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Statistical Description of the Complex-Boundary-Value Problem

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The statistical properties of the parameters of the statistical collision matrix defined in terms of the eigenstates of a complex-boundary-value problem is studied starting from the Hamiltonian of the system. It is shown that the random-matrix hypothesis can be used to calculate the statistical distribution of quantities such as the complex amplitude. An explicit calculation is carried out for the special case of two dimensions. As a check on the theoretical calculation, it is shown that the results of the real-boundary-value problem follow by suitably choosing a parameter.

I. INTRODUCTION

RECENTLY Moldauer[1] has developed a formalism to study the energy averages and the fluctuations of the nuclear collision cross sections. A statistical collision matrix is defined in terms of the eigenstates of a complex-boundary-value problem to calculate these averages and the fluctuations. No attempt has been made to study the statistical properties of the statistical collision matrix starting from the Hamiltonian of the system.

The random-matrix hypothesis[2] has been used in the past to study the joint probability distribution of the Hamiltonian matrix elements, expectation-value fluctuations, etc. We shall show that the random-matrix hypothesis can also be used to study the statistical properties of the eigenstates of a complex-boundary-value problem.

The starting point is the eigenvalue equation in the interior region of the configuration space[1]

$$HX_\mu = W_\mu X_\mu\,, \tag{1}$$

where H is the complete Hamiltonian of the system and X_μ, W_μ are its eigenstates and eigenvalues, respectively. The eigenvalue equation (1) is solved by specifying certain complex-boundary conditions at the dividing surface. It is assumed that the Hamiltonian H is invariant under rotations and under time reversal and that we are working with a submatrix of the total Hamiltonian matrix belonging to a particular symmetry type. As in the case of real-boundary conditions,[3] we expand the eigenfunctions in terms of a convenient orthonormal basis set

$$X_\mu = \sum_\nu a_{\nu\mu}\Phi_\nu\,, \tag{2}$$

where $a_{\nu\mu}(1\leqslant\nu\leqslant N)$ are the components of the eigenvector belonging to eigenvalue W_μ. It has been shown[2] that if the Hamiltonian is invariant under rotations as well as time reversal, then the basis set Φ_ν can be so chosen that

$$K\Phi_\nu = \Phi_\nu\,, \tag{3}$$

where K is the time-reversal operator.

We now show that the matrix formed from the coefficients $a_{\nu\mu}$ is a complex orthogonal matrix. To show this, we substitute the expansion of X_μ given by Eq. (2) in the relation[1]

$$\int_{\text{interior}} \tilde{X}_\mu{}^* X_\nu d\tau = \delta_{\mu\nu}\,, \tag{4}$$

where the wave function $\tilde{X}_\mu$ is obtained from X_μ using the time-reversal operator.[1] This together with relation (3) gives

$$\sum_\alpha \tilde{a}_{\mu\alpha} a_{\alpha\nu} = \delta_{\mu\nu}\,, \tag{5}$$

which proves the assertion.

The random-matrix hypothesis[2] used in the present case will imply that the matrix formed from the components $a_{\nu\mu}$ will be a random complex orthogonal matrix. Therefore, to find the statistical distribution of any quantity which is a function of $a_{\nu\mu}$, we need to know the volume element of the complex orthogonal matrix space. The quantity which is of particular interest in connection with the statistical collision matrix is the complex amplitude defined by[1]

$$\theta_{\mu c} = \left(\frac{\hbar^2}{2M_c a_c}\right)^{1/2} \int_{\text{surface}} \phi_c{}^* X_\mu dS\,, \tag{6}$$

where c denotes the channel, M_c the reduced mass, a_c the channel radius, and ϕ_c the channel wave function.

Putting the expansion given by the relation (2) into Eq. (6), we get

$$\theta_{\mu c} = \sum_\nu a_{\nu\mu} J_{\nu c}\,, \tag{7}$$

where $J_{\nu c}$ is given by

$$J_{\nu c} = \left(\frac{\hbar^2}{2M_c a_c}\right)^{1/2} \int_{\text{surface}} \phi_c{}^* \Phi_\nu dS\,. \tag{8}$$

From the fact that[1]

$$\tilde{\theta}_{\mu c}{}^* = \theta_{\mu c}\,, \tag{9}$$

it is easy to see that $J_{\nu c}$ is real.

[1] P. A. Moldauer, Phys. Rev. **135**, B642 (1964).
[2] C. E. Porter, *Statistical Theories of Spectra, Fluctuations* (Academic Press Inc., New York, 1965); N. Rosenzweig, *Brandeis University Summer Institute Lectures in Theoretical Physics, 1962 Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. 3, p. 91.
[3] N. Ullah, Nucl. Phys. **64**, 349 (1965).

We remark here that another way to study the statistical distribution of the parameters of the statistical collision matrix is to connect its parameters with the parameters of the real-boundary-value problem, the statistical properties of which have been studied in the past. Such an attempt has been made by Moldauer,[1] by introducing a transformation matrix T. The expression obtained by him for the complex amplitude $\theta_{\mu c}$ involving the T matrix looks similar to Eq. (7). However, the statistical properties of the T matrix have not been studied. It seems to be much simpler to study the statistical properties of the complex orthogonal matrix a which we have introduced.

II. CALCULATION OF THE PARAMETERS USING THE RANDOM COMPLEX ORTHOGONAL MATRIX

In this section we shall study the statistical properties of certain parameters of the statistical collision matrix. To illustrate the main points of the calculation we shall restrict outselves to a special case of a 2×2 random complex orthogonal matrix. This can be conveniently parametrized as

$$a=\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}, \qquad (10)$$

where

$$\omega=\omega_1+i\omega_2. \qquad (11)$$

The line element dS^2 is by definition[2]

$$dS^2=\mathrm{Tr}\; dada^\dagger. \qquad (12)$$

Using Eq. (10), it can be expressed as

$$dS^2=[\exp(2\omega_2)+\exp(-2\omega_2)](d\omega_1{}^2+d\omega_2{}^2). \qquad (13)$$

Therefore the volume element $\dot{a}$ can be written as

$$\dot{a}=[\exp(2\omega_2)+\exp(-2\omega_2)]d\omega_1 d\omega_2, \qquad (14)$$

where

$$-\pi\leqslant\omega_1\leqslant\pi;\quad -\infty\leqslant\omega_2\leqslant\infty. \qquad (15)$$

From the relations (14) and (15) we see that the total volume of the complex orthogonal space is not bounded and therefore the normalization integral will diverge. This means that we cannot simply take the probability density $P(a)$ proportional to $\dot{a}$ throughout the range given in (15). To get any meaningful results out of this theoretical formulation we should therefore introduce some kind of weight factor which ensures the convergence of the normalization integral. At this stage we do not want to go into the question of choosing a proper form of the weight function, which will be taken up when the formulation is generalized to N dimensions. For the sake of the present calculation we consider two kinds of weight factors: (a) unit step function in the range $-q\leqslant\omega_2\leqslant q$, and (b) the Gaussian weight factor $\exp(-p\omega_2{}^2)$. The parameter p or q will be obtained by comparing some theoretically predicted quantity with its experimental value.

Using Eqs. (7), (10), and (14) we find for case (b)

$$\langle|\theta_{\mu c}|^{2n}\rangle_\mu=(\tfrac{1}{2}J_c)^n\exp(-p^{-1})\sum_{s=0}^{n}\left\{\left[\binom{n}{s}\right]^2 \times\exp[(n-2s+1)^2p^{-1}]\right\}, \qquad (16)$$

where $\langle\,\rangle_\mu$ denotes the ensemble average,

$$J_c=\tfrac{1}{2}(J_{1c}{}^2+J_{2c}{}^2),\quad\text{and}\quad \binom{n}{s}$$

is the binominal coefficient.

For case (a), if $n=2m$, then we have

$$\langle|\theta_{\mu c}|^{4m}\rangle_\mu=(\tfrac{1}{2}J_c)^{2m}[\exp(2q)-\exp(-2q)]^{-1} \times\sum_{s=0}^{2m}\left\{\left[\binom{2m}{s}\right]^2\times(2m-2s+1)^{-1} [\exp[2q(2m-2s+1)] -\exp[-2q(2m-2s+1)]]\right\}, \qquad (17a)$$

and if $n=2m+1$, then

$$\langle|\theta_{\mu c}|^{4m+2}\rangle_\mu=\tfrac{1}{2}(\tfrac{1}{2}J_c)^{2m+1}[\exp(2q)-\exp(-2q)]^{-1} \times\left(\sum_{s=0}^{2m+1}{}'\left\{\left[\binom{2m+1}{s}\right]^2(m-s+1)^{-1} \times[\exp[4q(m-s+1)]-\exp[-4q(m-s+1)]]\right\}+8 \times\left[\binom{2m+1}{m+1}\right]^2 q\right), \qquad (17b)$$

where a prime on the summation over s indicates that the term $s=m+1$ has to be excluded.

As a check on our formulation we show that the limit $p\to\infty$ and Eq. (16) or the limit $q\to0$ and Eqs. (17a) and (17b) give the same width distribution as is obtained using the real-boundary conditions in the special case of two dimensions.[4] It is easy to see that both Eqs. (15) and (17) yield the same value of $\langle|\theta_{\mu c}|^{2n}\rangle_\mu$ if this limiting process is carried out. It is given by

$$\langle|\theta_{\mu c}|^{2n}\rangle_\mu=(\tfrac{1}{2}J_c)^n\binom{2n}{n}. \qquad (18)$$

Using the method of moments,[5] the width distribution is given by

$$P(x)dx=[\pi^2x(2-x)]^{-1/2}dx;\quad 0\leqslant x\leqslant 2, \qquad (19)$$

where $x=\Gamma_{\mu c}/\langle\Gamma_{\mu c}\rangle_\mu$ is the ratio of the width to the average width. This is the exact relation for the real-boundary conditions in the special case of two dimensions.[4]

[4] N. Ullah, J. Math. Phys. (to be published).

[5] M. G. Kendall, *The Advanced Theory of Statistics* (Charles Griffin and Company, Ltd., London, 1945), Vol. I, Chap. 4.

An application of this theoretical formulation enables us to express the two parameters B_c, A_c defined by Moldauer[1] in terms of a single parameter p or q. They are given by

$$B_c=\left[\frac{|\langle\theta_{\mu c}{}^2\rangle_\mu|}{\langle|\theta_{\mu c}|^2\rangle_\mu}\right]^2, \tag{20}$$

$$A_c=\frac{\langle|\theta_{\mu c}|^4\rangle_\mu}{[\langle|\theta_{\mu c}|^2\rangle_\mu]^2}. \tag{21}$$

Using Eqs. (17a) and (17b), we get for case (a)

$$B_c=16[\exp(2q)-\exp(-2q)]^2 \times[\exp(4q)-\exp(-4q)+8q]^{-2}, \tag{22a}$$

$$A_c=\tfrac{4}{3}[\exp(2q)-\exp(-2q)]^2 \times[\exp(4q)-\exp(-4q)+8q]^{-2} \times[\exp(4q)+\exp(-4q)+16]. \tag{22b}$$

For case (b), using Eq. (16), we get

$$B_c=4\exp(2p^{-1})[\exp(4p^{-1})+1]^{-2}, \tag{23a}$$

$$A_c=\exp(2p^{-1})[\exp(4p^{-1})+1]^{-2}[\exp(8p^{-1})+5]. \tag{23b}$$

It is interesting to note from Eqs. (22b) and (23b) that if we carry out the limiting process indicated earlier, then $A_c=1.5$, which is in agreement with the value of A_c which results from the exact distribution of the width in two dimensions, given by Eq. (19), for the case of real-boundary condition.

As a further application, we consider the average value of the normalization constant[1]

$$N_\mu=\int_{\text{interior}}|X_\mu|^2d\tau. \tag{24}$$

With the help of Eqs. (2), (10), and (14) we see that $\langle N_\mu\rangle_\mu$ for case (a) is given by

$$\langle N_\mu\rangle_\mu=\tfrac{1}{4}[\exp(4q)-\exp(-4q)+8q] \times[\exp(2q)-\exp(-2q)]^{-1}. \tag{25}$$

For case (b), it is given by

$$\langle N_\mu\rangle_\mu=\tfrac{1}{2}[\exp(4p^{-1})+1]\exp(-p^{-1}). \tag{26}$$

Using Eqs. (25) and (26) and the limit $p\to\infty$, $q\to 0$, we get $\langle N_\mu\rangle_\mu=1$, which checks with the result of the real-boundary condition.

An extension of this formulation to N dimensions is presented in the following paper, and detailed application to the fluctuations of cross sections will be presented in a later article.

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Statistical Description of the Complex-Boundary-Value Problem. II. Distribution of the Parameters of the Collision Matrix

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A detailed statistical study is made of the parameters of the statistical collision matrix using the N-dimensional random complex orthogonal matrix. It is shown that, even without a complete knowledge of weight function which has to be introduced for the convergence of the normalization integral, certain relations between the average values of the parameters of the statistical collision matrix can be obtained and a statement can be made that the channel correlations of the parameters are always positive. A suitable form of the weight function is guessed, and the distributions of the parameters are also given. It is shown that under certain conditions the distribution of the parameters is close to the Porter-Thomas distribution except for small values. The resonance correlations of the parameters are also studied. Excellent agreement has been obtained between the values predicted by the present theoretical formulation and those obtained by a numerical calculation using the parameters of the real-boundary-value problem and a certain transformation matrix.

I. INTRODUCTION

IN an earlier paper[1] we showed that the random-matrix hypothesis can be used to study the statistical properties of the statistical collision matrix introduced by Moldauer.[2] The earlier work[1] was intended to give the basic idea and its application to a simple two-dimensional case. In the applications of the statistical collision matrix to the study of the energy averages, fluctuations, and the correlations of the nuclear collision cross sections,[2] we need the N-dimensional generalization of the simple case discussed earlier. In this paper we shall study the distribution of the parameters of the statistical collision matrix. The results will be compared with the

[1] N. Ullah, preceding paper, Phys. Rev. **154**, 897 (1967).
[2] P. A. Moldauer, Phys. Rev. **135**, B642 (1964).