It is interesting to note that there is a close similarity between a superconductor and a laser and that many of the equations appearing here are analogous to the equations used in the description of lasers.[12] In that case the role of Δ is played by the electric field **E**. Both systems can in many respects be regarded as nonlinear oscillators.[13] At its operating point the impedance of a self-sustaining oscillator is zero and the vanishing of the resistance and the reactance correspond, respectively, to Eqs. (19) and (29).

[12] W. E. Lamb, Phys. Rev. **134**, A1429 (1964).
[13] M. Lax, Bull. Am. Phys. Soc. **11**, 111 (1966); and (to be published).

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Isospin Formulation of the Theory of a Granular Superconductor

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The properties of a granular superconductor are studied with the aid of the isospin formulation of the microscopic theory of superconductivity. The system consists of grains of homogeneous superconductor separated by insulating but tunnelable barriers (Josephson junctions). The general nonlinear equations of motion are set up for the isospins, "spin up" representing the absence, and "spin down" the presence, of a given Cooper pair. These equations are like torque equations for each isospin moving in an effective pseudomagnetic field due to all the other isospins. Linearized solutions result in various single-particle and collective excitations. A certain class of nonlinear solutions is shown to satisfy a Ginzburg-Landau–like differential equation. The effects of electric fields (within the junctions) and real magnetic fields are studied, one result being that there are bulk electromagnetic modes, analogous to the surface modes known to be associated with a single isolated Josephson junction. Consequences of changes in temperature and changes in effective electron-electron interaction are studied.

I. INTRODUCTION

In this paper we wish to examine the properties of a particular kind of granular superconductor; namely, one where each grain consists of a homogeneous superconductor, but at each grain boundary there is a thin insulating layer (e.g., oxide). Each layer is thin enough that it can be tunneled by the Cooper pairs of the superconductor; in other words, we have a Josephson junction[1] at each grain boundary. For simplicity, we assume that the junctions take up a negligible fraction of the total volume of material.

For such a superconductor, the energy density of the BCS theory[2] is augmented by a tunneling-energy density, the latter being directly proportional both to the linear density of tunnel junctions[3] and to the Cooper-pair transition amplitude for an average junction of unit area. We are free to imagine the tunneling-energy density as large or as small as we like, because of variations in the number of junctions per unit length. We cannot, however, let the tunneling energy be either too large or too small because of the tunneling transition probability. The upper limit is set by the limitation of second-order perturbation theory (the Cooper-pair tunneling being visualized as a two-step process,[4] the intermediate step involving the virtual state where only one of the two electrons composing the pair has tunneled). When the tunneling transition probability is too high, perturbation theory breaks down.

The lower limit to the tunneling transition probability is set by a physical process that has nothing to do with superconductivity per se; it is the value of the tunneling probability at which the normal-metal conductivity of the system (at temperatures where the normal phase is thermodynamically stable) switches over to *insulating* behavior, because there is a thermal activation energy associated with electron tunneling.[5] This activation energy is the energy required to change two neighboring,

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[1] B. D. Josephson, Advan. Phys. **14**, 419 (1965).
[2] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[3] By *linear* density, we mean the average number of junctions intersecting an arbitrarily oriented straight-line segment of unit length.
[4] P. W. Anderson, in *Lectures on the Many-Body Problem*, edited by E. R. Caianiello (Academic Press Inc., New York, 1964), Vol. 2, p. 113.
[5] C. A. Neugebauer and M. B. Webb, J. Appl. Phys. **33**, 74 (1962).

electrically neutral metallic grains into electrically charged grains, one charged $+e$, the other $-e$. Presumably, this change in electrical properties occurs suddenly at some critical minimum value of mean tunneling transition probability. This is analogous to the idea of the metal-insulator transition, as interatomic distance is varied, as was first discussed by Mott.[6]

Thin films appear to have a particularly favorable geometry for obtaining a granular metallic deposit of the type being considered here.[7] In fact, the microwave conductivity of thin films of tin and indium, as measured by Gittleman *et al.*,[8] strongly suggests a granular nature. This will be discussed further in Sec. V.

The general outline of this paper is as follows. In Sec. II, the BCS Hamiltonian for the grains of superconductor and the tunneling Hamiltonian for the junctions will be reformulated in terms of isotopic spin,[9] "spin up" indicating the absence and "spin down" the presence of Cooper pairs, just as was done by Wallace and Stavn[10] for the case of a single Josephson junction. This procedure makes our problem formally analogous to ferromagnetism; in particular, the tunneling Hamiltonian resembles an exchange Hamiltonian. This resemblance will then be exploited by making the continuum approximation to the tunneling Hamiltonian in exactly the same fashion that the exchange energy of the Heisenberg theory of ferromagnetism is approximated by a continuum model in micromagnetics, the continuum theory of ferromagnetism.[11] With the aid of the quantum-mechanical commutation relations satisfied by the isospin operators, the equations of motion and boundary conditions will next be obtained. Continuing the analogy with magnetism, these equations of motion are torque equations for each isospin precessing in an effective pseudomagnetic field due to the other spins.

In Sec. III, we begin the process of solving these equations of motion. As a first step, what is known as the semiclassical approximation in magnetism will be introduced: namely, treating the isospin vectors as classical quantities. (Note that we are not making the classical approximation in setting up the equations of motion, only in solving them.) Once this approximation is invoked, two solutions will be obtained almost by inspection; these are the time- and position-independent superconducting solution (BCS solution), and the time- and position-independent normal-metal solution. The tunneling energy vanishes in both cases.

To go beyond these two solutions is, in general, quite difficult because of the nonlinearity of the equations of motion. To get around this problem, we will *linearize* the equations of motion, i.e., assume that the classical spin vectors execute small-amplitude excursions with respect to one of the two known time- and position-independent solutions. There are two types of solutions to the linearized equations of motion: single-particle–like and collective. The former, being excited Cooper pairs, correspond to flipping over one isospin in the pseudomagnetic field due to the other spins. The latter correspond to collective excitations of all the isospins. Some of these, giving rise to nonequilibrium densities of conduction electrons, are disallowed on physical grounds. Other excitations, however, are consistent with charge neutrality, and are related to real physical phenomena. Examples of quantities which can thereby be calculated are: (1) the decay-rate of the time- and position-independent normal phase, which is energetically unstable relative to the superconducting phase; (2) the maximum velocity of a normal-superconducting interface.

In Sec. IV, we find a certain class of solutions to the equations of motion not requiring the linearization approximation. For this class (and only for this class) the problem can be re-expressed as a second-order nonlinear differential equation for an order parameter. This equation bears a strong resemblance to that of the Ginzburg-Landau phenomenological theory of superconductivity.[12] There are differences, however; for example, the nonlinear portion of the differential equation of Sec. IV has no power-series expansion, in contrast to that of Ginzburg and Landau. Although the equation is static (i.e., it does not contain any explicit time dependence), it applies to a certain type of time-dependent situation, that where *all* the isospins are precessing at the same constant rate. It will be shown that this angular frequency is proportional to *twice* the (time- and position-independent) electrochemical potential of the superconductor, in agreement with Josephson.[1]

In Sec. V, we will generalize to the situation where there is a constant electric potential within a given grain of superconductor, but that this potential changes as one moves from one grain to another. In other words, there are electric fields in the tunnel junctions, and *surface* charges on the individual grains. In the continuum model, this is represented by a finite effective electric field (but no bulk space charge) throughout the superconductor, with an electric energy density proportional to the square of this field. We will also generalize to take account of *real* magnetic fields (as distinguished from the pseudomagnetic fields already introduced) by inserting the magnetic vector potential into the tunneling Hamiltonian in the usual manner. The nonlinear differential equation of Sec. IV, in combination with Maxwell's equations, will then lead to a wave equation for

[6] N. F. Mott, Can. J. Phys. **34**, 1356 (1956).

[7] C. A. Neugebauer, in *Physics of Thin Films*, edited by G. Hass and R. E. Thun (Academic Press Inc., New York, 1964), Vol. II, p. 1.

[8] J. Gittleman, B. Rosenblum, T. E. Seidel, and A. W. Wicklund, Phys. Rev. **137**, A527 (1965).

[9] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

[10] P. R. Wallace and M. J. Stavn, Can. J. Phys. **43**, 411 (1965).

[11] L. Landau and E. Lifshitz, Physik Z. Sowjetunion **8**, 153 (1935); W. F. Brown, Jr., *Micromagnetics* (Interscience Publishers, Inc., New York, 1963); S. Shtrikman and D. Treves, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1963), Vol. III, Chap. 8.

[12] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

bulk electromagnetic modes of the superconductor. These are the bulk analogs of the two-dimensional junction electromagnetic modes predicted by Josephson.[1] The differences between the wave equation of Sec. V and that of Josephson is that: (1) The former involves the three-dimensional Laplacian rather than the two dimensional one; (2) the former is a linear, rather than a nonlinear, wave equation. This linearity is a consequence of the implicit assumption that the phase of the order parameter changes only slightly in passing through any one tunneling junction (introduced by treating the tunneling Hamiltonian on the continuum model).

In complete analogy with the usual Ginzburg-Landau theory, we will see that the nonlinear differential equation of Sec. V implies a critical dc current density, and that at currents less than critical, the microwave impedance is a function of this dc current density.

In Sec. VI, we will investigate the consequences of using a modified electron-electron interaction in the equations of motion for the isospins. The more complicated interaction is introduced for the usual reason: to better approximate the fact that electrons in a superconductor interact not only attractively with the comparatively long-range phonon-mediated force but also repulsively with the shorter ranged screened-Coulomb force. It is well known[13] that the latter repulsion is less effective in suppressing the former attraction than was originally thought to be the case in the BCS theory.[2] The Coulomb-repulsion matrix element appropriate for insertion in the BCS theory may be a factor of two or three smaller than the normal-metal value. We will see that the same effect may occur with many aspects of the present theory.

However, in one regard, the consequences of using a modified electron-electron interaction are drastically different for the granular superconductor than for the ideal BCS superconductor. We will see that, when the normal-metal Coulomb repulsive matrix element becomes greater than the phonon-induced attractive matrix element, new short-wavelength collective oscillations can occur (these being solutions to the linearized equations of Sec. III). These oscillations can have vanishingly small phase velocities and thus may quench any nonvanishing supercurrent in the granular superconductor.[14] The possibility thus exists that our granular system will lose superconductivity, despite the fact that the individual grains are still superconducting, and despite the fact that the tunneling between grains is not a limiting factor.

Finally, in Sec. VII, we will show how the formalism can be extended to finite temperatures. Concentrating on the limit $T \to T_c$, we will recalculate some of the quantities obtained in the previous sections at $T=0$.

II. EQUATIONS OF MOTION

Consider the isospin model of a Josephson junction as given by Wallace and Stavn.[10] In their model the Hamiltonian consists of

$$\mathcal{H}=\mathcal{H}_L+\mathcal{H}_R+\mathcal{H}_T. \tag{2.1}$$

$\mathcal{H}_L$ and $\mathcal{H}_R$ are BCS Hamiltonians[2] for the superconductors on the left- and right-hand sides of the junction, respectively. Each BCS Hamiltonian is reformulated in terms of isospins in the manner introduced by Anderson,[9]

$$\mathcal{H}_{\mathrm{BCS}}=-2\sum_k \epsilon_k s_{3k}-\sum_{kk'} V_{kk'}(s_{1k}s_{1k'}+s_{2k}s_{2k'}). \tag{2.2}$$

Here s_{1k}, s_{2k}, and s_{3k} are the x, y, and z components, respectively, of the isotopic spin vector $\mathbf{s}_k$, obeying the commutation relation

$$\mathbf{s}_k \times \mathbf{s}_{k'}=i\mathbf{s}_k\delta_{kk'}. \tag{2.3}$$

Spin up represents absence, spin down presence of a Cooper pair of wave vector $\mathbf{k}$. The Cooper-pair annihilation and creation operators are equal to s_{k+} and s_{k-}, respectively, where

$$s_{k+}=s_{k1}+is_{k2}\,,\qquad s_{k-}=s_{k1}-is_{k2}. \tag{2.4}$$

There is a set of spin operators $\mathbf{s}_{kL}$ associated with the left-hand superconductor and appearing in the BCS Hamiltonian $\mathcal{H}_L$, and another set $\mathbf{s}_{kR}$ associated with the right-hand superconductor and appearing in the BCS Hamiltonian $\mathcal{H}_R$. The two different sets of spin operators commute, i.e.,

$$\mathbf{s}_{kL}\times\mathbf{s}_{k'R}=0. \tag{2.5}$$

The tunneling Hamiltonian

$$\mathcal{H}_T=-\sum_{kk'} T'_{kk'}(s_{1kL}s_{1k'R}+s_{2kL}s_{2k'R}) \tag{2.6}$$

[13] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau Enterprises, Inc., New York, 1959), p. 83.

[14] The phase velocity of any collective oscillation will set an upper limit to the superfluid drift velocity of a superfluid system, provided there is a finite matrix element for the process of generating such an oscillation by transferring energy and momentum from the superfluid. The determination that such a matrix element is nonvanishing requires a detailed investigation that we will not attempt here. Once one assumes the matrix element to be finite, the limitation on the superfluid drift velocity follows by elementary arugments. See, e.g., I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), p. 6. *Note added in proof.* It has now been proved that these collective oscillations *cannot* quench superconductivity. The analysis of Secs. III and VI has been redone by linearizing with respect to the state of uniform, steady current flow (the state described at the end of Sec. V). It is found that, in general, all collective excitations now violate the condition of charge neutrality, and thus are disallowed physically. The only exceptions are those excitations whose propagation vectors $\boldsymbol{\kappa}$ are perpendicular to the direction of uniform current flow, but, of course, these are just the excitations which are unable to quench the steady supercurrent in any case. This restoration of the stability of superconductivity is especially reassuring because of the fact that our continuum model of a granular superconductor appears, at least in some respects, to be a reasonable model for a conventional dirty superconductor. The details of this work are now being written for publication.

describes the physical process of a Cooper pair tunneling through the junction. The matrix element $\mathcal{T}_{kk'}'$ is proportional to the square of the one-electron tunneling matrix element associated with the junction, and $\mathcal{T}_{kk'}'$ is inversely proportional to the intermediate-state excitation energy corresponding to an electron-like excitation on one side of the junction and a hole-like excitation on the other side.

We now visualize our bulk superconductor to consist of many small grains of pure superconductor with tunneling junctions at each grain boundary. We assume only a small change in orientation of $\mathbf{s}_k$ in isospin space as we move across any junction. Let the position vector $\mathbf{R}$ serve to designate which grain of superconductor we are considering, so that $\mathbf{s}$ is a function of both $\mathbf{k}$ and $\mathbf{R}$. Passing to the continuum limit in real space, we must replace Eqs. (2.3) and (2.5) by

$$\mathbf{s}_k(\mathbf{R})\times\mathbf{s}_{k'}(\mathbf{R}')=i\mathbf{s}_k(\mathbf{R})\delta_{kk'}\delta(\mathbf{R}-\mathbf{R}'). \quad (2.7)$$

Assuming that the tunneling junctions take up a negligible fraction of the total volume of the crystal, we can write the Hamiltonian of the system as

$$\mathcal{H}=\int\mathcal{H}(\mathbf{R})d^3R, \quad (2.8)$$

where the Hamiltonian density $\mathcal{H}(\mathbf{R})$ is composed of the two terms

$$\mathcal{H}(\mathbf{R})=\mathcal{H}_{\mathrm{BCS}}(\mathbf{R})+\mathcal{H}_T(\mathbf{R}), \quad (2.9)$$

$\mathcal{H}_{\mathrm{BCS}}(\mathbf{R})$ being given by Eq. (2.2) after substitution of the $\mathbf{R}$-dependent $\mathbf{s}_k$, and $\mathcal{H}_T(\mathbf{R})$ being given by

$$\mathcal{H}_T(\mathbf{R})=\sum_{kk'}\mathcal{T}_{kk'}\{\nabla_R s_{1k}\cdot\nabla_R s_{1k'}+\nabla_R s_{2k}\cdot\nabla_R s_{2k'}\}+\text{constant}. \quad (2.10)$$

The matrix element $\mathcal{T}_{kk'}$ is directly proportional to the matrix element $\mathcal{T}_{kk'}'$ of an individual junction, Eq. (2.6) (i.e., the reciprocal inductance per unit area of junction), suitably averaged over all types of junctions, and is inversely proportional to the mean linear dimension of the grains. The additive constant in Eq. (2.10), a constant which we will henceforth ignore, is what would remain of the tunneling energy if there were no change in orientation of $\mathbf{s}_k$ in passing through any junction.[14a] The first part of Eq. (2.10) represents the leading term associated with a change in orientation of $\mathbf{s}_k$. It is this term which plays the essential role in the equations of motion for $\mathbf{s}_k$ which we obtain presently. The passage from Eq. (2.6) to Eq. (2.10) is completely analogous to the replacement of the Heisenberg exchange Hamiltonian by terms proportional to the square of the spatial gradient of the components of the magnetization vector in micromagnetics,[11] the continuum theory of ferromagnetism.

Following Wallace and Stavn,[10] we make the approximation of treating $\mathcal{T}_{kk'}$ as a constant $\mathcal{T}$ independent of $\mathbf{k}$ and $\mathbf{k}'$. This allows us to rewrite Eq. (2.10) as

$$\mathcal{H}_T(\mathbf{R})=\mathcal{T}[|\nabla_R\sum_k s_{1k}|^2+|\nabla_R\sum_k s_{2k}|^2]. \quad (2.11)$$

(Here we have dropped the additive constant.) Note that we can now augment the right-hand side of (2.11) by the term

$$\mathcal{T}|\nabla_R\sum_k s_{3k}|^2$$

without modifying $\mathcal{H}_T(\mathbf{R})$. This follows from the fact that

$$n_0=-2\sum_k s_{3k} \quad (2.12)$$

is the total conduction-electron density, necessarily $\mathbf{R}$ independent in order to maintain electric charge neutrality. This modified form renders the analogy complete with micromagnetics[11] in that $\sum_k \mathbf{s}_k$ corresponds to the magnetization vector.[15] Similarly, in Eq. (2.6), if $\mathcal{T}_{kk'}'$ is replaced by a constant independent of $\mathbf{k}$ and $\mathbf{k}'$, then the tunneling Hamiltonian can be rewritten (aside from an additive constant) as

$$-\mathcal{T}'\sum_{kk'}\mathbf{s}_{kL}\cdot\mathbf{s}_{k'R},$$

the analog of the Heisenberg exchange Hamiltonian.

For the time being (until Sec. VI), we make the BCS approximation for the electron-electron interaction potential $V_{kk'}$, i.e.,

$$\begin{aligned}V_{kk'}&=V \quad \text{if} \quad |\epsilon_k|,\ |\epsilon_{k'}|<\hbar\omega,\\ &=0 \quad \text{otherwise}.\end{aligned} \quad (2.13)$$

(We are using a sign convention where a positive V represents an attractive interaction.) Under such conditions, $\mathbf{s}_k$ will be parallel with the z axis in isospin space for $\epsilon_k>\hbar\omega$, while $\mathbf{s}_k$ will be antiparallel with the z-axis for $\epsilon_k<-\hbar\omega$. This results from the one-electron terms $-2\epsilon_k s_{3k}$ in the energy density. Therefore the k sums in Eq. (2.11) have finite contributions only over the range of k space $|\epsilon_k|<\hbar\omega$, so that in effect $\mathcal{T}_{kk'}$ may be treated as though it has the same range as does $V_{kk'}$ in Eq. (2.13). In giving this argument, we have talked as though $\mathbf{s}_k$ were a classical spin vector, which is not correct. The argument can be recast in terms where only the three spins

$$\mathbf{S}_1\equiv\sum_{\epsilon_k<-\hbar\omega}\mathbf{s}_k,\quad \mathbf{S}_2\equiv\sum_{|\epsilon_k|<\hbar\omega}\mathbf{s}_k,\quad \mathbf{S}_3\equiv\sum_{\epsilon_k>\hbar\omega}\mathbf{s}_k$$

are assumed classical, an excellent approximation because of the large number of terms in each sum.

[14a] *Note added in proof.* It is possible that this negative constant, which represents an *attractive coupling* between Cooper-pair states on opposite sides of a junction, plays a role in the experimentally observed enhancement of transition temperature in thin-film granular superconductors. See W. Buckel and R. Hilsch, Z. Physik **138**, 109 (1954); O. F. Kammerer and M. Strongin, Phys. Letters **17**, 224 (1965); B. Abeles, R. W. Cohen, and G. W. Cullen, Phys. Rev. Letters **17**, 632 (1966).

[15] See R. H. Parmenter [Phys. Rev. **137**, A161 (1965)] for a discussion of a situation which differs from the present case only in that $\mathcal{T}_{kk'}$ is diagonal with respect to $\mathbf{k}$ and $\mathbf{k}'$ in the form of $\mathcal{H}_T$ containing all three components of $\mathbf{s}_k$.

We wish to set up the equations of motion for $\mathbf{s}_k(\mathbf{R})$, whose time derivative is given by

$$i\hbar(d\mathbf{s}_k/dt)=[\mathbf{s}_k,\mathcal{H}], \tag{2.14}$$

the right-hand side being the commutator of $\mathbf{s}_k$ and $\mathcal{H}$, the Hamiltonian of Eq. (2.8). With the aid of (2.7), this can be rewritten

$$\hbar(d\mathbf{s}_k/dt)=\mathbf{s}_k\times\mathbf{H}_k, \tag{2.15}$$

where

$$\mathbf{H}_k\equiv-(\delta\mathcal{H}/\delta\mathbf{s}_k)=-(\partial/\partial\mathbf{s}_k-\nabla_R\cdot\partial/\partial\nabla_R\mathbf{s}_k)\mathcal{H} \tag{2.16}$$

is, in suitable units, the effective *pseudo* magnetic field seen by the isospin vector $\mathbf{s}_k$. It should be understood that $\mathbf{H}_k$ has nothing to do with *real* magnetic fields that may be present and will be considered later (Sec. V). Note that $\mathbf{H}_k$ is defined as the negative of the so-called variational derivative[16] of $\mathcal{H}$ with respect to $\mathbf{s}_k$. The x, y, and z components of $\mathbf{H}_k$ are, respectively,

$$\begin{aligned} H_{1k}&=2[\sum_{k'} V_{kk'}s_{1k'}+\nabla_R^2\sum_{k'} \mathcal{T}_{kk'}s_{1k'}],\\ H_{2k}&=2[\sum_{k'} V_{kk'}s_{2k'}+\nabla_R^2\sum_{k'} \mathcal{T}_{kk'}s_{2k'}],\\ H_{3k}&=2\epsilon_k. \end{aligned} \tag{2.17}$$

It is convenient to define the two order parameters

$$\begin{aligned} \Delta_{1k}&\equiv\sum_{k'} V_{kk'}s_{1k'},\\ \Delta_{2k}&\equiv\sum_{k'} V_{kk'}s_{2k'}. \end{aligned} \tag{2.18}$$

Because of Eq. (2.13), $\Delta_{1k}=\Delta_1$, $\Delta_{2k}=\Delta_2$ are independent of $\mathbf{k}$ for $|\epsilon_k|<\hbar\omega$; otherwise $\Delta_{1k}=\Delta_{2k}=0$. In the former case, we have

$$\begin{aligned} H_1&=2[1+\xi^2\nabla_R^2]\Delta_1,\\ H_2&=2[1+\xi^2\nabla_R^2]\Delta_2,\\ H_{3k}&=2\epsilon_k, \end{aligned} \tag{2.19}$$

where we have dropped the k subscript from H_1 and H_2. The characteristic length ξ has been defined as

$$\xi^2=\mathcal{T}/V. \tag{2.20}$$

In calculating the commutator of $\mathbf{s}_k$ and $\mathcal{H}$ in order to obtain Eq. (2.15), one must perform a partial integration, with respect to $\mathbf{R}$, of the gradient terms in $\mathcal{H}$ in order to avoid having to evaluate the commutator of s_{ik} and $\nabla_R s_{ik}$ ($i=1$, 2). This partial integration[17] leads to the appearance of $\nabla_R^2 s_{ik'}$ in $\mathbf{H}_k$. But it also leads to a delta-function contribution to $\mathbf{H}_k$ on the surface of the superconductor,[18] given by

$$\begin{aligned} H_{1S}&=2\xi^2(\partial\Delta_1/\partial n)\delta_S,\\ H_{2S}&=2\xi^2(\partial\Delta_2/\partial n)\delta_S, \end{aligned} \tag{2.21}$$

δ_S being the surface delta function and $(\partial/\partial n)$ the gradient normal to the surface. In order to avoid pathological behavior of $(d\mathbf{s}_k/dt)$ at the surface, we must require

$$\mathbf{s}_k\times\mathbf{H}_S=0. \tag{2.22}$$

Because of the fact that $\mathbf{H}_s$ is restricted to the x-y plane of isospin space, the only way to satisfy (2.22) is to make $\mathbf{H}_s$ vanish. Thus the boundary conditions are that

$$\begin{aligned} \xi^2(\partial\Delta_1/\partial n)&=0,\\ \xi^2(\partial\Delta_2/\partial n)&=0. \end{aligned} \tag{2.23}$$

The obvious generalization of (2.23) at an interface between two different granular superconductors is that $\xi^2(\partial\Delta_i/\partial n)$ ($i=1$, 2) be continuous at the interface.

III. COLLECTIVE EXCITATIONS

In deriving Eq. (2.15), the equations of motion for $\mathbf{s}_k$, we have properly taken into account the fact that the components of $\mathbf{s}_k$ are noncommuting quantum-mechanical operators (or matrices). At this point we introduce the first of two approximations used in solving Eq. (2.15). Specifically, we replace $\mathbf{s}_k(\mathbf{R})$ by the classical quantity

$$\begin{aligned} s_{1k}&=\tfrac{1}{2}\sin\theta_k\cos\varphi_k,\\ s_{2k}&=\tfrac{1}{2}\sin\theta_k\sin\varphi_k,\\ s_{3k}&=\tfrac{1}{2}\cos\theta_k, \end{aligned} \tag{3.1}$$

so that $\mathbf{s}_k(\mathbf{R})$ is a classical vector of magnitude $\frac{1}{2}$ pointing in the direction denoted by $\theta_k(\mathbf{R})$ and $\varphi_k(\mathbf{R})$. In the theory of magnetism, this is known as the semiclassical approximation[19]; it leads to the consequence that the z component of total spin is not a constant of the motion. In the theory of superconductivity, this corresponds to the total number of electrons not being a constant of the motion.

The accuracy of this approximation in superconductivity theory is closely connected with the accuracy of the effective (or molecular) field approximation of magnetism,[20] as applied to superconductivity. The latter approximation is actually much more accurate in superconductivity than it is in magnetism, simply because the effective field acting on a given spin is due to many other spins in superconductivity, but due to only a few other spins in magnetism. Thus, in superconductivity, one can treat the effective field as a *classical* field, calculated classically. [In other words, substitution of Eq.

[16] See, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), p. 353.

[17] An analogous procedure is used in the second quantization of Schrödinger's equation. See, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 338.

[18] The idea of a delta-function surface magnetic field in micromagnetics was introduced by C. Kittel and C. Herring, Phys. Rev. **77**, 725 (1950).

[19] P. W. Anderson, Phys. Rev. **86**, 694 (1952).

[20] See, e.g., R. Kubo, *Statistical Mechanics* (North-Holland Publishing Company, Amsterdam, 1965), p. 302.

(3.1), via (2.18), into (2.19) does not lead to appreciable error in $\mathbf{H}_k$.] But this means that the semiclassical approximation is accurate, since the classical and quantum mechanical equations of motion for a spin in a classical magnetic field are identical.[21]

This first approximation, Eq. (3.1), we shall use throughout the remainder of the paper. A second approximation, to be used in this section, consists of linearizing the nonlinear equations of motion. Let

$$\theta_k=\theta_{k0}+\delta\theta_k\,,\qquad \varphi_k=\varphi_{k0}+\delta\varphi_k\,, \tag{3.2}$$

where both θ_k, φ_k and θ_{k0}, φ_{k0} are solutions. We assume $\delta\theta_k$, $\delta\varphi_k$ are small in the sense that we need keep only terms independent of, or linear in, some $\delta\theta_k$ or $\delta\varphi_k$ in the equations of motion. Furthermore, we take θ_{k0}, φ_{k0} to be the time- and position-independent solutions. (Without loss of generality, we may assume $\varphi_{k0}=0$.) Define

$$\Delta_0=\tfrac{1}{2}V\sum_{k'}\sin\theta_{k'0}\,,\qquad \delta\Delta_1=\tfrac{1}{2}V\sum_{k'}(\cos\theta_{k'0})\delta\theta_{k'}\,,\qquad \delta\Delta_2=\tfrac{1}{2}V\sum_{k'}(\sin\theta_{k'0})\delta\varphi_{k'}\,. \tag{3.3}$$

To the accuracy of the linearization procedure, we have

$$\Delta_1=\Delta_0+\delta\Delta_1\,,\qquad \Delta_2=\delta\Delta_2\,. \tag{3.4}$$

For θ_{k0} we have the equation of motion

$$\epsilon_k\sin\theta_{k0}-\Delta_0\cos\theta_{k0}=0\,,$$

so that

$$\sin\theta_{k0}=\Delta_0(\epsilon_k^2+\Delta_0^2)^{-1/2}\,,\quad \cos\theta_{k0}=\epsilon_k(\epsilon_k^2+\Delta_0^2)^{-1/2}. \tag{3.5}$$

Substituting into (3.3), we get

$$\Delta_0=\tfrac{1}{2}N(0)V\Delta_0\int_{-\hbar\omega}^{\hbar\omega}(\epsilon_k^2+\Delta_0^2)^{-1/2}d\epsilon_k\,, \tag{3.6}$$

$$\delta\Delta_1=\tfrac{1}{2}N(0)V\int_{-\hbar\omega}^{\hbar\omega}\epsilon_k(\epsilon_k^2+\Delta_0^2)^{-1/2}\delta\theta_k d\epsilon_k\,, \tag{3.7}$$

$$\delta\Delta_2=\tfrac{1}{2}N(0)V\Delta_0\int_{-\hbar\omega}^{\hbar\omega}(\epsilon_k^2+\Delta_0^2)^{-1/2}\delta\varphi_k d\epsilon_k\,, \tag{3.8}$$

where $N(0)$ is the one-electron density of allowed states per unit energy for a given electron spin at the Fermi level in the normal state.

Equation (3.6) can be solved for Δ_0, giving two solutions. The first is Case I:

$$\Delta_0=\epsilon_0\equiv 2\hbar\omega e^{-1/N(0)V}\,, \tag{3.9}$$

ϵ_0 being the BCS half-energy gap at the absolute zero of temperature. This is the time- and position-independent superconducting solution. The second is Case II:

$$\Delta_0=0\,,\quad\text{so that}\quad \theta_{k0}=0\ \text{ for }\ \epsilon_k>0\,,\qquad \theta_{k0}=\pi\ \text{ for }\ \epsilon_k<0\,. \tag{3.10}$$

This is the time- and position-independent normal solution.

Substituting Eqs. (2.19), (3.2), (3.4), and (3.5) into (2.15), keeping only terms linear in $\delta\theta_k$ or $\delta\varphi_k$, we get

$$(\epsilon_k^2+\Delta_0^2)\delta\theta_k-\epsilon_k[1+\xi^2\nabla_R^2]\delta\Delta_1=-\tfrac{1}{2}\hbar\Delta_0(d/dt)\delta\varphi_k\,, \tag{3.11}$$
$$\Delta_0\delta\varphi_k-[1+\xi^2\nabla_R^2]\delta\Delta_2=+\tfrac{1}{2}\hbar(d/dt)\delta\theta_k\,.$$

The latter equation comes from either the x or the z component of Eq. (2.15); the former equation comes from the y component.

First we consider the situation where there is only one nonvanishing $\delta\theta_k$ and $\delta\varphi_k$, so that in the limit of an infinite crystal, $\delta\Delta_1=\delta\Delta_2=0$. Rather than considering one single-particle excitation, we might equally well consider an incoherent superposition of single-particle excitations, where the signature of $\delta\theta_k$ and $\delta\varphi_k$ is a random function of ϵ_k. The random signature will insure that $\delta\Delta_1=\delta\Delta_2=0$. Under these conditions, $\delta\theta_k$ and $\delta\varphi_k$ are proportional[22] to $e^{-i\omega_0 t}$, as can be seen by inspection of Eq. (3.11), where

$$\hbar\omega_0=\pm2(\epsilon_k^2+\Delta_0^2)^{1/2}\,, \tag{3.12}$$

$$\delta\theta_k=\pm i\Delta_0(\epsilon_k^2+\Delta_0^2)^{-1/2}\delta\varphi_k\,. \tag{3.13}$$

For the case $\Delta_0=\epsilon_0$, $\hbar\omega_0$ is just the energy of an excited Cooper pair.[23] It should be noted that $\delta\theta_k=0$ for the case $\Delta_0=0$.

Next we consider the situation where there is a coherent superposition of many $\delta\theta_k$ and $\delta\varphi_k$. If each $\delta\theta_k$ and $\delta\varphi_k$ is proportional to some spherical harmonic ($l\neq0$) of the orientation of $\mathbf{k}$, then once again $\delta\Delta_1$ and $\delta\Delta_2$ will vanish and the single-particle solutions will ensue. If each $\delta\theta_k$ and $\delta\varphi_k$ is independent of the orientation of $\mathbf{k}$, then two possibilities exist, as can be seen by inspection of Eqs. (3.7), (3.8), and (3.11). If $\delta\theta_k$ is an odd function of ϵ_k, then $\delta\varphi_k$ must also be odd. This means that $\delta\Delta_1\neq0$, but $\delta\Delta_2=0$. On the other hand, if $\delta\theta_k$ is an even function of ϵ_k, then $\delta\varphi_k$ must also be even; and $\delta\Delta_1=0$, $\delta\Delta_2\neq0$.

The first possibility is given by

$$\delta\theta_k=\frac{\Delta_0\epsilon_k C}{\epsilon_k^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t)\,,\qquad \delta\varphi_k=\frac{-\frac{1}{2}i\hbar\omega_0\epsilon_k C}{\epsilon_k^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t)\,. \tag{3.14}$$

[21] See, e.g., P. W. Anderson, *Concepts in Solids* (W. A. Benjamin, Inc., New York, 1963), p. 164.

[22] Of course, $\delta\theta_k$ and $\delta\varphi_k$ are actually *real* quantities, but in a linearized theory it is perfectly acceptable and very convenient to think of them as *complex* variables.

[23] Avoid confusing the excitation energy $\hbar\omega_0$ with the cutoff energy $\hbar\omega$ of Eq. (2.13).

For simplicity, we restrict ourselves to situations where $\delta\theta_k$ and $\delta\varphi_k$ depend on only one spatial coordinate R. In (3.14), C, κ, and ω_0 are constants independent of $\mathbf{k}$, R, and t. Equation (3.11) and (3.14) are consistent only when κ and ω_0 satisfy the dispersion relation

$$N(0)V[1-(\xi\kappa)^2]\int_0^{\hbar\omega}\left(\frac{\epsilon^2}{\epsilon^2+\Delta_0^2}\right)\times\left(\frac{(\epsilon^2+\Delta_0^2)^{1/2}}{\epsilon^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\right)d\epsilon=1. \quad (3.15)$$

The second possibility is given by

$$\delta\theta_k=\frac{-\frac{1}{2}i\hbar\omega_0\Delta_0 C}{\epsilon_k^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t),$$
$$\delta\varphi_k=\frac{(\epsilon_k^2+\Delta_0^2)C}{\epsilon_k^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t). \quad (3.16)$$

The corresponding dispersion relation differs from Eq. (3.15) only by the absence of the factor $\epsilon^2/(\epsilon^2+\Delta_0^2)$ in the integrand.

At this point it is necessary to consider the possibility of charge unbalance due to the excitation. The net charge density is proportional to

$$\sum_k 2s_k=\sum_k \cos\theta_k\cong\sum_k \cos\theta_{k0}-\sum_k (\sin\theta_{k0})\delta\theta_k$$
$$=-\Delta_0\sum_k (\epsilon_k^2+\Delta_0^2)^{-1/2}\delta\theta_k. \quad (3.17)$$

This quantity vanishes when we have an incoherent superposition of many $\delta\theta_k$ and $\delta\varphi_k$, or when $\delta\theta_k$ is an odd function of ϵ_k as in the first of the above possibilities. For the second of the above possibilities, where $\delta\theta_k$ is an even function of ϵ_k, the net charge need not vanish. The resultant long-range Coulomb forces, not taken into account here, will drastically increase the characteristic frequencies over what would be inferred from our dispersion relation, converting them into conventional plasma oscillations with frequencies $\sim 10^4$ times larger than what the dispersion relation gives. We shall ignore such plasma oscillations, and thus must eliminate the second of the two above possibilities. Of course, the latter possibility gives no charge unbalance when $\Delta_0=0$, but then both possibilities have the same dispersion relation, since the factor $\epsilon^2/(\epsilon^2+\Delta_0^2)=1$.

Returning to Eq. (3.15), first consider Case I where $\Delta_0=\epsilon_0$. With the aid of (3.6), (3.15) can be rewritten in the form

$$(\xi\kappa)^2=-N(0)V[1-(\xi\kappa)^2][I(X)+I(-X)], \quad (3.18)$$

where we define

$$I(X)\equiv\tfrac{1}{2}X^{-1}(1-X^2)\int_0^{U'}[(x^2+1)^{1/2}-X]^{-1}\times(x^2+1)^{-1/2}dx, \quad (3.19)$$

$$X\equiv(\hbar\omega_0/2\epsilon_0), \quad (3.20)$$

$$U'\equiv(\hbar\omega/\epsilon_0). \quad (3.21)$$

Fortunately, the integral defining $I(X)$ can be evaluated analytically.

$$I(X)=\frac{(1-X^2)^{1/2}}{2X}\left\{\arcsin\left[X-\left(\frac{1-X^2}{(x^2+1)^{1/2}-X}\right)\right]\right\}_{x=0}^{x=U'}$$
$$\cong\frac{(1-X^2)^{1/2}}{2X}\left\{\tfrac{1}{2}\pi+\arcsin\left[X-\left(\frac{1-X^2}{U'-X}\right)\right]\right\}, \quad (3.22)$$

where, in the last line, we have assumed $U'\gg 1$. For the time being, we restrict ourselves to the case where $|X|\ll U'$, whereupon

$$I(X)+I(-X)=(1-X^2)^{+1/2}X^{-1}\arcsin X. \quad (3.23)$$

We shall make no attempt here to analyze exhaustively the properties of the various solutions to the dispersion relation; in particular, to investigate the stability properties (convective or absolute instabilities) using the Bers-Briggs technique.[24] Rather, we restrict ourselves to the case where both ω_0^2 and κ^2 are real (i.e., ω_0 and κ are each either real or pure imaginary). For the linearization procedure to be valid, $\delta\theta_k$ and $\delta\varphi_k$ must be small. Inspection of Eq. (3.14) shows that this can happen only if

$$(\tfrac{1}{2}\hbar\omega_0)^2\leqslant\Delta_0^2. \quad (3.24)$$

Thus, here, where $\Delta_0=\epsilon_0$, a real X^2 must be smaller in absolute value than unity. Equation (3.23) indicates that $I(X)+I(-X)$ is real and non-negative when $X^2\leqslant 1$. Combining this with (3.18), we have $(\xi\kappa)^2\leqslant 0$, or $\hbar\kappa$ is pure imaginary. Define

$$\xi'=\xi[-1+1/N(0)V]^{1/2}. \quad (3.25)$$

(Note that $N(0)V<1$.) For $i\hbar\kappa$ lying in the range zero to $\hbar/\xi'$, $\hbar\omega_0$ is real, being $2\epsilon_0$ at the lower end and zero at the upper end. For $i\hbar\kappa$ greater than $\hbar/\xi'$, $\hbar\omega_0$ is pure imaginary. In the limit of large $i\hbar\kappa$, $i\hbar\omega_0$ and $i\hbar\kappa$ are proportional, the ratio being the phase velocity

$$v_\infty\equiv\lim_{i\hbar\kappa\to\infty}(i\hbar\omega_0/i\hbar\kappa). \quad (3.26)$$

To calculate v_∞, we return to Eq. (3.15), getting

$$(\hbar/2\xi)^2 v_\infty^2=N(0)V\int_0^{\hbar\omega}(\epsilon^2+\Delta_0^2)^{-1/2}\epsilon^2 d\epsilon$$
$$\cong\tfrac{1}{2}N(0)V(\hbar\omega)^2,$$

so that

$$v_\infty=\omega\xi[2N(0)V]^{1/2}, \quad (3.27)$$

[24] R. J. Briggs, *Electron-Stream Interaction with Plasmas* (M.I.T. Press, Cambridge, Massachusetts, 1964), Chap. 2.

independent of whether $\Delta_0=\epsilon_0$ or zero. v_∞ represents the maximum velocity with which a disturbance, described by $\delta\theta_k$ and $\delta\varphi_k$, can move through the superconductor. An example is a moving normal-superconducting interface. On either side not too close to the instantaneous location of the interface, a linearized theory should be appropriate. Of course, in practice, moving normal-superconducting interfaces are usually associated with real magnetic fields, not taken into account here, which may limit the speed of motion to values many orders of magnitude smaller than Eq. (3.27).

Next we consider the solution Eq. (3.15) for Case II, where $\Delta_0=0$. We have

$$1=\tfrac{1}{2}N(0)V[1-(\xi\kappa)^2]\ln[1+(2\omega/i\omega_0)^2]. \quad (3.28)$$

As before, restricting ourselves to ω_0^2 and κ^2 real, we see that $\hbar\omega_0$ is always pure imaginary (i.e., never real, aside from zero). In the limit of large $i\hbar\omega_0$, $i\hbar\kappa$ is proportional, and Eq. (3.26) applies. When $i\hbar\omega_0\ll\hbar\omega$, (3.28) simplifies to

$$i\hbar\omega_0=\epsilon_0\exp\left[-\frac{1}{N(0)V}\left(\frac{(\xi\kappa)^2}{1-(\xi\kappa)^2}\right)\right]. \quad (3.29)$$

For $0\leqslant i\hbar\omega_0\leqslant\epsilon_0$, $\hbar\kappa$ is real, being $\hbar/\xi$ when $i\hbar\omega_0=0$ and zero when $i\hbar\omega_0=\epsilon_0$. For $i\hbar\omega_0\geqslant\epsilon_0$, $\hbar\kappa$ is pure imaginary. For the case $\hbar\kappa=0$, $i\hbar\omega_0=\epsilon_0$, we have the characteristic decay time[25]

$$\tau=\hbar/\epsilon_0. \quad (3.30)$$

This indicates that the normal-metal phase at $T=0$ is unstable against decay into the lower energy superconducting phase.

IV. GINZBURG-LANDAU-LIKE SOLUTIONS

Let us return to Eq. (2.15) and look for static solutions (i.e., $\mathbf{s}_k\times\mathbf{H}_k=0$) *without making the linearization approximation*. We continue to make the semiclassical approximation given by Eq. (3.1). By inspection, the z component of $\mathbf{s}_k\times\mathbf{H}_k$ will vanish if we take the angle $\varphi_k=\varphi$ to be independent of $\mathbf{k}$. We write[26]

$$\Delta\equiv\Delta_1+i\Delta_2=|\Delta|e^{+i\varphi}, \quad (4.1)$$

$$H\equiv H_1+iH_2=2[1+\xi^2\nabla_R^2]\Delta, \quad (4.2)$$

$$s_k\equiv s_{1k}+is_{2k}=\tfrac{1}{2}(\sin\theta_k)e^{+i\varphi}. \quad (4.3)$$

The vanishing of the x and y components of $\mathbf{s}_k\times\mathbf{H}_k$ can now be written

$$s_{3k}H-s_kH_{3k}=0, \quad (4.4)$$

or

$$\{(\cos\theta_k)[1+\xi^2\nabla_R^2]-(\epsilon_k/|\Delta|)\sin\theta_k\}\Delta=0. \quad (4.5)$$

We can eliminate the k dependence in this equation by defining g such that

$$\cos\theta_k=\epsilon_k(\epsilon_k^2+g^2)^{-1/2},\ \sin\theta_k=g(\epsilon_k^2+g^2)^{-1/2}, \quad (4.6)$$

whereupon we get

$$\xi^2\nabla_R^2\Delta+\Delta-g(|\Delta|)\Delta/|\Delta|=0. \quad (4.7)$$

Here we write $g=g(|\Delta|)$ as a function of $|\Delta|$. This functional dependence is given implicitly by the fact that

$$|\Delta|=N(0)V\int_0^{\hbar\omega}\sin\theta_k d\epsilon_k=N(0)Vg\ln(2\hbar\omega/g),$$

or, equivalently,

$$|\Delta|/\epsilon_0=(g/\epsilon_0)[1-N(0)V\ln(g/\epsilon_0)]. \quad (4.8)$$

If g were equal to $\epsilon_0^{-2}|\Delta|^3$, Eq. (4.7) would be the complete analog of the famous Ginzburg-Landau equation.[12] The fact that $g(|\Delta|)$ cannot be expanded as a power series in $|\Delta|$ shows that the two equations are not the same. There are qualitative similarities, however, as can be seen by comparing the two functions

$$F_1(|\Delta|)\equiv|\Delta|-g(|\Delta|),$$
$$F_2(|\Delta|)\equiv|\Delta|-\epsilon_0^{-2}|\Delta|^3. \quad (4.9)$$

F_1 and F_2 each rise from the origin at $|\Delta|=0$ with the same initial slope (unity), go through a maximum

$$[F_{1\,\max}=e^{-1}N(0)V\epsilon_0 \quad \text{at} \quad |\Delta|=e^{-1}\{1+N(0)V\}\epsilon_0,$$
$$F_{2\,\max}=(2/9)\sqrt{3}\epsilon_0 \quad \text{at} \quad |\Delta|=\tfrac{1}{3}\sqrt{3}\epsilon_0],$$

and return to the origin at $|\Delta|=\epsilon_0$ [at which point F_1 has a slope of $-N(0)V\{1-N(0)V\}^{-1}$, F_2 a slope of -2]. For reasonable values of $N(0)V$, $F_2\geqslant F_1$.

In general, if we define $\Delta=\Delta_1+i\Delta_2$, we can always write the tunneling Hamiltonian as

$$\mathcal{H}_T=V^{-1}\xi^2|\nabla_R\Delta|^2. \quad (4.10)$$

Similarly, the electron-electron interaction portion of $\mathcal{H}_{\rm BCS}$ can be written as $-V^{-1}|\Delta|^2$. The one part of $\mathcal{H}$ which cannot, in general, be written in terms of Δ is the portion $-2\sum_k\epsilon_k s_{3k}$. However, if we assume, as before, that $\varphi_k=\varphi$ is independent of $\mathbf{k}$ and that θ_k satisfies Eq. (4.6), then Eq. (4.8) implicitly gives g as a function of $|\Delta|$, while

$$-2\sum_k\epsilon_k s_{3k}=-2N(0)\int_0^{\hbar\omega}\epsilon^2(\epsilon^2+g^2)^{-1/2}d\epsilon$$
$$=-N(0)(\hbar\omega)^2+V^{-1}|\Delta|g(|\Delta|).$$

Thus the total Hamiltonian density is now, aside from an ignorable additive constant,

$$\mathcal{H}=V^{-1}[\xi^2|\nabla_R\Delta|^2-|\Delta|^2-|\Delta|g(|\Delta|)]. \quad (4.11)$$

If we now perform an operation analogous to that carried out by Ginzburg and Landau,[12] i.e., if we invoke

[25] See Ref. 13, p. 44.

[26] By defining $\Delta=\Delta_1+i\Delta_2$, we are making Δ a linear combination of Cooper-pair *annihilation* operators [see Eq. (2.4)], the second-quantized analog of a wave function. If we had defined $\Delta=\Delta_1-i\Delta_2$, it would have been the analog of the complex conjugate of the wave function. This distinction is important later when we introduce *real* magnetic fields.

the Euler-Lagrange equation

$$(\partial\mathcal{H}/\delta\Delta^*)\equiv[(\partial/\partial\Delta^*)-\nabla_R\cdot(\partial/\partial\nabla_R\Delta^*)]\mathcal{H}=0\,, \quad (4.12)$$

we do *not* recover Eq. (4.7). The discrepancy results from our putting in the constraint on θ_k described by Eq. (4.6) *before* performing the minimization, rather than performing the minimization of $\mathcal{H}$ with respect to arbitrary variations of the orientation of $\mathbf{s}_k$ (which leads to the equation $\mathbf{s}_k\times\mathbf{H}_k=0$). But, of course, if we had not introduced Eq. (4.6) before doing the variation, we would not have been able to express $\mathcal{H}$ as a function of Δ alone.

We next generalize the analysis of the first part of this section to the nonstatic situation where $\varphi_k=\varphi$, still independent of $\mathbf{k}$, now depends on time. The k independence of φ forces θ_k to be time-independent, as can be seen from inspection of the z component of Eq. (2.15). The x and y components of Eq. (2.15) can be written

$$s_{3k}H-s_k(H_{3k}+\hbar d\varphi/dt)=0\,. \quad (4.13)$$

This differs from Eq. (4.4) only in that $H_{3k}=2\epsilon_k$ is replaced by $H_{3k}+\hbar d\varphi/dt$.[27] In other words, as far as the equations of motion are concerned, the one-electron energy ϵ_k is replaced by $\epsilon_k+\frac{1}{2}\hbar d\varphi/dt$. Since the density of conduction electrons must remain fixed, the Fermi level must change in time exactly in step with $\frac{1}{2}\hbar d\varphi/dt$. Thus, in agreement with Josephson,[1] $\hbar d\varphi/dt$ is *twice* the electrochemical potential. In order that the range of the attractive electron-electron interaction $V_{kk'}$ be symmetrically placed with respect to the Fermi level, we must replace Eq. (2.13) by

$$\begin{aligned} V_{kk'}&=V \quad \text{if} \quad |\epsilon_k+\tfrac{1}{2}\hbar d\varphi/dt|\,,\ |\epsilon_{k'}+\tfrac{1}{2}\hbar d\varphi/dt|<\hbar\omega \\ &=0 \quad \text{otherwise}. \end{aligned} \quad (4.14)$$

As a consequence of (4.13) and (4.14), $d\varphi/dt$ disappears from the equations of motion, and we once again obtain Eq. (4.7) despite the fact that φ is time-dependent. This shows that our static Ginzburg-Landau–like equation is still appropriate under certain nonstatic conditions.

V. ELECTROMAGNETIC PROPERTIES

The electrostatic potential energy $-eU$ ($-e$ being the electronic charge) is, aside from an additive constant which we can ignore, equal to the negative of the electrochemical potential, i.e.,

$$+U=(\hbar/2e)(d\varphi/dt)\,. \quad (5.1)$$

For an ordinary superconductor, therefore, $d\varphi/dt$ should be independent of position $\mathbf{R}$. For the system considered in this paper, i.e., microscopic grains of superconductor with insulating barriers at all grain boundaries, this independence of U with respect to $\mathbf{R}$ will hold true over a given grain, but there can be differing values of U in neighboring grains so that there are electrostatic fields in the barriers (associated with surface charge on the two sides of each barrier). As before, we pass to the continuum model where U is position-dependent, $U(\mathbf{R})$ representing the electrostatic potential of the grain located at $\mathbf{R}$. The Coulomb energy associated with the electrically charged barriers leads to a Hamiltonian density

$$\mathcal{H}_C(\mathbf{R})=CE^2\,, \quad (5.2)$$

where

$$\mathbf{E}\equiv-\nabla_R U \quad (5.3)$$

is the effective, macroscopic electric field. The coefficient C is proportional to the mean capacitance per unit area of junction and inversely proportional to the mean linear dimension of the grains. The electric displacement $\mathbf{D}$ is defined, as usual, by

$$\mathbf{D}\equiv4\pi(\partial\mathcal{H}_C/\partial\mathbf{E})=8\pi C\mathbf{E}\,, \quad (5.4)$$

i.e., $8\pi C$ is the effective dielectric constant of the material.

In addition to the *pseudo* magnetic fields $\mathbf{H}_k$ discussed thus far, the presence of *real* magnetic fields can be included in the equations of motion by replacing ∇_R by $\nabla_R+i(2e/\hbar c)\mathbf{A}(\mathbf{R})$ in the tunneling Hamiltonian, $\mathbf{A}(\mathbf{R})$ being the magnetic vector potential at $\mathbf{R}$. Such a replacement should be made in Eqs. (2.10), (2.11), (2.17), (2.19), (2.21), (2.23), (4.2), (4.7), (4.10), and (4.11). In addition, Eq. (5.3) must be replaced by

$$\mathbf{E}=-\nabla_R U-c^{-1}d\mathbf{A}/dt\,. \quad (5.5)$$

The electrical current density $\mathbf{J}$ can be obtained from the relation

$$\mathbf{J}\equiv-c(\partial\mathcal{H}_T/\partial\mathbf{A})\,. \quad (5.6)$$

With the aid of Eq. (4.10), this becomes

$$\begin{aligned} \mathbf{J}=V^{-1}\xi^2(2e/\hbar)\{i[\Delta^*\nabla_R\Delta-\Delta\nabla_R\Delta^*] \\ -(4e/\hbar c)|\Delta|^2\mathbf{A}\}\,. \end{aligned} \quad (5.7)$$

Specializing to the case already considered, where $\varphi_k=\varphi$ is independent of $\mathbf{k}$,

$$\mathbf{J}=-V^{-1}\xi^2(4e/\hbar)|\Delta|^2[\nabla_R\varphi-(2e/\hbar c)\mathbf{A}]\,. \quad (5.8)$$

Similarly, the displacement current is

$$\begin{aligned} (4\pi)^{-1}(d\mathbf{D}/dt)&=2C(d\mathbf{E}/dt) \\ &=-(\hbar/e)C(d^2/dt^2)[\nabla_R\varphi-(2e/\hbar c)\mathbf{A}]\,, \end{aligned} \quad (5.9)$$

where we have used Eqs. (5.1) and (5.5).

Consider the special case where $|\Delta|$ is position independent. Using the fact that the *real* magnetic field $\mathbf{H}$ satisfies the equations

$$\mathbf{H}=\nabla_R\times\mathbf{A}\,, \quad (5.10)$$

$$\nabla_R\times\mathbf{H}=(4\pi/c)[\mathbf{J}+(4\pi)^{-1}(d\mathbf{D}/dt)]\,, \quad (5.11)$$

[27] This is equivalent to the situation in magnetic resonance where a dynamic problem is reduced to a static one by shifting to a rotating coordinate system. See, e.g., I. I. Rabi, N. F. Ramsey, and J. Schwinger, Rev. Mod. Phys. 26, 167 (1954).

we now get

$$[\nabla_R^2 - v^{-2}(d^2/dt^2)]\mathbf{H} = \lambda^{-2}\mathbf{H}, \tag{5.12}$$

where we have defined

$$v = c(8\pi C)^{-1/2}, \tag{5.13}$$

$$\lambda = (V/2\pi)^{1/2}(\hbar c/4e\xi|\Delta|). \tag{5.14}$$

The solutions of Eq. (5.12) represent bulk electromagnetic modes of our granular superconductor. They are analogous to the two-dimensional electromagnetic modes of an ordinary Josephson junction. Josephson's wave equation [Eq. (3.12) of Ref. 1] differs from (5.12) only in that the Laplacian is two-dimensional and the equation is nonlinear. As has been mentioned in the Introduction, the linearity of (5.12) results from the continuum approximation of Sec. II. The characteristic velocity v of the wave equation is simply the velocity of light c divided by the square root of the effective dielectric constant $8\pi C$. The length λ can be rewritten in terms of the London penetration depth λ_0 and the Pippard coherence distance ξ_0 of the superconducting material forming the grains of our system. Since

$$\lambda_0 = (mc^2/4\pi n_0 e^2)^{1/2}, \tag{5.15}$$

$$\xi_0 = (\hbar v_F/\pi\epsilon_0), \tag{5.16}$$

we can express λ as

$$\lambda = \pi[\tfrac{1}{6}\pi N(0)V]^{1/2}(\lambda_0\xi_0/\xi)(\epsilon_0/|\Delta|). \tag{5.17}$$

The coefficient $\pi[\frac{1}{6}\pi N(0)V]^{1/2}$ is of the order of magnitude unity; for example, it is 1.6 when $N(0)V = 0.5$.

In the static case, Eq. (5.12) is London's equation. The only difference is that the effective penetration depth λ may be much larger than that appropriate to any ordinary superconductor. It is only necessary to make the effective coherence distance ξ of our granular superconductor sufficiently small, something which can always be done by decreasing the density of tunneling junctions. The fact that λ/ξ can be made much greater than unity suggests that the granular superconductor has type-II behavior.[28] This is only partly true. The analysis of this paper is appropriate only so long as one can ignore circulating currents *within* a grain (as contrasted with currents, either conduction or displacement, from one grain to another). This means that, no matter how large λ/ξ is, one cannot hope to maintain the individual grains superconducting above that magnetic field where an isolated grain would by itself be superconducting. In principle, there should be an additional kinetic-energy term in the Hamiltonian density, Eq. (2.9), this term resulting from current flow *within* a grain. Throughout this paper we have been implicitly assuming that this term was negligible, relative to $\mathcal{H}_{BCS}$, because of the smallness of the current densities. When the only currents flowing are those passing through tunneling junctions, this is an excellent approximation. But the approximation may break down when appreciable currents are circulating in the individual grains.

The position-independent solutions of Eq. (5.12) are electromagnetic oscillations with a characteristic frequency

$$\tilde{\omega} \equiv v/\lambda = (2e|\Delta|/\hbar V)(\mathcal{T}/C)^{1/2}. \tag{5.18}$$

The ratio $(\mathcal{T}/C)$ is a property of the average junction in the superconductor; it is proportional to the ratio of reciprocal inductance per unit area of junction to capacitance per unit area of junction. The inductance is associated with the tunneling energy $\mathcal{H}_T$, the capacitance with the Coulomb energy $\mathcal{H}_C$.

Equation (5.12) was derived under the assumption that $|\Delta|$ is position-independent. The simplest instance of this is when $\Delta = \epsilon_0$ is position-independent, but there can be cases where $|\Delta|$ is position-independent although Δ is not. This occurs when there is a uniform drift velocity of the Cooper pairs, appropriate for a suitably thin film, whereupon $|\Delta| < \epsilon_0$. As the drift velocity increases, $|\Delta|$ decreases, so that $\tilde{\omega}$ decreases and λ increases. This corresponds to an increase in the effective inertia of the Cooper pairs, or to an increase in the incremental inductance (imaginary part of the microwave impedance). The connection between changes in λ and changes in inertia can be seen very quickly by writing the London equation

$$\mathbf{E} = (4\pi/c^2)\lambda^2 d\mathbf{J}/dt, \tag{5.19}$$

this being a direct consequence of Eqs. (5.1), (5.5), (5.8), and (5.14). The coefficient on the right-hand side of (5.19) is a measure of the inertia of the Cooper pairs.

Gittleman *et al.*[8] have measured a dependence of microwave reactance on dc current density in thin superconducting films at temperatures too small for quasiparticle excitations[29] (normal carriers) at the currents used. These authors suggested that the source of the dependence was the junctions of oxide or "dirt" in the films, in agreement with the present theory.

The dc current density (in a suitably thin film where we can ignore the vector potential) is given by

$$\mathbf{J}_{dc} = -(4e/\hbar V)\xi^2|\Delta|^2\nabla_R\varphi.$$

But for the case $|\Delta|$ position-independent, our nonlinear differential equation for Δ, Eq. (4.7), gives us $\nabla_R\varphi$ as a function of $|\Delta|$, since

$$-\xi^2|\nabla_R\varphi|^2 + 1 - |\Delta|^{-1}g(|\Delta|) = 0.$$

Thus

$$J_{dc} = \pm(4e\xi/\hbar V)|\Delta|^2[1 - |\Delta|^{-1}g(|\Delta|)]^{1/2}, \tag{5.20}$$

[28] For a description of type-II superconductors, see P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), Chap. 3.

[29] Both the real and the imaginary part of the microwave impedance can depend on dc current density whenever there are appreciable numbers of quasiparticle excitations, whose distribution function is rearranged with changes in Cooper-pair drift velocity. See R. H. Parmenter, RCA Rev. 23, 323 (1962).

and we have the functional connection between J_{dc} and $|\Delta|$. By differentiating J_{dc} with respect to $|\Delta|$, we find that there is a maximum value of J_{dc}, namely

$$J_{\rm crit}=(4e\xi/\hbar V)\epsilon_0{}^2\mathcal{C}. \quad (5.21)$$

The numerical factor $\mathcal{C}$, of order-of-magnitude unity, is a rather complicated algebraic function of $N(0)V$ that can be determined straightforwardly from Eq. (4.8), the defining relation for $g(|\Delta|)$.

VI. MODIFIED INTERACTION POTENTIAL

Thus far, we have considered only a particularly simple model for the interaction potential $V_{kk'}$, that of Eq. (2.13). We now wish to consider a slightly more general model, namely

$$\begin{aligned} V_{kk'} &= V_a - V_b \quad \text{if} \quad |\epsilon_k|,|\epsilon_{k'}| \quad \text{are both} \quad <\hbar\omega_a \\ &= -V_b \quad \text{if} \quad |\epsilon_k|,|\epsilon_{k'}| \quad \text{are both} \quad <\hbar\omega_b \\ &\quad\ \text{but not both} \quad <\hbar\omega_a \\ &= 0 \quad \text{otherwise}. \end{aligned} \quad (6.1)$$

Here $-V_b$ represents the Coulomb-repulsion contribution to the electron-electron interaction. Corresponding to the fact that it is short range in real space (the Debye screening length being about 0.5 Å in a typical metal), $-V_b$ is rather long range in momentum space, and $\hbar\omega_b$ is comparable with the Fermi energy. V_a represents the phonon-induced attractive contribution to the interaction. The mean phonon energy $\hbar\omega_a$ is much smaller than $\hbar\omega_b$, corresponding to the fact that V_a is shorter-ranged in momentum space, and longer-ranged in real space. Of course, in actuality the phonon-induced attraction is short-range in space but time-retarded, rather than being longer-range in space but instantaneous in time. The latter should be thought of as a suitably time-averaged approximation to the former. It appears that both forms of phonon-induced attraction lead to substantially the same results.[30]

Although we are modifying $V_{kk'}$, we shall continue to assume $\mathcal{T}_{kk'}=\mathcal{T}$ independent of ϵ_k and $\epsilon_{k'}$ as before. (This approximation will be discussed later in this section.) We define

$$\begin{aligned} &\Delta_{1k}=\Delta_{1a}, \quad \Delta_{2k}=\Delta_{2a}, \quad 0\leqslant|\epsilon_k|\leqslant\hbar\omega_a; \\ &\Delta_{1k}=\Delta_{1b}, \quad \Delta_{2k}=\Delta_{2b}, \quad \hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b; \end{aligned} \quad (6.2)$$

and

$$\xi_a{}^2=\mathcal{T}/(V_a-V_b), \quad \xi_b{}^2=\mathcal{T}/V_b. \quad (6.3)$$

In terms of these quantities we can write the effective pseudomagnetic field $\mathbf{H}_k$. For $|\epsilon_k|\leqslant\hbar\omega_a$,

$$\begin{aligned} H_{1k}&=2[1+\xi_a{}^2\nabla_R{}^2]\Delta_{1a}, \\ H_{2k}&=2[1+\xi_a{}^2\nabla_R{}^2]\Delta_{2a}, \\ H_{3k}&=2\epsilon_k; \end{aligned} \quad (6.4)$$

while for $\hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b$,

$$\begin{aligned} H_{1k}&=2[1-\xi_b{}^2\nabla_R{}^2]\Delta_{1b}, \\ H_{2k}&=2[1-\xi_b{}^2\nabla_R{}^2]\Delta_{2b}, \\ H_{3k}&=2\epsilon_k. \end{aligned} \quad (6.5)$$

The semiclassical approximation, Eq. (3.1), allows us to write

$$\begin{aligned} 2\Delta_{1a}&=N(0)V_a\int_{-\hbar\omega_a}^{\hbar\omega_a}\sin\theta_k\cos\varphi_k d\epsilon_k-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_k\cos\varphi_k d\epsilon_k, \\ 2\Delta_{2a}&=N(0)V_a\int_{-\hbar\omega_a}^{\hbar\omega_a}\sin\theta_k\sin\varphi_k d\epsilon_k-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_k\sin\varphi_k d\epsilon_k, \end{aligned} \quad (6.6)$$

$$\begin{aligned} 2\Delta_{1b}&=-N(0)V_b\int_{-\hbar\omega_a}^{\hbar\omega_b}\sin\theta_k\cos\varphi_k d\epsilon_k, \\ 2\Delta_{2b}&=-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_k\sin\varphi_k d\epsilon_k. \end{aligned} \quad (6.7)$$

Initially, let us repeat the procedure of Sec. III and linearize the equations of motion according to Eq. (3.2). We have

$$\Delta_{1a,b}=\Delta_{0a,b}+\delta\Delta_{1a,b}, \quad \Delta_{2a,b}=\delta\Delta_{2a,b}, \quad (6.8)$$

where

$$\begin{aligned} 2\Delta_{0a}&=N(0)V_a\int_{-\hbar\omega_a}^{\hbar\omega_a}\sin\theta_{k0}d\epsilon_k-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_{k0}d\epsilon_k, \\ 2\Delta_{0b}&=-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_{k0}d\epsilon_k, \end{aligned} \quad (6.9)$$

[30] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).

$$2\delta\Delta_{1a}=N(0)V_a\int_{-\hbar\omega_a}^{\hbar\omega_a}(\cos\theta_{k0})\delta\theta_k d\epsilon_k-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}(\cos\theta_{k0})\delta\theta_k d\epsilon_k,$$
$$2\delta\Delta_{1b}=-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}(\cos\theta_{k0})\delta\theta_k d\epsilon_k, \quad (6.10)$$

$$2\delta\Delta_{2a}=N(0)V_a\int_{-\hbar\omega_a}^{\hbar\omega_a}\sin\theta_{k0}\delta\varphi_k d\epsilon_k-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_{k0}\delta\varphi_k d\epsilon_k,$$
$$2\delta\Delta_{2b}=-N(0)V_b\int_{-\hbar\omega_b}^{\hbar\omega_b}\sin\theta_{k0}\delta\varphi_k d\epsilon_k. \quad (6.11)$$

For θ_{k0} we have the time- and position-independent solutions given by the equations of motion

$$\epsilon_k\sin\theta_{k0}-\Delta_{0a}\cos\theta_{k0}=0,\quad 0\leqslant|\epsilon_k|\leqslant\hbar\omega_a,\quad \epsilon_k\sin\theta_{k0}-\Delta_{0b}\cos\theta_{k0}=0,\quad \hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b,$$

which, when substituted into Eqs. (6.9), give

$$\Delta_{0a}=N(0)[V_a-V_b]\Delta_{0a}\int_0^{\hbar\omega_a}(\epsilon^2+\Delta_{0a}^2)^{-1/2}d\epsilon-N(0)V_b\Delta_{0b}\int_{+\hbar\omega_a}^{\hbar\omega_b}(\epsilon^2+\Delta_{0b}^2)^{-1/2}d\epsilon,$$
$$\Delta_{0b}=-N(0)V_b\Delta_{0a}\int_0^{\hbar\omega_a}(\epsilon^2+\Delta_{0a}^2)^{-1/2}d\epsilon-N(0)V_b\Delta_{0b}\int_{+\hbar\omega_a}^{\hbar\omega_b}(\epsilon^2+\Delta_{0b}^2)^{-1/2}d\epsilon. \quad (6.12)$$

Since Δ_{0a} and $|\Delta_{0b}|$ are both much smaller than $\hbar\omega_a$ or $\hbar\omega_b$, the above equations simplify to

$$1=N(0)[V_a-V_b]\ln(2\hbar\omega_a/\Delta_{0a})-N(0)V_b(\Delta_{0b}/\Delta_{0a})\ln(\omega_b/\omega_a),$$
$$(\Delta_{0b}/\Delta_{0a})=-N(0)V_b\ln(2\hbar\omega_a/\Delta_{0a})-N(0)V_b(\Delta_{0b}/\Delta_{0a})\ln(\omega_b/\omega_a).$$

Defining the effective potentials

$$V_b'\equiv V_b[1+N(0)V_b\ln(\omega_b/\omega_a)]^{-1}, \quad (6.13)$$

$$V\equiv V_a-V_b', \quad (6.14)$$

we get

$$\Delta_{0a}=2\hbar\omega_a e^{-1/N(0)V}, \quad (6.15)$$

$$\Delta_{0b}=-(V_b'/V)\Delta_{0a}. \quad (6.16)$$

This represents the time- and position-independent superconducting solution,[13] and will be called Case I in analogy with Eq. (3.9). The other solution of Eq. (6.12), namely, the time- and position-independent normal-metal solution $\Delta_{0a}=\Delta_{0b}=0$, is analogous to Eq. (3.10) and will be called Case II.

Substituting the results for θ_{k0} into (6.10) and (6.11), we get

$$2\delta\Delta_{1a}=N(0)[V_a-V_b]\int_{-\hbar\omega_a}^{+\hbar\omega_a}\epsilon_k(\epsilon_k^2+\Delta_{0a}^2)^{-1/2}\delta\theta_k d\epsilon_k-N(0)V_b\left[\int_{-\hbar\omega_b}^{-\hbar\omega_a}+\int_{+\hbar\omega_a}^{+\hbar\omega_b}\right]\epsilon_k(\epsilon_k^2+\Delta_{0b}^2)^{-1/2}\delta\theta_k d\epsilon_k,$$
$$2\delta\Delta_{1b}=-N(0)V_b\int_{-\hbar\omega_a}^{+\hbar\omega_a}\epsilon_k(\epsilon_k^2+\Delta_{0a}^2)^{-1/2}\delta\theta_k d\epsilon_k-N(0)V_b\left[\int_{-\hbar\omega_b}^{-\hbar\omega_a}+\int_{+\hbar\omega_a}^{+\hbar\omega_b}\right]\epsilon_k(\epsilon_k^2+\Delta_{0b}^2)^{-1/2}\delta\theta_k d\epsilon_k, \quad (6.17)$$

$$2\delta\Delta_{2a}=N(0)[V_a-V_b]\Delta_{0a}\int_{-\hbar\omega_a}^{+\hbar\omega_a}(\epsilon_k^2+\Delta_{0a}^2)^{-1/2}\delta\varphi_k d\epsilon_k-N(0)V_b\Delta_{0b}\left[\int_{-\hbar\omega_b}^{-\hbar\omega_a}+\int_{+\hbar\omega_a}^{+\hbar\omega_b}\right](\epsilon_k^2+\Delta_{0b}^2)^{-1/2}\delta\varphi_k d\epsilon_k,$$
$$2\delta\Delta_{2b}=-N(0)V_b\Delta_{0a}\int_{-\hbar\omega_a}^{+\hbar\omega_a}(\epsilon_k^2+\Delta_{0a}^2)^{-1/2}\delta\varphi_k d\epsilon_k-N(0)V_b\Delta_{0b}\left[\int_{-\hbar\omega_b}^{-\hbar\omega_a}+\int_{+\hbar\omega_a}^{+\hbar\omega_b}\right](\epsilon_k^2+\Delta_{0b}^2)^{-1/2}\delta\varphi_k d\epsilon_k. \quad (6.18)$$

In analogy with Eq. (3.11), we can now write the linearized equations of motion for $\delta\theta_k$ and $\delta\varphi_k$. If $0\leqslant|\epsilon_k|$

$\leqslant \hbar\omega_a$, then

$$(\epsilon_k^2+\Delta_{0a}^2)\delta\theta_k-\epsilon_k[1+\xi_a^2\nabla_R^2]\delta\Delta_{1a}=-\tfrac{1}{2}\hbar\Delta_{0a}(d/dt)\delta\varphi_k,\quad \Delta_{0a}\delta\varphi_k-[1+\xi_a^2\nabla_R^2]\delta\Delta_{2a}=+\tfrac{1}{2}\hbar(d/dt)\delta\theta_k,\quad (6.19)$$

while if $\hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b$, then

$$(\epsilon_k^2+\Delta_{0b}^2)\delta\theta_k-\epsilon_k[1-\xi_b^2\nabla_R^2]\delta\Delta_{1b}=-\tfrac{1}{2}\hbar\Delta_{0b}(d/dt)\delta\varphi_k,\quad \Delta_{0b}\delta\varphi_k-[1-\xi_b^2\nabla_R^2]\delta\Delta_{2b}=+\tfrac{1}{2}\hbar(d/dt)\delta\theta_k.\quad (6.20)$$

As before, if there is only one nonvanishing $\delta\theta_k$ and $\delta\varphi_k$, then $\delta\Delta_{1a,b}=\delta\Delta_{2a,b}=0$, and $\delta\theta_k$, $\delta\varphi_k$ are proportional to $\exp-(i\omega_0 t)$, where

$$\hbar\omega_0=\pm2(\epsilon_k^2+\Delta^2_{0a,b})^{1/2},\qquad (6.21)$$

$$\delta\theta_k=\pm i\Delta_{0a,b}(\epsilon_k^2+\Delta^2_{0a,b})^{-1/2}\delta\varphi_k.\qquad (6.22)$$

In these last two equations, we insert Δ_{0a} if $|\epsilon_k|<\hbar\omega_a$, Δ_{0b} if $\hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b$.

If there is a coherent superposition of many $\delta\theta_k$ and $\delta\varphi_k$, all independent of the orientation of **k**, then there are two possibilities compatible with Eqs. (6.17)–(6.20), just as in Sec. III. Either $\delta\theta_k$ and $\delta\varphi_k$ are both *odd* functions of ϵ_k (whence $\delta\Delta_{1a,b}\neq0$, $\delta\Delta_{2a,b}=0$), or both $\delta\theta_k$ and $\delta\varphi_k$ are *even* functions of ϵ_k (whence $\delta\Delta_{1a,b}=0$, $\delta\Delta_{2a,b}\neq0$). Actually, we can ignore the second possibility for exactly the same reason as before (charge unbalance). The first possibility is given by

$$\delta\theta_k=\frac{\Delta_{0a,b}\epsilon_k C_{a,b}}{\epsilon_k^2+\Delta_{0a,b}^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t),$$
$$\delta\varphi_k=\frac{-\frac{1}{2}i\hbar\omega_0\epsilon_k C_{a,b}}{\epsilon_k^2+\Delta_{0a,b}^2-(\frac{1}{2}\hbar\omega_0)^2}\exp i(\kappa R-\omega_0 t).\qquad (6.23)$$

The coefficient C_a goes with $0\leqslant|\epsilon_k|\leqslant\hbar\omega_a$; the coefficient C_b with $\hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b$. Equation (6.23) solves Eqs. (6.19) and (6.20) provided that

$$\begin{aligned}\Delta_{0a}C_a&=N(0)[V_a-V_b][1-(\xi_a\kappa)^2]F_a\Delta_{0a}C_a\\&\quad-N(0)V_b[1-(\xi_a\kappa)^2]F_b\Delta_{0b}C_b,\\ \Delta_{0b}C_b&=-N(0)V_b[1+(\xi_b\kappa)^2]F_a\Delta_{0a}C_a\\&\quad-N(0)V_b[1+(\xi_b\kappa)^2]F_b\Delta_{0b}C_b,\end{aligned}\qquad (6.24)$$

where we have defined the integrals

$$F_a\equiv\int_0^{\hbar\omega_a}\left(\frac{\epsilon^2}{\epsilon^2+\Delta_{0a}^2}\right)\left(\frac{(\epsilon^2+\Delta_{0a}^2)^{1/2}}{\epsilon^2+\Delta_{0a}^2-(\frac{1}{2}\hbar\omega_0)^2}\right)d\epsilon,\qquad (6.25)$$

$$F_b\equiv\int_{\hbar\omega_a}^{\hbar\omega_b}\left(\frac{\epsilon^2}{\epsilon^2+\Delta_{0b}^2}\right)\left(\frac{(\epsilon^2+\Delta_{0b}^2)^{1/2}}{\epsilon^2+\Delta_{0b}^2-(\frac{1}{2}\hbar\omega_0)^2}\right)d\epsilon.\qquad (6.26)$$

Equations (6.24) can be rewritten

$$-\left(\frac{\Delta_{0b}C_b}{\Delta_{0a}C_a}\right)=\frac{1-N(0)[V_b-V_a][1-(\xi_a\kappa)^2]F_a}{N(0)V_b[1-(\xi_a\kappa)^2]F_b}=\frac{N(0)V_b[1+(\xi_b\kappa)^2]F_a}{1+N(0)V_b[1+(\xi_b\kappa)^2]F_b}.\qquad (6.27)$$

Finally, let us define

$$V_b''\equiv V_b\{1+N(0)V_b[1+(\xi_b\kappa)^2]F_b\}^{-1},\qquad (6.28)$$

$$V_{\text{eff}}\equiv V_a-V_b''.\qquad (6.29)$$

Now (6.27) can be solved for the dispersion relation connecting ω_0 and κ, the result being

$$1=N(0)V_{\text{eff}}[1-(\xi_a\kappa)^2]F_a.\qquad (6.30)$$

This is analogous to Eq. (3.15), appropriate for the simpler form of interaction potential. One essential difference between (3.15) and (6.30) is that, in the latter, the interaction (V_a-V_b) enters in determining the coherence distance ξ_a [Eq. (6.3)] whereas the interaction (V_a-V_b'') enters in the factor $N(0)V_{\text{eff}}$. In (3.15), on the other hand, the same interaction enters in both places. Since $\hbar\omega_a$ and $\hbar\omega_b$ are much bigger than $|\Delta_{0b}|$, we can take

$$F_b\cong\int_{\hbar\omega_a}^{\hbar\omega_b}\frac{\epsilon\,d\epsilon}{\epsilon^2-(\frac{1}{2}\hbar\omega_0)^2}=\ln\left(\frac{\omega_b^2-\frac{1}{4}\omega_0^2}{\omega_a^2-\frac{1}{4}\omega_0^2}\right)^{1/2}.\qquad (6.31)$$

Under usual conditions $|\omega_0|$ will be much smaller than ω_a or ω_b, whereupon $F_b\cong\ln(\omega_b/\omega_a)$. Thus, in the long-wavelength limit where $\kappa\ll\xi_b^{-1}$, V_b'' goes into V_b' [Eq. (6.13)], the effective Coulomb repulsion for the time- and position-independent solution, and V_{eff} becomes the V of Eq. (6.14).

For the moment, let us restrict ourselves to the case $V_b\leqslant V_a\leqslant 2V_b$. It follows from Eq. (6.3) that $\xi_a\geqslant\xi_b$. This inequality will tend to be enhanced by the fact that $\mathcal{T}_{kk'}$ is not really independent of ϵ_k and $\epsilon_{k'}$, but rather decreases as $|\epsilon_k|$, $|\epsilon_{k'}|$ increase in size primarily because of the energy denominator associated with the virtual intermediate state. [See the discussion directly following Eq. (2.6).] Thus, if $\xi_a^2=\mathcal{T}/(V_a-V_b)$, one might reasonably take $\xi_b^2\cong(2\epsilon_0/\hbar\omega_a)(\mathcal{T}/V_b)$, so that ξ_b could be an order of magnitude smaller than ξ_a. This means that $|\kappa|$ may be comparable with ξ_a^{-1} and still much smaller than ξ_b^{-1}. Under these conditions, the discussion of Eq. (3.15) is still applicable. In particular, for Case I we have $\hbar\kappa$ pure imaginary. For $i\hbar\kappa$ lying in the range zero to $(\hbar/\xi_a)[-1+1/N(0)V]^{-1/2}$, $\hbar\omega_0$ is real, being $2\epsilon_0$ at the lower end and zero at the upper end. For larger $i\hbar\kappa$, $\hbar\omega_0$ is pure imaginary. For large $i\hbar\kappa$, the ratio $(i\hbar\omega_0/i\hbar\kappa)$ approaches the limiting phase velocity $v_\infty=\omega_a\xi_a[2N(0)V]^{1/2}$ (still subject to the proviso $|\kappa|\ll\xi_b^{-1}$).

In contrast, consider the case where V_a is *slightly less* than V_b, such that V is still attractive and such that $|\xi_a|$ is still much larger than $|\xi_b|$. Now ξ_a becomes pure imaginary and $\hbar\kappa$ becomes real. We thus have, for Case I, a set of collective excitations for which both $\hbar\omega_0$ and $\hbar\kappa$ are real (i.e., collective *oscillations*, bounded in space and time). In particular, there is one such mode characterized by vanishing phase velocity, $(\omega_0/\kappa)=0$. But, as has already been pointed out in the Introduction, this implies that persistent currents may be unstable,[14] with consequent loss of superconductivity in the literal sense of the word, despite the fact that the individual grains of our system are still in the thermodynamic superconductive phase. Nor can this possible breakdown of superconductivity be blamed on the tunneling junctions having exceeded their current capacity.

Let us return to the case previously considered, where ξ_a and ξ_b are both real, with $\xi_a \gg \xi_b$. For either Case I or II, for $i\hbar\kappa$ sufficiently large, $i\hbar\omega_0$ becomes proportional to $i\hbar\kappa$, v_∞ being the proportionality constant. However, as $i\hbar\omega_0$ continues increasing, F_b gets smaller [see Eq. (6.31)]; while as $i\hbar\kappa$ increases, $[1+(\xi_b\kappa)^2]$ gets smaller. Both these occurrences enhance the effective Coulomb repulsion V_b'', and thus decrease the net effective attractive interaction, V_{eff}, of Eq. (6.29). Thus $i\hbar\omega_0$ actually goes through a maximum and starts decreasing as $i\hbar\kappa$ continues to increase. At a certain value of $i\hbar\kappa$, $i\hbar\omega_0$ will vanish. This occurs at that $i\hbar\kappa$ which makes $V_{\text{eff}}=0$ for Case II; for Case I the corresponding $i\hbar\kappa$ is somewhat smaller. At still larger $i\hbar\kappa$, $\hbar\omega_0$ once again becomes real.

In the remainder of this section, we will indicate how the modified interaction potential of Eq. (6.1) affects the Ginzburg-Landau–like equation of Sec. IV. As before, we take $\varphi_k=\varphi$ independent of $\mathbf{k}$. Define

$$\begin{aligned} \Delta_a &\equiv \Delta_{1a}+i\Delta_{2a}\equiv \Delta_{a0}e^{+i\varphi}, \\ \Delta_b &\equiv \Delta_{1b}+i\Delta_{2b}\equiv \Delta_{b0}e^{+i\varphi}, \end{aligned} \tag{6.32}$$

where

$$\Delta_{a0}=N(0)V_a\int_0^{\hbar\omega_a}\sin\theta_k d\epsilon_k -N(0)V_b\int_0^{\hbar\omega_b}\sin\theta_k d\epsilon_k, \tag{6.33}$$

$$\Delta_{b0}=-N(0)V_b\int_0^{\hbar\omega_b}\sin\theta_k d\epsilon_k.$$

The two quantities Δ_{a0} and Δ_{b0} are real (i.e., not complex) but not necessarily positive. Define g_a, g_b such that

$$\begin{aligned} g_a\cos\theta_k &= \epsilon_k\sin\theta_k, \quad 0\leqslant|\epsilon_k|\leqslant\hbar\omega_a; \\ g_b\cos\theta_k &= \epsilon_k\sin\theta_k, \quad \hbar\omega_a\leqslant|\epsilon_k|\leqslant\hbar\omega_b. \end{aligned} \tag{6.34}$$

Equation (6.34), when substituted into (6.33), gives

$$\Delta_{0a}=N(0)[V_a-V_b]g_a\ln(2\hbar\omega_a/g_a) -N(0)V_bg_b\ln(\omega_b/\omega_a),$$

$$\Delta_{b0}=-N(0)V_b[g_a\ln(2\hbar\omega_a/g_a)+g_b\ln(\omega_b/\omega_a)],$$

which can be rewritten

$$g_a\ln(2\hbar\omega_a/g_a)=[N(0)V_a]^{-1}[\Delta_{a0}-\Delta_{b0}], \tag{6.35}$$

$$g_b\ln(\omega_b/\omega_a)=-[N(0)V_a]^{-1}\times[\Delta_{a0}-\Delta_{b0}(1-V_a/V_b)]. \tag{6.36}$$

Thus g_a is a nonlinear function of $(\Delta_{a0}-\Delta_{b0})$, while g_b is a linear function of Δ_{a0} and Δ_{b0}. The equations of motion, $\mathbf{s}_k\times\mathbf{H}_k=0$, now give the pair of coupled differential equations

$$\begin{aligned} [\xi_a^2\nabla_R^2+1-\Delta_{a0}^{-1}g_a(\Delta_{a0}-\Delta_{b0})]\Delta_a &= 0, \\ [-\xi_b^2\nabla_R^2+1-\Delta_{b0}^{-1}g_b(\Delta_{a0},\Delta_{b0})]\Delta_b &= 0. \end{aligned} \tag{6.37}$$

We will not attempt to discuss these coupled equations in this paper.

VII. FINITE TEMPERATURES

Thus far we have considered only $T=0$, the absolute zero of temperature. Within the context of the effective-field approximation[20] [closely related to the semiclassical approximation, Eq. (3.1), as has already been discussed at the beginning of Sec. III], it is easy to generalize the isospin formulation of superconductivity theory to finite temperatures. In the Hamiltonian density, one merely multiplies each factor of $\mathbf{s}_k$ (and $\nabla_R\mathbf{s}_k$) by the statistical factor $(1-2f_k)$, where f_k is the thermodynamic probability of occupancy of the quasiparticle (one-electron) excited state indexed by wave vector $\mathbf{k}$. Thus

$$(1-2f_k)=\tanh(\tfrac{1}{2}\beta E_k), \tag{7.1}$$

where $\beta=1/k_BT$, and E_k is the quasiparticle excitation energy.

This statistical treatment of the tunneling Hamiltonian [i.e., by multiplying each term in the double sum of (2.6) or (2.10) by the factor $(1-2f_k)\times(1-2f_{k'})$] is completely equivalent to the statistical treatment of the electron-electron interaction Hamiltonian of the BCS theory.[2] Since both interaction Hamiltonians have the same mathematical structure with respect to the operators $\mathbf{s}_k$ [compare the double sums in (2.2) and (2.6)], this procedure is eminently reasonable. There is, however, one difference between the two Hamiltonians which must be kept in mind. The phonon-induced electron-electron matrix element $V_{kk'}$ is inversely proportional to an energy denominator equal to the energy of the typical virtual phonon involved in the intermediate state. Since this energy is much greater than that of the energies of the virtual quasiparticle excitations involved in the intermediate state, it is an excellent approximation to assume that $V_{kk'}$ is independent of (1) any temperature dependence in these quasiparticle energies; (2) whether or not, in BCS terminology, ground pairs or excited pairs are involved. Both assumptions were used by BCS to demonstrate that $(1-2f_k)\times(1-2f_{k'})$ is the appropriate factor to use in the double sum of Eq. (2.2). In contrast, neither assumption is true in the case of the tunneling matrix element $\mathcal{T}_{kk'}$ (or $\mathcal{T}_{kk'}'$), simply because quasiparticle energies alone are involved in the energy denominator involved in $\mathcal{T}_{kk'}$, this energy denominator being temperature-dependent and a function of whether or not ground pairs or excited pairs are tunneling.[4] As a consequence, in addition to the *explicit* temperature-dependent factor $(1-2f_k)$, we must assume that $\mathcal{T}$, and thus ξ, is *implicitly* temperature-dependent. We will not attempt to calculate the temperature dependence of ξ here, other than to point out that ξ will diverge as T approaches T_c, the superconducting transition temperature, as a consequence of the vanishing energy denominator. Thus the behavior of ξ, as $T\rightarrow T_c$, is similar to that of the conventional Ginzburg-Landau coherence distance.[12] (But is it *not* similar to the behavior of the Pippard coherence distance, which stays finite as $T\rightarrow T_c$.)

Our prescription for introducing the statistical factor means that formally the equations of motion [Eqs. (2.15) and (2.19)] are unchanged. It is only in the defining equations for the various order parameters [e.g., Eq. (2.18)] that the factor $(1-2f_k)$ appears.

Let us first consider the nonlinear situation described in Sec. IV. Equation (4.7), the Ginzburg-Landau–like differential equations, is still appropriate. Only now $g(|\Delta|)$ is a function of temperature. To find g, we insert $(1-2f_k)$ into the integral defining $|\Delta|$ [the right-hand side of the equation preceding Eq. (4.8)], i.e.,

$$|\Delta|=N(0)Vg\int_0^{\hbar\omega}(1-2f_k)(\epsilon_k^2+g^2)^{-1/2}d\epsilon_k. \quad (7.2)$$

For this case, the quasiparticle excitation energy E_k is

$$E_k=\epsilon_k\cos\theta_k+|\Delta|\sin\theta_k=(\epsilon_k^2+|\Delta|g)(\epsilon_k^2+g^2)^{-1/2}. \quad (7.3)$$

Defining the ratio

$$\alpha\equiv g/|\Delta|, \quad (7.4)$$

we can now rewrite (7.2) as

$$1=N(0)V\alpha\int_0^{\hbar\omega}(\epsilon^2+|\Delta|^2\alpha^2)^{-1/2}$$
$$\times\tanh\{\tfrac{1}{2}\beta[(\epsilon^2+|\Delta|^2\alpha)(\epsilon^2+|\Delta|^2\alpha^2)^{-1/2}]\}d\epsilon. \quad (7.5)$$

In the limit $|\Delta|\rightarrow 0$, this becomes

$$1=N(0)V\alpha\int_0^{(1/2)\beta\hbar\omega}x^{-1}\tanh x dx, \quad (7.6)$$

which implies that α is finite in this limit (taking $T\neq 0$). Thus g vanishes linearly with $|\Delta|$ when $T>0$. For example, for T near T_c, Eq. (7.6) gives

$$\lim_{|\Delta|\rightarrow 0}g(|\Delta|)=|\Delta|[1-N(0)V(1-T/T_c)]. \quad (7.7)$$

Despite this, there is no power-series expansion of $g(|\Delta|)$ versus $|\Delta|$ at any temperature. This is a consequence of the fact that the integral of Eq. (7.5) cannot be expanded in $|\Delta|$. To see this, rewrite (7.5) as

$$1=\tfrac{1}{2}\beta N(0)V\alpha\int_0^{\hbar\omega}\left(\frac{\epsilon^2+|\Delta|^2\alpha}{\epsilon^2+|\Delta|^2\alpha^2}\right)\left(\frac{\tanh y}{y}\right)d\epsilon, \quad (7.8)$$

where

$$y\equiv\tfrac{1}{2}\beta(\epsilon^2+|\Delta|^2\alpha)(\epsilon^2+|\Delta|^2\alpha^2)^{-1/2}. \quad (7.9)$$

Now $y^{-1}\tanh y$ is an analytic function of y^2 (at $y=0$), but

$$(2/\beta)^2y^2=\epsilon^2+|\Delta|^2\alpha^2+2|\Delta|^2\alpha(1-\alpha)$$
$$+|\Delta|^4\alpha^2(1-\alpha)^2(\epsilon^2+|\Delta|^2\alpha^2)^{-1},$$

when expanded in powers of $|\Delta|^2$, gives rise to a term proportional to $|\Delta|^4\epsilon^{-2}$, causing the above integral to diverge. Still more seriously, the factor $(\epsilon^2+|\Delta|^2\alpha)/(\epsilon^2+|\Delta|^2\alpha^2)$, when expanded in powers of $|\Delta|^2$, gives rise to a term proportional to $|\Delta|^2\epsilon^{-2}$, also causing the integral to diverge. This nonanalyticity of $g(|\Delta|)$ versus $|\Delta|$ can be traced to the functional form of the quasiparticle excitation energy, Eq. (7.3).

In cases where the linearization approximation is to be made, it is appropriate to take the quasiparticle excitation spectrum to be that associated with the unperturbed time- and position-independent situation, i.e.,

$$E_k=(\epsilon_k^2+\Delta_0^2)^{1/2}. \quad (7.10)$$

As before, $\Delta_0=\epsilon_0(T)$ or zero, depending on whether we are considering Case I or Case II, only now $\epsilon_0(T)$ is the finite-temperature BCS half-energy gap. As far as the dispersion relations for the collective excitations are concerned, the only change from $T=0$ is the insertion of $(1-2f_k)$ in the appropriate integrand. For the simpler $V_{kk'}$, this is either Eq. (3.15) or (3.19), since the derivation of Eq. (3.18) from (3.15) still applies at finite T. For the more complicated $V_{kk'}$, this is F_a, defined by Eq. (6.25) and appearing in Eq. (6.30). [The statistical factor can be approximated by unity, and thus ignored, in Eq. (6.26), the defining equation for F_b.]

For simplicity, we now restrict ourselves to the simpler $V_{kk'}$, where the dispersion relation is

$$N(0)V[1-(\xi\kappa)^2]\int_0^{\hbar\omega}\left(\frac{\epsilon^2}{\epsilon^2+\Delta_0^2}\right)\left(\frac{(\epsilon^2+\Delta_0^2)^{1/2}}{\epsilon^2+\Delta_0^2-(\frac{1}{2}\hbar\omega_0)^2}\right)$$
$$\times\tanh[\tfrac{1}{2}\beta(\epsilon^2+\Delta_0^2)^{1/2}]d\epsilon=1. \quad (7.11)$$

First we consider Case II, where $\Delta_0=0$. Then

$$1=N(0)V[1-(\xi\kappa)^2]\int_0^{\hbar\omega}[\epsilon^2-(\tfrac{1}{2}\hbar\omega_0)^2]^{-1}\epsilon$$
$$\times\tanh(\tfrac{1}{2}\beta\epsilon)d\epsilon. \quad (7.12)$$

Making use of the fact that $\hbar\omega\gg k_BT$, and assuming that

$\hbar\omega \gg |\hbar\omega_0|$, we can rewrite (7.12) as

$$(\xi\kappa)^2 = N(0)V[1-(\xi\kappa)^2] \times \left\{ (\tfrac{1}{4}\beta\hbar\omega_0)^2 \int_0^\infty [x^2-(\tfrac{1}{4}\beta\hbar\omega_0)^2]^{-1} x^{-1} \times \tanh x dx + \ln(\beta/\beta_c) \right\}. \quad (7.13)$$

Let us consider only the limit $|\hbar\omega_0| \ll 4k_BT$, so that we can replace $\tanh x$ by x in the above integral. The dispersion relation becomes

$$[1-(\xi\kappa)^2]^{-1}(\xi\kappa)^2/N(0)V - \ln(T_c/T) + (\tfrac{1}{8}\pi)(i\hbar\omega_0/k_BT) = 0. \quad (7.14)$$

In the static limit where $i\hbar\omega_0 = 0$, we get

$$\xi\kappa = \{1+[N(0)V \ln(T_c/T)]^{-1}\}^{-1/2}, \quad (7.15)$$

valid over the whole temperature range. This shows that the static κ goes from ξ^{-1} at $T=0$ to zero at $T=T_c$. In the position-independent limit where $\kappa=0$, we are limited to temperatures close to T_c (in order that $|\hbar\omega_0| \ll 4k_BT$). Here we get

$$i\hbar\omega_0 = (8/\pi)k_BT \ln(T_c/T), \quad (7.16)$$

this being proportional to the square of $\epsilon_0(T)$ near $T=T_c$. Finally, at $T=T_c$, for small $|\xi\kappa|$, we get

$$i\hbar\omega_0 = [8k_BT_c/\pi N(0)V](i\xi\kappa)^2 \quad (7.17)$$

(i.e., both ω_0 and κ are pure imaginary here).

Next we consider Case I, where $\Delta_0 = \epsilon_0(T)$. When $\kappa=0$, we can solve Eq. (7.11) by inspection, getting

$$\hbar\omega_0 = 2\epsilon_0(T) \quad (7.18)$$

for all temperatures. In the same fashion that Eq. (3.18) follows from Eq. (3.15) at $T=0$, we can rewrite Eq. (7.11) as

$$(\xi\kappa)^2 = -N(0)V[1-(\xi\kappa)^2][J(X)+J(-X)], \quad (7.19)$$

where we define

$$J(X) \equiv \left(\frac{1-X^2}{2X}\right) \int_0^{U'} \frac{\tanh[\gamma(x^2+1)^{1/2}]}{[(x^2+1)^{1/2}-X](x^2+1)^{1/2}} dx,$$

$$X \equiv \hbar\omega_0/2\epsilon_0(T), \quad (7.20)$$

$$U' \equiv \hbar\omega/\epsilon_0(T),$$

$$\gamma \equiv \tfrac{1}{2}\beta\epsilon_0(T).$$

Just as in the case at $T=0$, we see that $X^2 \leqslant 1$ and that $(\xi\kappa)^2 \leqslant 0$. We restrict ourselves to temperatures close to T_c (so that $\gamma \ll 1$). If we also take $|\hbar\omega_0| \ll 4k_BT_c$, then

$$J(X)+J(-X) \cong \tfrac{1}{2}\pi\gamma(1-X^2)^{1/2}. \quad (7.21)$$

In the limit $|\xi\kappa| \gg 1$, this gives the dispersion relation

$$(i\xi\kappa)^2 = [\pi N(0)V/8k_BT_c]\{[2\epsilon_0(T)]^2-[\hbar\omega_0]^2\}^{1/2}. \quad (7.22)$$

In calculating the finite-temperature collective excitations, we have assumed, for both Cases I and II, that $i\hbar\omega_0 \ll \hbar\omega$. If we consider the opposite limit, we return to the same result obtained in Eqs. (3.26) and (3.27), namely, that $i\hbar\omega_0$ is proportional to $i\hbar\kappa$, the proportionality constant being $v_\infty = \omega\xi[2N(0)V]^{1/2}$ independent of which case we are considering. Note that the temperature enters only implicitly through its influence on ξ (so that, for example, $v_\infty \to \infty$ as $T \to T_c$).

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