It is noted that this is the same as the corresponding expression for the Schrödinger equation except for the factor of[8] $2k$ and of course the presence of the B-T radial wave function u_0 instead of the Schrödinger wave function. The first Born approximation is of course

$$\tan\delta_0 = -2k'^2 \int_0^\infty dr'\, r'[j_0(k'r')]^2 V(r').$$

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Decay Modes $\eta \to \pi^+\pi^-\pi^0\gamma$ and $\eta \to \pi^0\gamma\gamma$*

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We calculate rates and photon-energy spectra for the decay modes $\eta \to \pi^+\pi^-\pi^0\gamma$ and $\eta \to \pi^0\gamma\gamma$, with the aid of a new model. The relevance of the decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$ to the possibility of C violation in electromagnetic interactions is discussed.

I. INTRODUCTION

In this article, we propose a mechanism which can account for a partial decay rate $\eta \to \pi^0\gamma\gamma$ comparable in magnitude to the other major η decays. We also make a detailed analysis of the decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$, which has not been treated previously and for which no experimental data are yet available. The ratio of these two decays is predicted by the model. The γ-ray energy spectra, whose knowledge is helpful in the experimental detection of these decays, are also presented.

A recent experiment of DiGiugno *et al.*[1] indicates that (37.5 ± 3.6) percent of the neutral decay products of η consist of the decay mode $\eta \to \pi^0\gamma\gamma$. They also obtain for the ratio to the $\eta \to \gamma\gamma$ decay, $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to 2\gamma)] = 0.9\pm0.1$. In apparent disagreement, Wahlig *et al.*[2] report $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)] < 0.5$. Nevertheless, two other experiments seem to confirm the abundant occurrence of the $\eta \to \pi^0\gamma\gamma$ decay. Strugalski *et al.*[3] obtain $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)] = 0.86\pm0.40$, while Grunhaus[4] gives (27 ± 9) percent for the percentage of $\pi^0\gamma\gamma$ among the neutral decay products of eta.

A copious rate for $\eta \to \pi^0\gamma\gamma$ causes some theoretical embarrassment. The ratio of the other two detected radiative decays of eta, namely $\eta \to \pi\pi\gamma$ and $\eta \to \gamma\gamma$ has been successfully accounted for[5,6] by using a basic trilinear vector-vector-pseudoscalar-meson interaction, followed by transitions $\rho \to 2\pi$ and (vector meson) $\to \gamma$. This rho-dominance model also fits very well[7] the photon-energy spectrum in the decay $\eta \to \pi^+\pi^-\gamma$. If this is assumed to be the mechanism for all η radiative decays, then $\eta \to \pi^0\gamma\gamma$ is expected to be very small. Roughly, we estimate

$$(\eta \to \pi^0\gamma\gamma)/(\eta \to \pi^+\pi^-\gamma) \simeq (\rho^0 \to \pi^0\gamma)/(\rho^0 \to \pi^+\pi^-)$$

by using an effective $\eta\rho\gamma$ vertex, which gives $<1\%$ for this ratio. Alles, Baracca, and Ramos[8] have calculated $(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)$ with this model including all possible vector-meson intermediate states and obtain $(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma) = 1.06\times10^{-3}$. When considering also η-X mixing they show that this number cannot be significantly improved without badly damaging the $(\eta \to \pi^+\pi^-\gamma)/(\eta \to \gamma\gamma)$ ratio.

II. FORMULATION OF MODEL AND CALCULATIONS

As the trilinear meson interactions VVP and VPP (V is the vector-meson nonet, P the pseudoscalar-meson octet) fail to account for $\eta \to \pi\gamma\gamma$ by a factor of 10^3, it is reasonable to expect that improvements like form factors, etc. will not change this factor significantly. We suggest therefore that the large rate for $\eta \to \pi^0\gamma\gamma$ is related to quadrilinear meson interactions. A well-known example of this kind is the $\lambda(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2$ term of the interaction Lagrangian. It is natural to enlarge this

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[1] C. DiGiugno, R. Querzoli, G. Troise, F. Vanoli, M. Giorgi, P. Schiavon, and V. Silvestrini, Phys. Rev. Letters **16**, 767 (1966).

[2] M. A. Wahlig, E. Shibata, and I. Manelli, Phys. Rev. Letters **17**, 221 (1966).

[3] Strugalski *et al.*, in Proceedings of the 13th International Conference on High-Energy Physics, Berkeley, California, 1966 (to be published).

[4] J. Grunhaus, Ph.D. thesis, Columbia University, New York, 1966 (unpublished), and (private communication).

[5] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[6] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); **16**, 424 (1966).

[7] F. S. Crawford, Jr., and LeRoy R. Price, Phys. Rev. Letters **16**, 333 (1966).

[8] W. Alles, A. Baracca, and A. T. Ramos, Nuovo Cimento **45**, A272 (1966).

term to a $MMMM$ interaction, where M stands for a vector- or pseudoscalar-meson multiplet. The quadrilinear meson interaction has in fact been already discussed in connection with higher symmetries.[9,10] Here, let us postulate the existence of a direct $VVPP$ term in the interaction Lagrangian, and we take for it the simplest form consistent with Lorentz covariance and gauge invariance. The last requirement permits us to introduce the electromagnetic coupling to neutral vector mesons by direct transitions[11] (vector meson) $\leftrightarrow\gamma$ in a gauge-invariant manner. Then, the term relevant for η decay is in an obvious notation

$$\mathcal{L}_{\text{int}}=(g/2\mu^2)\eta\boldsymbol{\pi}\cdot\boldsymbol{\rho}_{\mu\nu}\chi^{\mu\nu}\,, \tag{1}$$

with $\rho_{\mu\nu}=\partial_\mu\rho_\nu-\partial_\nu\rho_\mu$, etc. The $\boldsymbol{\pi}\cdot\boldsymbol{\rho}$ indicates isotopic-spin structure and g is a dimensionless strong-coupling constant. The arbitrary mass appearing in (1) is chosen to equal the pion mass (μ) because this gives a reasonable strong-interaction range. χ denotes the appropriate combination of the ω and φ fields which has octet transformation properties under SU_3 and thus couples to photon, i.e.,

$$\chi=(1/\sqrt{3})\omega+(\sqrt{2}/\sqrt{3})\varphi\,.$$

Introducing the electromagnetic interaction and the coupling of ρ to pions, we obtain from (1) the two decays $\eta\to(\pi+\rho+\chi)\to\pi+\pi\pi+\gamma$ and $\eta\to(\pi+\rho+\chi)\to\pi+\gamma+\gamma$. The strength of the transitions (vector meson) $\leftrightarrow\gamma$ is given[11] by $f_{\rho\gamma}=em_\rho^2/f_\rho$ and $f_{\chi\gamma}=em_\chi^2/f_\chi$, and $g_{\rho\pi\pi}^2/4\pi=2.4$ from the experimental ρ-meson width. The invariant matrix elements for the decays (1) $\eta\to\pi^+\pi^-\pi^0\gamma$ and (2) $\eta\to\pi^0\gamma\gamma$ are then

$$\begin{aligned}(egg_{\rho\pi\pi}/f_\chi\mu^2)^{-1}M_{(1)}=&\frac{1}{(p^{(+)}+p^{(-)})^2-m_\rho^2}[\epsilon^{(\gamma)}\cdot(p^{(+)}+p^{(-)})k\cdot(p^{(+)}-p^{(-)})-\epsilon^{(\gamma)}\cdot(p^{(+)}-p^{(-)})k\cdot(p^{(+)}+p^{(-)})]\\&+\frac{1}{(p^{(0)}+p^{(+)})^2-m_\rho^2}[\epsilon^{(\gamma)}\cdot(p^{(0)}+p^{(+)})k\cdot(p^{(0)}-p^{(+)})-\epsilon^{(\gamma)}\cdot(p^{(0)}-p^{(+)})k\cdot(p^{(0)}+p^{(+)})]\\&+\frac{1}{(p^{(-)}+p^{(0)})^2-m_\rho^2}[\epsilon^{(\gamma)}\cdot(p^{(-)}+p^{(0)})k\cdot(p^{(-)}-p^{(0)})-\epsilon^{(\gamma)}\cdot(p^{(-)}-p^{(0)})k\cdot(p^{(-)}+p^{(0)})]\end{aligned} \tag{2}$$

$$(e^2g/f_\rho f_\chi\mu^2)^{-1}M_{(2)}=(1/\sqrt{2})[(\epsilon^{(\gamma)}\cdot\epsilon^{(\gamma')})(k\cdot k')-(\epsilon^{(\gamma)}\cdot k')(\epsilon^{(\gamma')}\cdot k)]\,, \tag{3}$$

where k, $\epsilon^{(\gamma)}$, k', $\epsilon^{(\gamma')}$, $p^{(+)}$, $p^{(-)}$, $p^{(0)}$, are the polarization and momentum four-vectors of photons and pions. The decay rates are found to be

$$\Gamma(\eta\to\pi^+\pi^-\pi^0\gamma)=\frac{g^2}{4\pi}\frac{g_{\rho\pi\pi}^2}{4\pi}\frac{\alpha}{(2\pi)^3\mu^4(f_\chi^2/4\pi)}\int_0^{k_{\max}}dk_0\frac{k_0^3}{M_\eta-2k_0}F(k_0)\,, \tag{4}$$

$$\begin{aligned}F(k_0)=\int_\mu^{\omega_{\max}}d\omega\Bigg\{&\frac{[(\omega^2-\mu^2)(Q^2-4\mu^2)Q^2]^{1/2}}{Q^2-m_\rho^2}\left[\frac{Q^2-4\mu^2}{Q^2-m_\rho^2}+\frac{2(t^2-2t\omega-2\mu^2+2m_\rho^2)}{t^2}\right]\\&+\frac{1}{t^3(Q^2-m_\rho^2)}[Q^2(t\omega+\mu^2-m_\rho^2)(t^2-2t\omega-2\mu^2+2m_\rho^2)-t^2(Q^2-4\mu^2)(t\omega-2\omega^2+\mu^2)]\ln\frac{1-\beta}{1+\beta}\Bigg\}\,,\end{aligned} \tag{5}$$

where

$$\beta=\frac{t(\omega^2-\mu^2)^{1/2}(Q^2-4\mu^2)^{1/2}}{(Q^2)^{1/2}(t\omega+\mu^2-m_\rho^2)};\qquad t=(M_\eta^2-2M_\eta k_0)^{1/2}$$

$$Q^2=t^2-2t\omega+\mu^2;\qquad \omega_{\max}=\frac{t^2-3\mu^2}{2t} \tag{6}$$

and

$$\Gamma(\eta\to\pi^0\gamma\gamma)=\frac{g^2}{4\pi}\frac{\alpha^2M_\eta}{6\pi^2(f_\chi^2/4\pi)(f_\rho^2/4\pi)\mu^4}\int_0^{\bar{k}_{\max}}dk_0\frac{k_0^3(\bar{k}_{\max}-k_0)^3}{(M_\eta-2k_0)^3}\,, \tag{7}$$

where

$$\bar{k}_{\max}=\frac{M_\eta}{2}-\frac{\mu^2}{2M_\eta}\,. \tag{8}$$

[9] I. S. Gerstein, Phys. Rev. Letters 14, 453 (1965).
[10] D. Griffiths and D. Welling, Phys. Rev. Letters 14, 874 (1965).

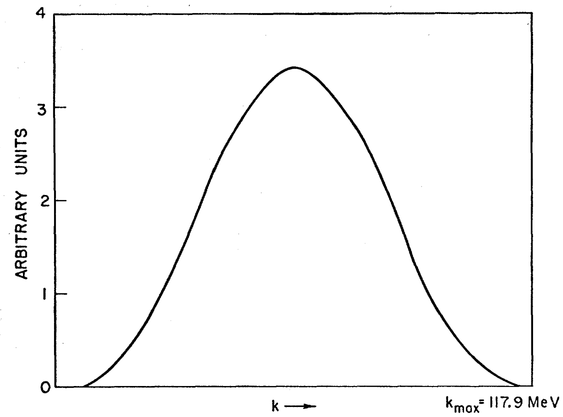

FIG. 1. The calculated γ-ray energy spectrum in the decay $\eta\rightarrow\pi^+\pi^-\pi^0\gamma$ in the η rest frame.

α is the fine-structure constant. The photon energy spectra for the two decays in the η rest system are given in Figs. 1 and 2.

The ratio of the two decays is independent of the values of g, f_X and the choice of the pion mass in the definition of (1). Using for f_ρ the value given by the ρ dominance of the pion form factor $f_\rho = g_{\rho\pi\pi}$,[11] one obtains

$$\Gamma(\eta\rightarrow\pi^+\pi^-\pi^0\gamma)/\Gamma(\eta\rightarrow\pi^0\gamma\gamma)=0.23\%. \tag{9}$$

This ratio involves only the direct decay $\eta\rightarrow\pi^+\pi^-\pi^0\gamma$ which occurs to order α. One has also a similar inner bremstrahlung decay which occurs to order α^3, where the γ ray is emitted by π^+ or π^- from the process $\eta\rightarrow\pi^+\pi^-\pi^0$. This is expected to be significant only for the very low-energy end of the γ spectrum.

In order to check whether (7) can account for the observed rate of $\eta\rightarrow\pi^0\gamma\gamma$, we calculate the required g^2 strength. There is no experimental measurement of the

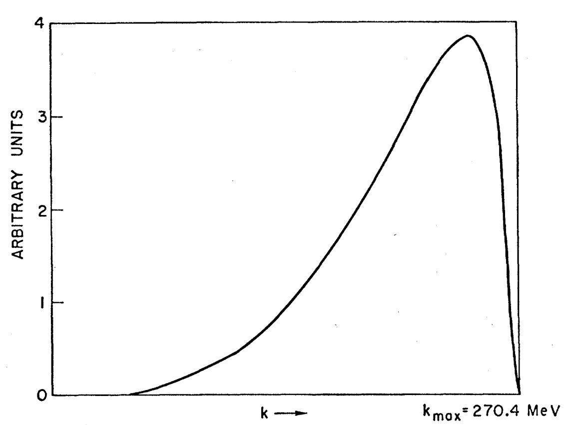

FIG. 2. The calculated γ-ray energy spectrum in the decay $\eta\rightarrow\pi^0\gamma\gamma$ in the η rest frame.

[11] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

η life-time and theoretical estimates[12,13] give Γ_η (total) to range between 0.2–2 keV. Then, if we take $\eta^0\rightarrow\pi^0\gamma\gamma$ to amount for 25%[1,3,4] of the neutral decay modes and use a ratio of neutral to charged η-decay products of 2.3, one needs from (7) a strength

$$g^2/4\pi=3.5\text{–}35 \tag{10}$$

to accommodate the decay within the estimated η life-time.[14] Although somewhat on the higher side, this is a reasonable number.

III. DISCUSSION

(i) The decay $\eta\rightarrow\pi^0\gamma\gamma$ has been indeed predicted by Bronzan and Low[15] to be as frequent as $\eta\rightarrow\gamma\gamma$, on the basis of A-quantum-number considerations. They assume that A-forbidden processes are inhibited by a factor $\epsilon\simeq0.01$ and roughly compare two-body to three-body phase space by another factor of $\simeq100$. If one estimates in the same manner the $\eta\rightarrow\pi^+\pi^-\pi^0\gamma$ A-allowed process, using again a factor $a\simeq100$ for the three- to four-particle phase space, one would have[16]

$$(\eta\rightarrow\pi^+\pi^-\gamma)/(\eta\rightarrow\pi^+\pi^-\pi^0\gamma)=\epsilon a\simeq1$$

or

$$(\eta\rightarrow\pi^+\pi^-\pi^0)/(\eta\rightarrow\pi^+\pi^-\pi^0\gamma)\simeq\alpha a\simeq1\,.$$

Hence, A-quantum-number considerations indicate $\eta\rightarrow\pi^+\pi^-\pi^0\gamma$ should be comparable to the other main η decays. Our model however predicts it to be appreciably smaller.

(ii) A key feature of our calculation is the ρ dominance for the $\pi\pi$ versus γ in comparing the two decays. In fact, one could write the simplest effective vertex for $\eta\rightarrow\rho\pi\gamma$, from which if one derives the ratio of the decays, one obtains the result given in (9). Hence, this result is more general than the specific assumption of the underlying interaction (1). As the ρ-dominance model accounts very well for $(\omega\rightarrow\pi\gamma)/(\omega\rightarrow3\pi)$,[5] and $(\eta\rightarrow\pi\pi\gamma)/(\eta\rightarrow\gamma\gamma)$,[5,6] one expects (9) to be a faithful estimate. Moreover, the equivalence of the vector dominance model with the current algebra results for these cases has been demonstrated recently in several papers.[17]

(iii) The transition $\eta\rightarrow\pi^+\pi^-\pi^0\gamma$ proceeds to the most favorable final-state configuration by an electric dipole transition. This configuration is with two pions in a P wave, while the third pion is in an S wave relative to the dipion. The three-pion state is an eigenstate of the charge-conjugation operator, with $C=-1$ and has zero

[12] F. A. Berends and P. Singer, Phys. Letters **19**, 249, 616 (1965).
[13] M. Veltman and J. Yellin, Phys. Rev. **154**, 1469 (1967).
[14] We use SU_3 to relate $f_X=\sqrt{3}f_\rho$.
[15] J. B. Bronzan and F. Low, Phys. Rev. Letters **12**, 522 (1964).
[16] $\eta\rightarrow3\pi$, $\eta\rightarrow\pi^0\gamma\gamma$ are A-allowed; $\eta\rightarrow\pi^+\pi^-\gamma$, $\eta\rightarrow\gamma\gamma$ are A-forbidden.
[17] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255, 384 (1966); W. W. Wada, *ibid.* **16**, 956 (1966); M. Ademollo and R. Gatto, Nuovo Cimento **44**, A282 (1966); M. Ademollo, *ibid.* **46**, A156 (1966).

isotopic spin. The total angular momentum and parity of this three-pion state is 1^+. Hence, if there is an axial-vector meson with $T=0$, $C=-1$, it could mediate the decay through a primary trilinear PVA vertex. So far, no reliable experimental evidence for such a meson is available.

(iv) If the C violation is connected with electromagnetic processes,[18,19] the $\eta\to\pi\pi\pi\gamma$ decay offers new possibilities for its investigation. The lowest configuration obtained in a C violating $\eta\to\pi\pi\pi\gamma$ transition is again with three pions in an 1^+ state, belonging however to a $T=1$ state and having $C=+1$. It consists of a dipion of zero spin with a third pion in a P-wave relative to it. Now, the observation of C violation in $\eta\to\pi^+\pi^-\pi^0$ and $\eta\to\pi^+\pi^-\gamma$ is hampered by the fact that the C violating transitions have to occur against large angular-momentum barriers, being therefore inhibited by a factor difficult to estimate. The $\eta\to\pi\pi\pi\gamma$ transition which would violate C has no more angular-momentum barriers that the C-conserving transition. If a sufficient number of $\eta\to\pi^+\pi^-\pi^0\gamma$ decays is observed, one expects a large π^+/π^- asymmetry if there are electromagnetic C-violating interactions. In such a case, the decay $\eta\to\pi^0\pi^0\pi^0\gamma$ will also occur.

One should however keep in mind that in this decay, the possible C-violating transition is of isovector nature. (It is isoscalar in the C-conserving decay.) Hence, no C-violating effects will show up in $\eta\to\pi\pi\pi\gamma$ if the C violating transition is of isoscalar character, as required by certain models.[20]

[18] J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. **139**, B1650 (1965).

[19] S. Barshay, Phys. Letters **17**, 78 (1965).

[20] T. D. Lee, in Proceedings of the 13th International Conference on High-Energy Physics, Berkeley, California, 1966 (to be published).

(v) If the $VVPP$ interaction produces the decays discussed here, it should also have other detectable effects. Directly related to the decays in question are the radiative decays $\rho\to\eta\pi\gamma$ and $\varphi,\omega\to\eta\pi\gamma$, which turn out to be rare and their detection is a matter for the future. Other more favorable transitions, like $\varphi,\omega\to\pi^+\pi^-\gamma$ are not simply related to $\eta\to\pi\gamma\gamma$ due to the SU_3 structure of $VVPP$. One can express an SU_3-invariant $VVPP$ interaction in the following way

$$\mathcal{L}_{\mathrm{int}}=g_1\,\mathrm{Tr}(VV)\,\mathrm{Tr}(PP)+g_2\,\mathrm{Tr}(VP)\,\mathrm{Tr}(VP)+g_3\,\mathrm{Tr}(VVPP)+g_4\,\mathrm{Tr}(VPVP). \quad (11)$$

The $\omega\omega\boldsymbol{\pi}\cdot\boldsymbol{\pi}$ vertex giving rise to $\omega\to\pi^+\pi^-\gamma$ has no contribution from g_2, while $\boldsymbol{\pi}\cdot\boldsymbol{\rho}\omega\eta$ giving rise to $\eta\to\pi^0\gamma\gamma$ has no term with g_1.

One might remark, that in discussing quadrilinear meson interactions in higher symmetry schemes, Gerstein[9] gives arguments for the magnitude of the coupling involved, obtaining it equal to the $g_{\rho\pi\pi}$ coupling. This compares very favorably with our Eq. (10). Our number however relies on the choice of the pion mass in defining the interaction in Eq. (1). Nevertheless, this is justified also by the fact that the average of the momenta appearing in the Lagrangian is indeed approximately μ in the decays discussed here.

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