PHYSICAL REVIEW VOLUME 154, NUMBER 2 10 FEBRUARY 1967

Nonlocal Damping of Helicon Waves

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(Received 16 August 1966)

Nonlocal damping of a helicon wave propagating at an angle to a static magnetic field is analyzed. It is shown that in metals such damping is primarily caused by transit-time effects rather than by "Landau damping." An expression for the damping is derived for all values of the electron mean free path, provided the helicon wavelength is much longer than the cyclotron radius.

INTRODUCTION

WHEN a helicon wave propagates in a metal (with a spherical Fermi surface) along a static magnetic field, it can be damped by two processes[1]: collisional damping and absorption caused by Doppler-shifted cyclotron resonance. The ever-present collisional damping is caused by the scattering of electrons by impurities, defects, and phonons. Its measure is the quantity $\omega_c\tau$, where $\omega_c=eB/mc$ is the carrier cyclotron frequency and τ the relaxation time. Collisional damping is small when $\omega_c\tau$ is large.

The second type of damping occurs only under suitable conditions, namely, when there exist carriers within the metal whose velocity along the magnetic field is such that they experience a Doppler-shifted cyclotron resonance.[2] Since such carriers are very effective in extracting energy from the wave, the damping, when it exists, is extremely severe. When the electron distribution function is the Fermi distribution, the onset of the damping (as the frequency of the wave or the magnetic field are varied) is precipitous. For metallic densities, the quantity kR is a measure of this effect. Here k is the wave vector of the helicon wave, and $R=v_F/\omega_c$ is the cyclotron radius where v_F is the Fermi velocity. When $kR<1$, there is no cyclotron damping, and the wave, except for collisional damping, propagates unattenuated. When $kR\gtrsim 1$, the damping is so severe that the helicon ceases to be a well-defined excitation.

When the helicon wave propagates at an angle to the magnetic field ($0<\Delta<\pi/2$,) it is no longer a purely transverse wave and two other sources of damping can exist. The existence of a (small) longitudinal electric field leads to so-called Landau damping which is quite analogous to the (Landau) damping of longitudinal plasma oscillations. We show, however, that this form of damping is negligibly small. In addition, the magnetic field of the wave is now such that when added to the static magnetic field, it causes the carriers to experience a moving periodic mirror field, which, in turn, leads to what we shall call *magnetic* Landau damping.[3] We will show that this is the dominant damping mechanism.

A measure of the effectiveness of both types of damping is the quantity kl, where $l=v_F\tau$ is the mean free path for collisions. The larger kl is, the more effective is the resonant damping. Kaner and Skobov[4] have analyzed the damping of helicon waves (which they called "Landau damping") in the limit $kl\rightarrow\infty$. Unfortunately, under most experimental conditions kl is not very large since, by definition, $kl=(\omega_c\tau)(kR)$. Thus, kl must be smaller than $\omega_c\tau$ because kR must be kept less than unity if cyclotron damping is to be avoided.

In the present paper we present a derivation of the helicon dispersion relation in the limit $kR<1$, $\omega_c\tau>1$, but for arbitrary values of kl. In the limit $kl\rightarrow\infty$, our results differ from those of Kaner and Skobov for reasons which are discussed later in the paper.

HELICON DISPERSION RELATION

The propagation of a plane monochromatic wave of the form $\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r}]$ in an infinite translationally invariant metal is governed by Maxwell equations which reduce to

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E})+k_0^2\boldsymbol{\varepsilon}(k,\omega)\mathbf{E}=0\,,\tag{1}$$

where

$$\epsilon_{\alpha\beta}(\mathbf{k},\omega)=\delta_{\alpha\beta}+\frac{4\pi\sigma_{\alpha\beta}(\mathbf{k},\omega,B_0)}{i\omega}\,.\tag{2}$$

The quantity $\sigma_{\alpha\beta}$ is the wave-number– and wave-frequency–dependent conductivity of the medium. The vector $\mathbf{E}$ is the high-frequency electric field, and B_0 is the applied static magnetic field. In a metal the delta function in (2) can be neglected.

Setting the determinant of the coefficients in Eq. (1) equal to zero yields the dispersion relation

$$Ak^4-Bk^2+C=0\,,\tag{3}$$

[1] See, for example, *Proceedings of the Symposium on Plasma Effects in Solids, Paris, 1964*, edited by J. Bok (Academic Press Inc., New York, 1965).

[2] P. B. Miller and R. R. Haering, Phys. Rev. **128**, 126 (1962); P. M. Platzman and S. J. Buchsbaum, *ibid.* **132**, 2 (1963); C. C. Grimes and S. J. Buchsbaum, Phys. Rev. Letters **12**, 357 (1964); M. T. Taylor, Phys. Rev. **137**, A1145 (1965).

[3] T. H. Stix [*The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), p. 206] calls this damping "transit-time damping." We believe that the name "magnetic Landau damping" which was recently suggested to us by Dr. Stix, is more appropriate.

[4] E. A. Kaner and V. G. Skobov, Zh. Eksperim. i Teor. Fiz. **45**, 610 (1963) [English transl.: Soviet Phys.—JETP **18**, 419 (1964)].

where

$$A=\epsilon_{zz}\cos^2\Delta+2\epsilon_{xz}\sin\Delta\cos\Delta+\epsilon_{xx}\sin^2\Delta\,, \tag{4a}$$

$$B=(\omega^2/c^2)[\epsilon_{zz}(\epsilon_{xx}+\cos^2\Delta\epsilon_{yy})+(\epsilon_{xy}\sin\Delta-\epsilon_{yz}\cos\Delta)^2 +(\epsilon_{xx}\epsilon_{yy}\sin^2\Delta+2\epsilon_{yy}\epsilon_{xz}\sin\Delta\cos\Delta)-\epsilon_{xz}^2]\,, \tag{4b}$$

$$C=(\omega^4/c^4)\det|\epsilon|\,. \tag{4c}$$

The conductivity tensor $\boldsymbol{\sigma}$, for a free-electron gas whose distribution function is the Fermi distribution, has been evaluated using the Boltzmann equation by others.[5] In a Cartesian coordinate system with $\mathbf{B}_0$ along the z axis and the propagation vector $\mathbf{k}$ in the x-z plane, the elements of $\boldsymbol{\sigma}$ are given by

$$\sigma_{xx}=\frac{N}{i}\sum_{n=0}^{\infty}n^2\int_0^{\pi}\frac{a}{b^2}\frac{J_n^2(b)\sin^3\theta d\theta}{(a^2-n^2)}\,, \tag{5a}$$

$$\sigma_{yy}=\frac{N}{i}\sum_{n=0}^{\infty}\int_0^{\pi}\frac{a|J_n'(b)|^2\sin^3\theta d\theta}{(1+\delta_{n0})(a^2-n^2)}\,, \tag{5b}$$

$$\sigma_{zz}=\frac{N}{i}\sum_{n=0}^{\infty}\int_0^{\pi}\frac{aJ_n^2(b)\cos^2\theta\sin\theta d\theta}{(1+\delta_{n0})(a^2-n^2)}\,, \tag{5c}$$

$$\sigma_{xy}=-\sigma_{yx}=N\sum_{n=0}^{\infty}\int_0^{\pi}\frac{a^2}{b}\frac{J_n(b)J_n'(b)\sin^3\theta d\theta}{(1+\delta_{n0})(a^2-n^2)}\,, \tag{5d}$$

$$\sigma_{yz}=-\sigma_{zy}=-N\sum_{n=0}^{\infty}\int_0^{\pi}\frac{aJ_n(b)J_n'(b)\sin^2\theta\cos\theta d\theta}{(1+\delta_{n0})(a^2-n^2)}\,, \tag{5e}$$

$$\sigma_{xz}=\sigma_{zx}=\frac{N}{i}\sum_{n=0}^{\infty}n^2\int_0^{\pi}\frac{J_n^2(b)\sin^2\theta\cos\theta d\theta}{b(a^2-n^2)}\,. \tag{5f}$$

Here

$$N=(3ne^2/m\omega_c)\,,$$
$$a=(\omega-i\nu_c-kv_F\cos\theta\cos\Delta)/\omega_c\,,$$
$$b=(kv_F\sin\theta\sin\Delta)/\omega_c\,,$$

where n is the electron density, v_F is the Fermi velocity, and Δ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$.

In the limit $\omega_c\tau\gg1$, and $kv_F/\omega_c=kR\ll1$, the Bessel functions can be expanded and the integration over θ performed. We then find, for $\omega\ll\omega_c$,

$$\sigma_{xx}=\frac{ne^2}{m\omega_c}\left\{\frac{1}{\omega_c\tau}[1+\mathcal{O}(k^2R^2)]\right\}\,, \tag{6a}$$

$$\sigma_{yy}=\frac{ne^2}{m\omega_c}\left\{\frac{1}{\omega_c\tau}+\tfrac{3}{8}F(\epsilon)|kR|(\cos\Delta)\tan^2\Delta +\mathcal{O}((kR)^3)+\mathcal{O}\left(\frac{1}{\omega_c\tau}(kR)^2\right)\right\}\,, \tag{6b}$$

$$\sigma_{zz}=\frac{3ne^2}{m\omega_c}\frac{1}{(\omega_c\tau)(kR\cos\Delta)^2}[H(\epsilon)+\mathcal{O}((kR)^2)]\,, \tag{6c}$$

$$\sigma_{xy}=-\frac{ne^2}{m\omega_c}[1+\mathcal{O}(kR)^2]\,, \tag{6d}$$

$$\sigma_{xz}=\mathcal{O}\left(\frac{ne^2}{m\omega_c}\frac{(k\mathrm{R})^2}{\omega_c\tau}\right)\,, \tag{6e}$$

$$\sigma_{yz}=-\frac{ne^2}{m\omega_c}\frac{3}{4}(\tan\Delta)G(\epsilon)[1+\mathcal{O}((kR)^2)]\,, \tag{6f}$$

where

$$\epsilon=(kl\cos\Delta)^{-1}$$

and

$$F(\epsilon)=\int_{-1}^{1}\frac{\epsilon(1-x^2)^2}{x^2+\epsilon^2}dx =2(1+\epsilon^2)^2\tan^{-1}(1/\epsilon)-2\epsilon(5/3+\epsilon^2)\,, \tag{7a}$$

$$F(\epsilon\to\infty)=\frac{16}{15}\frac{1}{\epsilon}-\frac{16}{105}\frac{1}{\epsilon^3}\,,$$

$$F(\epsilon\to0)=\pi-\frac{16}{3}\epsilon+2\pi\epsilon^2-\cdots\,,$$

$$G(\epsilon)=\int_{-1}^{1}\frac{x^2(1-x^2)}{x^2+\epsilon^2}dx =\tfrac{4}{3}+2\epsilon^2-2\epsilon(1+\epsilon^2)\tan^{-1}(1/\epsilon)\,, \tag{7b}$$

$$G(\epsilon\to\infty)=\frac{4}{15}\frac{1}{\epsilon^2}-\frac{4}{35}\frac{1}{\epsilon^4}+\cdots\,,$$

$$G(\epsilon\to0)=\tfrac{4}{3}-\pi\epsilon+4\epsilon^2+\cdots\,,$$

$$H(\epsilon)=1-\epsilon\tan^{-1}(1/\epsilon)\,,$$

$$H(\epsilon\to\infty)=\frac{1}{3}\frac{1}{\epsilon^2}-\frac{1}{5}\frac{1}{\epsilon^4}\,, \tag{7c}$$

$$H(\epsilon\to0)=1-\pi\epsilon+\epsilon^2+\cdots\,.$$

It is clear from Eqs. (6a)–(6f) that if we drop terms or order $(kR)^2$ and smaller, the coefficients A, B, and C [Eq. (4)] simplify considerably. We find to order kR that it is sufficient to set

$$A=\epsilon_{zz}\cos^2\Delta\,, \tag{8a}$$

$$B=(\omega^2/c^2)[\epsilon_{zz}(\epsilon_{xx}+\cos^2\Delta\epsilon_{yy}) +(\epsilon_{xy}\sin\Delta-\epsilon_{yz}\cos\Delta)^2]\,, \tag{8b}$$

$$C=(\omega^4/c^4)\epsilon_{zz}\epsilon_{xy}^2\,. \tag{8c}$$

It is now a simple matter to find the dispersion relation for the helicon wave and, in particular, its damping. We find

$$k^2\cong k_H^2(1-i\Gamma)\,, \tag{9}$$

where

$$k_H^2\equiv\frac{\omega^2}{c^2}\frac{\omega_p^2}{\omega\omega_c}\frac{1}{\cos\Delta} \tag{10}$$

[5] M. Cohen, M. Harrison, and W. Harrison, Phys. Rev. **117**, 937 (1960).

is the well-known dispersion relation for helicon-wave propagation in the local limit, and

$$\Gamma \cong \frac{B}{2Ak_H^2} \cong \frac{1}{\omega_c\tau}\frac{[1+\cos^2\Delta]}{2\cos\Delta}+\frac{3}{16}F(\epsilon)\sin^2\Delta|kR| + \frac{\cos\Delta\sin^2\Delta}{6H(\epsilon)}(kR)^2(\omega_c\tau)[\tfrac{3}{4}G(\epsilon)-1]^2. \quad (11)$$

In our notation, the expression for Γ which Kaner and Skobov derived in the limit $kl \gg 1$ is

$$\Gamma_{\mathrm{KS}} = (\omega_c\tau\cos\Delta)^{-1}+(3\pi/16)\sin^2\Delta|kR|. \quad (12)$$

It is not clear how these authors arrived at the above expression, but presumably they replaced $F(\epsilon)$ and $G(\epsilon)$ by their limiting values π and $\frac{4}{3}$, respectively. Unfortunately this procedure is not quite correct. For example, in the limit of large (kl), $[G(\epsilon)-\frac{4}{3}]$ is of order $(1/kl)^2$ so that the third term in Eq. (11) is of order $(\omega_c\tau)^{-1}$, that is, of the same order as the first, the collisional damping, term. The same remarks apply to the $F(\epsilon)$ term. Thus, we find in the limit $\epsilon\rightarrow 0$, i.e., $kl\cos\Delta \gg 1$,

$$\Gamma = \frac{1}{\omega_c\tau\cos\Delta}\left[1-\frac{3}{2}\left(1-\frac{\pi^2}{16}\right)\sin^2\Delta\right] + \frac{3\pi}{16}\sin^2\Delta|kR|. \quad (13)$$

Similarly, in the nearly local limit ($\epsilon\rightarrow\infty$, $kl\cos\Delta \ll 1$),

$$\Gamma = \frac{1}{\omega_c\tau\cos\Delta}\left[1+\frac{3}{10}(kl\sin\Delta\cos\Delta)^2\right]. \quad (14)$$

Equation (14) is actually quite accurate for fairly large kl ($kl\cos\Delta \simeq 1$). For example, when $kl\cos\Delta = 1$, we find from the exact Eq. (11)

$$\Gamma_{\epsilon=1} = \frac{1}{\omega_c\tau\cos\Delta}[1+0.25\sin^2\Delta], \quad (15)$$

which is nearly what Eq. (14) would predict.

In Fig. (1) we have plotted the quantity $\chi(\epsilon)$ where

$$\Gamma(\epsilon) \equiv \frac{1}{\omega_c\tau\cos\Delta}[1+\chi(\epsilon)\sin^2\Delta]. \quad (16)$$

The comparison between experimentally measured absorption, for finite kl,[6,7] at angles other than zero and theory should be re-examined in the light of these results.

[6] C. C. Grimes, Bull. Am. Phys. Soc. 11, 570 (1966).
[7] J. R. Houck and R. Bowers, Bull. Am. Phys. Soc. 11, 256 (1966).

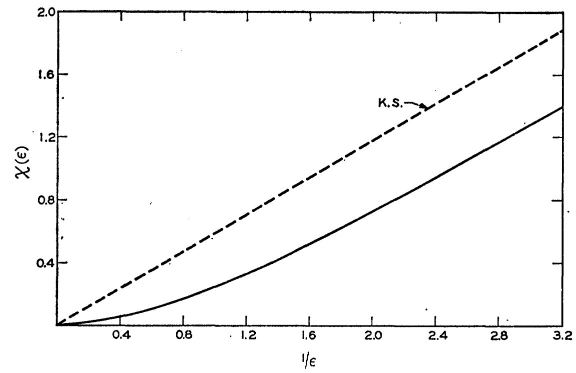

FIG. 1. The solid curve is a plot of the function $\chi(\epsilon)$ as defined in Eq. (16) of the text. The dashed line represents the damping as given by Kaner and Skobov.

SIMPLE MODEL FOR THE DAMPING

What is the origin of the damping represented by the various terms in Eq. (11)? In this section we show that it is, to a good approximation, caused by "magnetic Landau damping" mentioned in the Introduction.

Consider the z-component of the equation of motion of a charge carrier of mass m and charge e:

$$m\frac{dv_z}{dt} = F_z = eE_z + \frac{e}{c}(\mathbf{v}\times\mathbf{B})_z. \quad (17)$$

The force F_z which acts on the particle in the z direction, that is, along the static magnetic field, is made up of two parts: a part due to the small but finite E_z and a magnetic part $(e/c)(\mathbf{v}\times\mathbf{B})_z$. The magnetic force, when the field varies slowly in space, can, to a good approximation, be written as[8]

$$(e/c)(\mathbf{v}\times\mathbf{B})_z \cong \mu\frac{\partial B_z}{\partial z}. \quad (18)$$

Here the quantity $\mu = \frac{1}{2}mv_\perp^2/B_0$ is the magnetic moment of the carrier whose velocity at right angles to $\mathbf{B}_0$ is $v_\perp$. The physical origin of the force which Eq. (18) represents is well known. It is the force which a charged particle with magnetic moment μ experiences when it moves *adiabatically* in a magnetic field whose strength varies slowly with position; for example, a mirror field. When a helicon wave propagates at an angle to the static magnetic field, it ceases to be a purely torsional wave. That is, there is then a finite, first-order component of magnetic field, B_z, which alternately adds to and subtracts from the static magnetic field $\mathbf{B}_0$. The result is that a gyrating particle finds itself in a moving periodic magnetic mirror, and its interaction with that mirror will be particularly strong when the velocity of the particle along z just matches the velocity of the mirror along z, i.e., when $v_z = \omega/k_z$. Those par-

[8] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), pp. 7–11.

ticles for which v_z is just less than ω/k_z will be speeded up by the mirror and will extract energy from the wave. Those for which v_z is just greater than ω/k_z will be slowed down. Since in thermal equilibrium there are more slow-moving particles than fast-moving particles $(\partial f_0/\partial v_z<0)$, the processes just described will damp the wave.

The reader will have noticed, of course, that the physical description of the damping is similar to conventional Landau damping which exists when E_z is different from zero. Mathematically the two are identical. It is easy to show, however, that for helicon waves in a metal, magnetic Landau damping is by far the predominant damping mechanism. The ratio of the two forces in Eq. (17) is given by

$$\frac{eE_z}{\mu(\partial B_z/\partial z)}=\frac{eE_z}{\mu k_z B_z}=\frac{e\omega}{\mu k_z k_x}\frac{E_z}{E_y}. \tag{19}$$

The ratio (E_z/E_y) can be obtained from the co-factors of the dispersion relation. Inserting the ratio E_z/E_y into Eq. (19), we find[9]

$$eE_z/(\mu\partial B_z/\partial z)\simeq\omega/kv_F. \tag{20}$$

The Landau damping is completely negligible compared with magnetic Landau damping. In view of the fact that a helicon in a metal is almost all magnetic field, it is not surprising that we find the electric effects (Landau damping) small compared to the magnetic effects (magnetic Landau damping).

It is possible to obtain the actual magnitude of the magnetic Landau damping (in the large-kl limit) from the simple equation of motion (17). Stix[3] and Jackson[10] have shown that the rate at which particles gain energy from the field as a result of a one-dimensional force F in the z direction [see Eq. (17)] is given by

$$\frac{d}{dt}\langle\mathcal{E}\rangle_{z_0,v_0}=-\frac{\pi}{2mk_z^2}\omega nF^2\left[\frac{\partial f_0}{\partial v}\right]_{v=\omega/k_z}. \tag{21}$$

The rate of change of particle energy $(\langle\mathcal{E}\rangle_{z_0,v_0})$ has been suitably averaged over the initial position z_0 and velocity v_0 of the particles. For a simple metal (spherical Fermi surface) at low temperatures, f_0 is the "one-dimensional" Fermi distribution

$$f_0=\frac{3}{4v_F}\left[1-\frac{v_z^2}{v_F^2}\right],\quad v_z\leq v_F$$
$$=0,\qquad v_z\geq v_F.$$

If we substitute $F=\mu(\partial B_z/\partial z)$ into Eq. (21), setting $\mu=\frac{1}{2}mv_F^2/B_0$, we find

$$\frac{d}{dt}\langle\mathcal{E}\rangle_{z_0,v_0}=\frac{B_0^2}{8\pi}\left[\frac{3\pi}{8}(kR)\sin^2\Delta\right]. \tag{22}$$

In deriving Eq. (22) we made use of the local helicon dispersion relation, Eq. (10).

We can now set the rate of gain of average particle kinetic energy equal to the rate of decrease of electromagnetic energy in the helicon. Neglecting the electric energy relative to magnetic energy in the wave, we find

$$\frac{dW}{dt}\equiv-2\omega_i W=-W\left[\frac{3\pi}{8}kR\sin^2\Delta\right], \tag{23}$$

where $W=B_0^2/8\pi$ and ω_i is the imaginary part of the frequency of the helicon wave. Thus

$$\frac{\omega_i}{\omega}=\frac{3\pi}{16}kR\sin^2\Delta. \tag{24}$$

This is precisely the term found by Kaner and Skobov and is the predominant term in our Eq. (11) in the limit $kl\cos\Delta\to\infty$. The additional damping terms in our equation must arise from the fact that at finite $\omega_c\tau$ particle motion is not quite adiabatic.

Note added in manuscript. Conclusions similar to ours concerning the origin of the damping were reached independently by J. Walpole and A. McWhorter [J. Walpole, Ph.D. dissertation, M.I.T., 1966 (unpublished); J. Walpole and A. McWhorter, Phys. Rev. (to be published)]. These authors consider, in addition, tilted ellipsoidal Fermi surfaces. Then even when the wave propagates along the magnetic field, tilted cyclotron orbits can cause magnetic Landau-like damping.

[9] This was pointed out to us by G. A. Pearson.
[10] J. D. Jackson, J. Nucl. Energy, Pt. C, 1, 171 (1960).