculations on nuclear reactions to approximate the overlap integral by a single-particle wave function in an effective one-body potential whose parameters are adjusted to give the correct separation energy for the particular reaction and, since nuclear-structure calculations indicate the nonsphericity of the Hartree-Fock potential, we should also use a nonspherical one-body potential.

For the $1p$-shell nuclei, Volkov uses the wave functions

$$\phi_0=C_0\beta^{1/2}ze^{-\frac{1}{2}\beta z^2}e^{-\frac{1}{2}\alpha(x^2+y^2)},$$
$$\phi_{\pm1}=C_{\pm1}\alpha^{1/2}(x\pm iy)e^{-\frac{1}{2}\alpha(x^2+y^2)}e^{-\frac{1}{2}\beta z^2},$$

where α,β are related to the deformation parameter ϵ through the relation

$$\frac{\alpha}{\beta}=\frac{1+\frac{1}{3}\epsilon}{1-\frac{2}{3}\epsilon}.$$

These wave functions can be rewritten in the form

$$\phi_0=C_0(\beta/\alpha)^{1/2}\alpha^{1/2}ze^{-\frac{1}{2}\alpha(x^2+y^2+z^2)}e^{\frac{1}{2}(\alpha-\beta)z^2},$$
$$\phi_{\pm1}=C_{\pm1}\alpha^{1/2}(x\pm iy)e^{-\frac{1}{2}\alpha(x^2+y^2+z^2)}e^{\frac{1}{2}(\alpha-\beta)z^2},$$

or, expanding the last factor,

$$\phi_\nu=C_\nu(\beta/\alpha)^{(1-|\nu|)/2}\alpha^{1/2}e^{-\frac{1}{2}\alpha r^2}\sum_n A_n r^{2n+1}(Y_1^0)^{2n}Y_1^\nu,$$

from which it can be seen that the effect of deformation

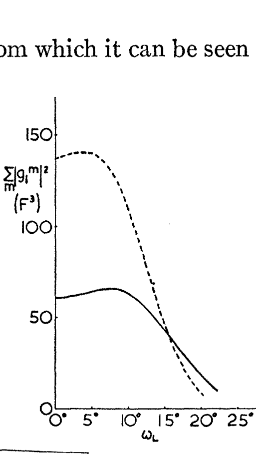

FIG. 3. The nuclear matrix element for noncoplanar scattering calculated using distorted waves. The target is Li^6, the incident energy is 185 MeV, and the optical potential parameters are taken from Ref. 9. The scattering angle θ_L is 35.5°. For the full line the length parameter is the same as in Fig. 2, while for the dashed line the length parameter has been increased to 2.24 F.

[10] W. T. Pinkston and G. R. Satchler, Nucl. Phys. **72**, 641 (1965); N. Austern, Phys. Rev. **136**, B1743 (1964).

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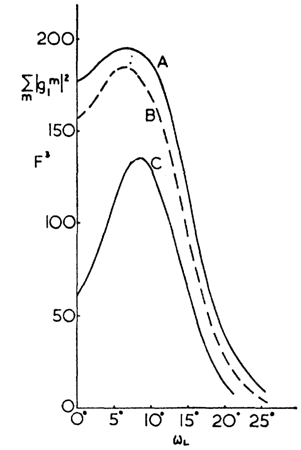

FIG. 4. The effect of deformation on noncoplanar scattering calculated using plane waves. The incident energy is 185 MeV and the parameters are given in Table I. The scattering angle θ_L is 35.5°.

is to change the normalization and to introduce admixtures of higher shell-model configurations. The wave functions of Volkov, which are the same as those in Appendix A of Nilsson's paper,[12] are defined in the body-fixed system which must be rotated into the space-fixed system before the matrix element can be calculated. This involves no essential complication. A lowest-order estimate, using plane waves, of the effect of deformation for noncoplanar scattering is given in Fig. 4, and this indicates that the effect is sizeable. From this estimate we may conclude that a calculation using spherical wave functions for a nucleus which has positive deformation will lead to an underestimate for the magnitude of the cross section and therefore an overestimate of the spectroscopic factor, and vice versa in the case of a nucleus of negative deformation. A similar conclusion has been reached by Rost[13] in a detailed distorted-wave analysis of the stripping reaction $O^{18}(He^3,d)$.

For nonspherical nuclei it is no longer true that a knowledge of one component of the overlap integral determines the other components since the simple representation given by Eq. (4) is not valid, but a combined study of coplanar and noncoplanar scattering in the (*p*,2*p*) reaction can provide a powerful method of investigating the spatial distribution of bound protons in nonspherical nuclei. The oscillator functions we have used to obtain the results shown in Figs. 2, 3, and 4 do not have the correct asymptotic behavior for the overlap integral and must be replaced by functions whose asymptotic behavior is related to the proton separation energy.[3,10] It is encouraging, therefore, that methods are now being developed for obtaining wave functions in a deformed finite potential.[13,14] In addition, the distorted-wave analysis of the (*p*,2*p*) reaction has reached the stage at which it is feasible to attempt the determination of spectroscopic factors.[15] What is now required is an experimental study of coplanar and noncoplanar scattering on suitably selected spherical and nonspherical nuclei with the emphasis on an accurate determination of the absolute magnitude of the cross section and resolution of protons leaving the residual nucleus in the ground and low-lying excited states.

The requirement that impulse approximation should be valid places a rather fundamental restriction on the energy region in which the proposed experiments should be carried out. This restriction arises because it is essential to the discussion presented above that the proton-proton scattering take place in singlet even states. However, the calculations of Lim and McCarthy[16] using their distorted-wave *t*-matrix approximation indicate that contributions from triplet odd scattering may occur when full account is taken of the finite range of the two-body potential, its exchange character, and of antisymmetrization. These authors conclude that the triplet scattering decreases with increasing energy of the incident proton and quote the ratio of the triplet to singlet scattering at 143 MeV as 4%. We may therefore estimate with reasonable confidence that the conclusions reached here on the basis of distorted-wave impulse approximation are valid for experiments carried out with incident protons with energies of 150 MeV or above. The extension of the investigation to lower energies requires a more exact treatment of the interaction.

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[13] E. Rost, Phys. Letters **21**, 87 (1966).

[14] E. Rost and G. E. Brown, Bull. Am. Phys. Soc. **10**, 487 (1965); P. Röper (unpublished report).

[15] B. K. Jain and D. F. Jackson (to be published).

[16] K. L. Lim and I. E. McCarthy, Phys. Rev. Letters **13**, 446 (1964); Nucl. Phys. **88**, 433 (1966).