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Bakamjian-Thomas Theory and the Serber Model*

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The Bakamjian-Thomas theory is used to obtain relativistic scattering equations for the relative motion of two particles. These are presented in a squared and in an unsquared form, and both forms are realized in Cartesian coordinates and reduced to partial-wave equations for the case of an isotropic potential dependent only on the Bakamjian-Thomas relative coordinate. These equations are shown to bear a resemblance to the Klein-Gordon equation with Serber's potential A. The first Born approximation of the scattering amplitude computed from the Bakamjian-Thomas equations and that computed from the Klein-Gordon equation are shown to agree as $k\to\infty$. Differences between these approaches are expected to show themselves mainly at large-angle scattering, since the (unsquared) Bakamjian-Thomas Green's function differs appreciably from that of the Klein-Gordon equation only for small r.

I. INTRODUCTION

THE Bakamjian-Thomas (B-T) model for describing interaction relativistically but with potential functions is employed to derive a state-vector equation of motion similar to that postulated in Serber's model. The equation is presented in "squared" and "unsquared" versions; each is realized in Cartesian coordinates, and on assuming a potential independent of the relative momentum, each is then separated into partial-wave radial equations. Computations of the relevant kernels and Green's functions are presented. The squared and unsquared B-T equations are compared with Serber's Klein-Gordon (K-G) equation with regard to Born approximations and Green's functions.

II. A REVIEW OF B-T THEORY

Bakamjian and Thomas[1] have formulated a relativistic theory, which instead of resorting to field operators, is based on a relativistic Schrödinger equation which acts directly on a Hilbert space. A potential energy V is incorporated in the theory in such a way that the ten generators of the proper inhomogeneous Lorentz group satisfy the usual commutation relations of this group.[2] Comparison of the B-T equation and Serber's KG equation will show that V corresponds roughly to the "V" in the $(V,0,0,0)$ four-vector-potential inserted into the K-G equation. Comments on the appropriateness of Serber's imaginary-isotropic V are given in Appendix I.

Recently, the question of defining scattering in the B-T theory has been investigated by Fong and Sucher[3] and by Coester.[4] For a bibliography the reader is referred to these two papers.

The ten infinitesimal generators corresponding to two free spinless particles, space translation (three momentum), time translation (Hamiltonian), space-time rotations (velocity boosts), and spatial rotations (angular momentum), are

$$H_0=H_1+H_2\,,$$
$$\mathbf{P}_0=\mathbf{p}_1+\mathbf{p}_2\,,$$
$$\mathbf{K}_0=\tfrac{1}{2}(\mathbf{r}_1H_1+H_1\mathbf{r}_1)+\tfrac{1}{2}(\mathbf{r}_2H_2+H_2\mathbf{r}_2)\,,$$
$$\mathbf{J}_0=\mathbf{r}_1\times\mathbf{p}_1+\mathbf{r}_2\times\mathbf{p}_2\,,$$

where $H_i=(m_i^2+\mathbf{p}_i^2)^{1/2}$. The subscript 0 indicates no interaction, and $\mathbf{r}_i$ and $\mathbf{p}_i$ are the usual canonically conjugate single-particle position and momentum operators. H_0, $\mathbf{P}_0$, $\mathbf{K}_0$, $\mathbf{J}_0$ trivially satisfy the commutation relations of the inhomogeneous Lorentz group.

To introduce an interaction without disturbing Lorentz invariance it is necessary to modify these operators in such a way that the commutation relations are still satisfied. As usual this is done in the "instant" form of interaction, but without extending the two-particle Hilbert space. Instead of creating and annihilating particles, a "potential" is introduced in a way reminiscent of nonrelativistic quantum mechanics.

The theory is formulated in terms of center-of-mass variables obtained by a canonical transformation from $\mathbf{r}_1$, $\mathbf{p}_1$, $\mathbf{r}_2$, $\mathbf{p}_2$ to center-of-mass variables $\mathbf{R}$, $\mathbf{P}$, $\mathbf{x}$, $\mathbf{k}$, where $\mathbf{R}$ is conjugate to $\mathbf{P}$, $\mathbf{x}$, to $\mathbf{k}$. The transformation is defined by

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2 \quad\text{and}\quad \mathbf{k}=(\epsilon_1\mathbf{p}_1-\epsilon_2\mathbf{p}_2)/(\epsilon_1+\epsilon_2)\,,$$

where

$$E_i=(m_i^2+\mathbf{p}_i^2)^{1/2},\quad w_i=(m_i^2+\mathbf{k}^2)^{1/2},\quad \epsilon_i=\tfrac{1}{2}(E_i+w_i)\,,$$

$i=1$, 2, and $\mathbf{k}^2$ is obtained from

$$w_1(\mathbf{k}^2)+w_2(\mathbf{k}^2)=(P_i^2)^{1/2}\,,$$

and where P_i is the total-four-momentum vector. Although the expression for $\mathbf{k}$ appears formidable, it is simply the momentum of particle 1 in the center-of-mass system which is traveling with a velocity $\mathbf{u}=(\mathbf{p}_1+\mathbf{p}_2)/(E_1+E_2)$ (and, of course, $-\mathbf{k}$ is the momentum of

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[1] B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).
[2] The V here is equal to the $v/2$ of Fong and Sucher and should not be confused with their $V=H-H_0$.
[3] R. Fong and J. Sucher, J. Math. Phys. 5, 956 (1964); see also their bibliography.
[4] F. Coester, Helv. Phys. Acta 38, 7 (1965); see also his bibliography.

potential 2). It is easy to show that

$$H_0=H_1+H_2=(\mathbf{P}^2+h_0^2)^{1/2},$$

where

$$h_0^2=(m_1^2+\mathbf{k}^2)^{1/2}+(m_2^2+\mathbf{k}^2)^{1/2}.$$

A potential $v(\mathbf{k},\mathbf{x})$ which is a rotationally invariant function of $\mathbf{k}$ and $\mathbf{x}$ may be introduced into the infinitesimal generators H and $\mathbf{K}$ so that the commutation relations are still satisfied:

$$h_0\rightarrow h_0+v(\mathbf{k},\mathbf{x})\equiv h,$$

so that

$$\begin{aligned}H&=(h^2+\mathbf{P}^2)^{1/2},\\ \mathbf{K}&=\tfrac{1}{2}(\mathbf{R}H+H\mathbf{R})-\mathbf{l}\times\mathbf{P}(h+H)^{-1},\end{aligned}\tag{1}$$

where $\mathbf{l}$ is the internal angular momentum $\mathbf{x}\times\mathbf{k}$. It is this representation which was discovered by Bakamjian and Thomas in 1953. Fong and Sucher have shown that the S matrix associated with the B-T Hamiltonian is "asymptotically covariant" for a potential $v(\mathbf{k},\mathbf{x})$ which is rotationally invariant and is sufficiently localized that scattering boundary conditions may be applied. It should be noted that this leaves considerable freedom for defining even an energy-independent potential; see Appendix I. Fong and Sucher further argue that "$\cdots$ if a two-body covariant S matrix has a 'potential' origin $\cdots$ then there always exists a B-T type of Hamiltonian which yields the same S matrix."

III. THE SQUARED B-T EQUATION

A. Derivation of the Squared Equation

For convenience the B-T equation is written in the center-of-mass system. Since the three components of $\mathbf{P}$ commute, a basis on which they are simultaneously diagonal exists. If the state vector is an eigenstate of $\mathbf{P}$, it will remain one with the same eigenvalue since $[H,\mathbf{P}]=0$. Furthermore, any eigenstate of $\mathbf{P}$ can be velocity-transformed to a state with $\mathbf{P}=0$. The scattering problem will be carried through for such states, i.e., "in the center-of-mass system."

In terms of the single-particle energy E, the B-T equation becomes

$$H|\psi\rangle=(\sqrt{h^2})|\psi\rangle=2E|\psi\rangle.\tag{2}$$

The squared equation follows from this by operating on both sides from the left with H:

$$\begin{aligned}HH|\psi\rangle&=2HE|\psi\rangle=2EH|\psi\rangle,\\ H^2|\psi\rangle&=(2E)^2|\psi\rangle.\end{aligned}$$

On substituting from Eqs. (1), and putting $m_1=m_2\equiv m$, this reduces to the explicit form

$$(-\mathbf{k}^2+\mathbf{k}'^2)|\psi\rangle=[(m^2+\mathbf{k}^2)^{1/2}v/2+(v/2)(m^2+\mathbf{k}^2)^{1/2}+(v/2)^2]|\psi\rangle,$$

where $\mathbf{k}'^2=E^2-m^2$. Notice that, because $v\rightarrow 0$ as $|\mathbf{x}|\rightarrow\infty$, $\mathbf{k}'^2$ may be regarded as an "eigenvalue" of $\mathbf{k}^2$ in the state $|\psi\rangle$ for large $|\mathbf{x}|$. It is convenient to remove the factor $\frac{1}{2}$ from these equations by introducing $V=v/2$:

$$(-\mathbf{k}^2+\mathbf{k}'^2)|\psi\rangle=[(m^2+\mathbf{k}^2)^{1/2}V+V(m^2+\mathbf{k}^2)^{1/2}+V^2]|\psi\rangle.\tag{3}$$

B. Realization in Cartesian Coordinates

To realize this operator equation in a representation with $\mathbf{x}$ diagonal (keeping $\mathbf{P}=0$), complete sets of states are introduced. (Notice that $\mathbf{x}$ is used as the eigenvalue of the operator $\mathbf{x}$, in order to save primes.)

$$\begin{aligned}\int d^3x''\,\langle\mathbf{x}|-\mathbf{k}^2+\mathbf{k}'^2|\mathbf{x}''\rangle\langle\mathbf{x}''|\psi\rangle
&=\int d^3x'd^3x''\,[\langle\mathbf{x}|(m^2+\mathbf{k}^2)^{1/2}|\mathbf{x}'\rangle\langle\mathbf{x}'|V|\mathbf{x}''\rangle\\
&\quad+\langle\mathbf{x}|V|\mathbf{x}'\rangle\langle\mathbf{x}'|(m^2+\mathbf{k}^2)^{1/2}|\mathbf{x}''\rangle]\langle\mathbf{x}''|\psi\rangle\\
&\quad+\int d^3x''\,\langle\mathbf{x}|V^2|\mathbf{x}''\rangle\langle\mathbf{x}''|\psi\rangle.\end{aligned}$$

The possible $\mathbf{k}^2$ and $\mathbf{k}\cdot\mathbf{x}$ dependences of V (Appendix I) are now excluded, as in the case of Serber's potential:

$$V=V(|\mathbf{x}|).$$

Then,

$$(\nabla^2+\mathbf{k}'^2)\psi(\mathbf{x})=\int d^3x''\,[K(|\mathbf{x}-\mathbf{x}''|)V(|\mathbf{x}''|)+V(|\mathbf{x}|)K(|\mathbf{x}-\mathbf{x}''|)]\psi(\mathbf{x}'')+V^2(|\mathbf{x}|)\psi(\mathbf{x}),\tag{4}$$

where

$$K(|\mathbf{x}-\mathbf{x}'|)=\langle\mathbf{x}|(m^2+\mathbf{k}^2)^{1/2}|\mathbf{x}'\rangle.$$

$K(|\mathbf{x}-\mathbf{x}'|)$ is computed by inserting a complete set of states:

$$\begin{aligned}K(|\mathbf{x}-\mathbf{x}'|)&=\int d^3k'd^3k''\,\langle\mathbf{x}|\mathbf{k}'\rangle\times\langle\mathbf{k}'|(m^2+\mathbf{k}^2)^{1/2}|\mathbf{k}''\rangle\langle\mathbf{k}''|\mathbf{x}'\rangle\\
&=\frac{1}{(2\pi)^3}\int d^3k'\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}(m^2+\mathbf{k}'^2)^{1/2}\\
&=\frac{1}{(2\pi)^2}\int_0^\infty dk\,k^2(m^2+k^2)^{1/2}\times\int_{\theta=0}^{\pi}d(\cos\theta)e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta},\end{aligned}$$

where $k=|\mathbf{k}'|$ and θ has for convenience been chosen as the angle between $(\mathbf{x}-\mathbf{x}')$ and $\mathbf{k}$. The θ integration yields

$$\begin{aligned}K(|\mathbf{x}-\mathbf{x}'|)&=\frac{-2}{(2\pi)^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty dk\,k(m^2+k^2)^{1/2}\sin k|\mathbf{x}-\mathbf{x}'|\\
&=\frac{m^2}{2\pi^2|\mathbf{x}-\mathbf{x}'|^2}K_2(m|\mathbf{x}-\mathbf{x}'|),\end{aligned}\tag{5}$$

where K_2 is the second-order modified Bessel function of the second kind. It should be noted that the integral in Eq. (5) does not literally converge but may be defined by introducing a cutoff function of the form $e^{-\Lambda k}$ and after the integration letting $\Lambda\rightarrow 0$.[5]

Introducing Eq. (5) into Eq. (4) yields

$$(\nabla^2+\mathbf{k}'^2)\psi(\mathbf{x})=\int d^3x'\frac{m^2}{2\pi^2|\mathbf{x}-\mathbf{x}'|^2}K_2(m|\mathbf{x}-\mathbf{x}'|)\times[V(|\mathbf{x}|)+V(|\mathbf{x}'|)]\psi(\mathbf{x}')+V^2(|\mathbf{x}|)\psi(\mathbf{x}). \quad (6)$$

It should be noted that nonlocality enters this equation not only implicitly in the use of an instantaneous-action-at-a-distance potential but also explicitly in the way this potential is incorporated into the equation.

C. Partial-Wave Reduction

The squared B-T equation may be realized in spherical coordinates in terms of an indicated integral. To accomplish this it is helpful to find the action of any smooth function $F(\mathbf{k}^2)$ of the operator $\mathbf{k}^2$ on a partial-wave expansion

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\phi)g_l(r).$$

It is shown in Appendix II that

$$F(\mathbf{k}^2)[\sum_{l,m} Y_l^m(\theta,\phi)g_l(r)]=\sum_{l,m} Y_l^m(\theta,\phi)F(k_l^2)[g_l(r)], \quad (7)$$

where

$$k_l=\left[-\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{l(l+1)}{r^2}\right]^{1/2} \quad (8)$$

and also that for any smooth function $g(r)$,

$$F(k_l^2)[g(r)]=\int_0^\infty dr'\,g(r')\times\left[\frac{2}{\pi}r'^2\int_0^\infty dk\,k^2F(k^2)j_l(kr)j_l(kr')\right]. \quad (9)$$

Equation (3) is now realized in spherical coordinates by applying Eqs. (7) and (9) with $g=R_l$ and $g=VR_l$:

$$(-k_l^2+k'^2)R_l(r)=\int_0^\infty dr'\,V(r')R_l(r')\times\left[\frac{2}{\pi}r'^2\int_0^\infty dk\,k^2(m^2+k^2)^{1/2}j_l(kr)j_l(kr')\right]+V(r)\int_0^\infty dr'\,R(r')\left[\frac{2}{\pi}r'^2\int_0^\infty dk(m^2+k^2)^{1/2}\times j_l(kr)j_l(kr')\right]+V^2(r)R_l(r).$$

[5] A. Erdélyi, *Table of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 1.

Substitution of the usual radial wave function $u_l=rR_l(r)$ yields the partial-wave equation

$$\left(\frac{d^2}{dr^2}+k'^2-\frac{l(l+1)}{r^2}\right)u_l(r)=\int_0^\infty dr'\,\kappa(r,r';l)u_l(r')\times[V(r')+V(r)]+V^2(r)u_l(r), \quad (10)$$

where

$$\kappa(r,r';l)=\frac{2rr'}{\pi}\int_0^\infty dk\,k^2(m^2+k^2)^{1/2}j_l(kr)j_l(kr').$$

D. Zero-Mass Approximations

From Eq. (5),

$$\lim_{m\to 0}K(|\mathbf{x}-\mathbf{x}'|)\equiv K_{m=0}(|\mathbf{x}-\mathbf{x}'|)=\frac{1}{\pi^2|\mathbf{x}-\mathbf{x}'|^4}.$$

It is noted that this result is identical with that obtained by replacing $\langle\mathbf{x}|(m^2+\mathbf{k}^2)^{1/2}|\mathbf{x}'\rangle$ by $\langle\mathbf{x}||\mathbf{k}||\mathbf{x}'\rangle$.

The radial function $\kappa(r,r';l)$ has been evaluated in closed form only for zero mass:

$$\lim_{m\to 0}\kappa(r,r';l)\equiv\kappa_{m=0}(r,r';l)=\frac{2rr'}{\pi}\int_0^\infty dk\,k^3j_l(kr)j_l(kr')=\frac{\pi}{2r'^{1/2}(rr')^{1/2}}\int_0^\infty dk\,k^2[k^{-1/2}J_{l+\frac{1}{2}}(kr)]\times J_{l+\frac{1}{2}}(kr')(kr')^{1/2},$$

and

$$\int_0^\infty dk\,k^{-1/2}J_{l+\frac{1}{2}}(kr)J_{l+\frac{1}{2}}(kr')(kr')^{1/2}=\frac{1}{\pi r^{1/2}}Q_l\left(\frac{r^2+r'^2}{2rr'}\right),$$

so finally

$$\kappa_{m=0}(r,r';l)=\frac{1}{2rr'^{l+1}}\left(\frac{d}{r'dr'}\right)^2\left[r'^{l+2}Q_{l+2}\left(\frac{r^2+r'^2}{2rr'}\right]\right).$$

IV. THE UNSQUARED B-T EQUATION

A. Derivation of the Unsquared Equation

Squaring the B-T equation has been shown to yield Eqs. (6) and (10), three-dimensional and radial K-G equations with potential terms on their right-hand sides which involve convolutions with a kernel. Because the scattering Green's functions for the Cartesian and radial K-G equations are known, the application of either Eq. (6) or (10) to scattering is, except for the complexity of their right-hand sides, straightforward.

To avoid the complications of these convolution integrals, the unsquared B-T equation is considered directly. The scattering Green's function however becomes more complicated. From Eqs. (1) and (2), recalling that $v=2V$, and again putting the masses equal

$$H|\psi\rangle=\{[2(m^2+\mathbf{k}^2)^{1/2}+2V(|\mathbf{x}|)]^2\}^{1/2}|\psi\rangle=2E|\psi\rangle.$$

It is first shown that this equation may be replaced by

$$H'|\psi\rangle=[2(m^2+\mathbf{k}^2)^{1/2}+2V(|\mathbf{x}|)]|\psi\rangle=2E|\psi\rangle, \quad (11)$$

for any complex potential $V(|\mathbf{x}|)$ which decays sufficiently rapidly. Then $H=H'$, because any possible ambiguity of signs which might have arisen on squaring and then extracting the square root does not arise. The scattering wave function $\psi(\mathbf{x})$ is approximated asymptotically as $|\mathbf{x}|\to\infty$ by

$$\psi_A(\mathbf{x})=(1/\mathrm{Vol}^{1/2})\{e^{i\mathbf{k}'\cdot\mathbf{x}}+[f(\theta)/r]e^{i|\mathbf{k}'||\mathbf{x}|}\},$$

where Vol≡volume in three-space. $\langle H'\rangle$ is given by the inner product

$$(\psi(\mathbf{x}), [2(m^2+\mathbf{k}^2)^{1/2}+2V(|\mathbf{x}|)]\psi(\mathbf{x})).$$

Because of the $1/\mathrm{Vol}^{1/2}$ in the normalization of the wave function, the contribution to $\langle H'\rangle$ from any fixed volume tends to zero as Vol$\to\infty$. ψ may therefore be replaced by ψ_A in a volume which starts at some large radius so that even the $e^{i|\mathbf{k}||\mathbf{x}|}/r$ terms do not contribute to this volume integral. Thus

$$\langle H'\rangle\to\frac{1}{\mathrm{Vol}}[e^{i\mathbf{k}'\cdot\mathbf{x}}, 2(m^2+\mathbf{k}^2)^{1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}]$$
$$=\frac{2}{\mathrm{Vol}}\int d^3x\, e^{i\mathbf{k}'\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}}(m^2+\mathbf{k}'^2)^{1/2}$$
$$=2(m^2+\mathbf{k}'^2)^{1/2}=2E,$$

the positive total energy in the center-of-mass system. It is noted that this result does not depend on the Hermiticity of $V(|\mathbf{x}|)$.

B. Realization in Cartesian Coordinates

To realize Eq. (11) in Cartesian three-space, complete sets of states are inserted into the inverse equation

$$|\psi\rangle=-\frac{1}{(m^2+\mathbf{k}^2)^{1/2}-E}V(|\mathbf{x}|)|\psi\rangle+|\mathbf{k}'\rangle, \quad (12)$$

yielding

$$\psi(x)=-\int d^3x'\, G(|\mathbf{x}-\mathbf{x}'|)V(|\mathbf{x}'|)\psi(\mathbf{x}')+(2\pi)^{-3/2}e^{i\mathbf{k}'\cdot\mathbf{x}}. \quad (13)$$

Here

$$G(|\mathbf{x}-\mathbf{x}'|)=\langle\mathbf{x}|\frac{1}{(m^2+\mathbf{k}^2)^{1/2}-E}|\mathbf{x}'\rangle$$

which becomes, after transforming to spherical coordinates and on performing the angular integrations,

$$G(|\mathbf{x}-\mathbf{x}'|)=\frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty dk\frac{k\sin k|\mathbf{x}-\mathbf{x}'|}{(m^2+\mathbf{k}^2)^{1/2}-E}. \quad (14)$$

(Again $k=|\mathbf{k}'|$.) This integral, which does not converge as a real integral, becomes a Green's function if the contour of integration is moved away slightly from the singularity at $k=(E^2-m^2)^{1/2}$. The quadrature in Eq. (14) was not performed; the branch points of the integrand at $k=\pm im$ cause some difficulty.

It should be mentioned that $G(|\mathbf{x}-\mathbf{x}'|)$ reduces to the usual[6] nonrelativistic expression in the limit $k\ll m$: Abbreviating $|\mathbf{x}-\mathbf{x}'|\equiv r$,

$$G(r)=\frac{1}{2\pi^2 r}\int_0^\infty dk\frac{k\sin kr}{(m^2+k^2)^{1/2}-(m^2+k'^2)^{1/2}}$$
$$\underset{k\ll m}{\longrightarrow}\frac{1}{2\pi^2 r}\int_0^\infty dk\frac{k\sin kr}{m+k^2/2m-m-k'^2/2m}$$
$$=\frac{m}{\pi^2 r}\int_0^\infty dk\frac{k\sin kr}{k^2-k'^2},$$

the integral representation of the nonrelativistic-scattering Green's function, where the "$2m$" is included in G instead of in U.

C. Partial-Wave Reduction

On resolving the wave function into partial waves

$$\psi(r,\theta,\phi)=\sum_{l,m}Y_l^m(\theta,\phi)R_l(r).$$

and using Eqs. (7) and (9), Eq. (12) becomes

$$R_l(r)=-\int_0^\infty dr'\left(\frac{r'}{r}\right)g(r,r';l)V(r')R_l(r')+(2\pi)^{-3/2}i^l(2l+1)j_l(kr),$$

where

$$g(r,r';l)=\frac{2rr'}{\pi}\int_0^\infty dk\frac{k^2j_l(kr)j_l(kr')}{(m^2+k^2)^{1/2}-E}. \quad (15)$$

This is rewritten in terms of the usual radial wave function

$$u_l(r)=(2\pi)^{3/2}i^{-l}(2l+1)^{-1}rR_l(r),$$
$$u_l(r)=-\int_0^\infty dr'\, g(r,r';l)V(r')u_l(r')+rj_l(kr). \quad (16)$$

Although the correct contour for the outgoing-wave scattering problem is the contour C_+ of the following section, a variety of contours may be employed to determine the phase shifts, δ_l; see Appendix III.

[6] E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Chap. 12.

D. Zero-Mass Approximations

The Green's-function integral Eq. (13) may be approximated

$$\lim_{m\to 0} G(|\mathbf{x}-\mathbf{x}'|)\equiv G_{m=0}(|\mathbf{x}-\mathbf{x}'|) = \frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty dk\frac{k\,\sin k|\mathbf{x}-\mathbf{x}'|}{k-k'}.$$

This expression has a pole at $k=k'$, but is adequately evaluated by displacing the contour below or above the pole, which corresponds, respectively, to choosing outgoing (+) or incoming (−) scattering boundary condition. This correspondence is established by Eq. (18), and the definitions of these contours applies retroactively to Eq. (12).

The integral G with $m=0$ may be reduced to Si and Ci functions[7]:

$$G_{m=0}^{(\pm)}(r)=-\frac{1}{2\pi^2 r}\frac{d}{dr}\int_{C\pm} dk\frac{\cos kr}{k-k'},$$

where

$$r=|\mathbf{x}-\mathbf{x}'|;$$

$C_\pm$ are shown in Fig. 1. $G_{m=0}^{(\pm)}$ is reduced by writing it as the sum of the corresponding principal-value integral plus or minus one-half the residue at the pole $k=k'$, i.e.,

$$\int_{C\pm} dk\frac{\cos kr}{k-k'}=\mathrm{P.V.}\int_0^\infty dk\frac{\cos kr}{k-k'}\pm\frac{1}{2}\oint dk\frac{\cos kr}{k-k'},$$

so that

$$G_{m=0}^{(\pm)}(r)=\frac{1}{2\pi^2 r}\frac{d}{dr}[\cos k'r\,\mathrm{Ci}(k'r)+\tfrac{1}{2}\pi\sin k'r +\sin k'r\,\mathrm{Si}(k'r)\pm i\pi\cos k'r].$$

From the definitions of Si and Ci,

$$\frac{d}{dr}\mathrm{Si}(k'r)=\frac{\sin k'r}{r},\quad \frac{d}{dr}\mathrm{Ci}(k'r)=\frac{\cos k'r}{r},$$

it follows that

$$G_{m=0}^{(\pm)}(r)=(k'/2\pi^2 r)[-\sin k'r\,\mathrm{Ci}(k'r)+\cos k'r\,\mathrm{Si}(k'r) +(1/k'r)+\tfrac{1}{2}\pi\cos k'r\pm i\pi\sin k'r]. \quad (17)$$

For asymptotically large r, $G_{m=0}^{(\pm)}$ simplifies: since

$$\lim_{z\to\infty}\mathrm{Si}(z)=\tfrac{1}{2}\pi \quad\text{and}\quad \lim_{z\to\infty}\mathrm{Ci}(z)=0,$$

it follows that

$$G_{m=0}^{(\pm)}(r)\underset{r\to\infty}{\sim}\frac{2k'}{4\pi}\frac{e^{\pm ik'r}}{r}.$$

[7] *Handbook of Mathematical Functions*, edited by A. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55, Chap. 5.

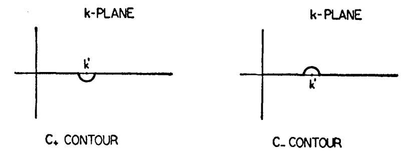

FIG. 1. C_+ and C_- contours.

Similarly,

$$G_{m=0}^{(\pm)}(r)\underset{|\mathbf{x}|\to\infty}{\sim}\frac{2k'}{4\pi}\frac{e^{\pm ik'|\mathbf{x}|}}{|\mathbf{x}|}e^{\mp i\mathbf{k}'\cdot\mathbf{x}'}. \quad (18)$$

The unsquared B-T equation, Eq. (13), with G replaced by $G_{m=0}^{(+)}$, becomes, as $|\mathbf{x}|\to\infty$,

$$\psi^{(+)}(\mathbf{x})\sim\frac{1}{(2\pi)^{3/2}}e^{ik'z}-2k\frac{e^{+ik'|\mathbf{x}|}}{4\pi|\mathbf{x}|}\int d^3x' \times e^{-i\mathbf{k}'\cdot\mathbf{x}'}V(|\mathbf{x}'|)\psi^{(+)}(\mathbf{x}').$$

This is in the usual asymptotic form of a scattering equation:

$$\psi^{(+)}(\mathbf{x})\sim\frac{1}{(2\pi)^{3/2}}\left[e^{ik'z}+\frac{f^{(+)}(\hat{k}')}{r}e^{ik'|\mathbf{x}|}\right],$$

where (19)

$$f_{m=0}^{(+)}(\hat{k}')=-\frac{(2\pi)^{3/2}}{4\pi}2k'\int d^3x'\,e^{-i\mathbf{k}'\cdot\mathbf{x}'}V(|\mathbf{x}'|)\psi^{(+)}(\mathbf{x}').$$

$f_{m=0}^{(+)}$ looks like the expression for the nonrelativistic Schrödinger scattering amplitude except that the multiplicative factor $2k$ replaces $2m$. Note that the $\psi^{(+)}(x')$ must be found by solving the B-T equation. The wave function normalization is the same as that given by Merzbacher.[6]

Using the identities

$$\mathrm{Si}(z)\underset{z\to 0}{\sim} z \quad\text{and}\quad \mathrm{Ci}(z)\underset{z\to 0}{\sim}\gamma+\ln z,$$

where γ is Euler's constant,[7] the singularity at $r=0$ is given by

$$G_{m=0}^{(\pm)}(r)\underset{r\to 0}{\sim}\frac{1}{2\pi^2 r^2};$$

this limit is independent of k and of the choice of contour $C_\pm$.

The correction introduced into Eq. (17) by a nonnegligible mass m can be estimated:

$$G=\frac{1}{2\pi^2 r}\int_0^\infty dk\frac{k\sin kr}{(m^2+k^2)^{1/2}-E} =\frac{1}{2\pi^2 r}\left[\int_0^\infty dk\frac{k(m^2+k^2)^{1/2}\sin kr}{k^2-k'^2}+E\int_0^\infty dk\frac{k\sin kr}{k^2-k'^2}\right],$$

where $k'^2=E^2-m^2$. The contours $C_\pm$ are used as before, and the integrals are each written as a sum of a principal-value integral and a residue, as before. The residues are found without approximation, but

$(m^2+k^2)^{1/2}$ is replaced by $k+m^2/2k$ in the principalvalue integrals. These are

$$\frac{\text{P.V.}}{2\pi^2 r}\left[\int_0^\infty dk\frac{k^2\sin kr}{k^2-k'^2}+\tfrac{1}{2}m^2\int_0^\infty dk\frac{\sin kr}{k^2-k'^2}+E\int_0^\infty dk\frac{k\sin kr}{k^2-k'^2}\right]=\frac{\text{P.V.}}{2\pi^2 r}\left[-\frac{d^2}{dr^2}\int_0^\infty dk\frac{\sin kr}{k^2-k'^2}+\tfrac{1}{2}m^2\int_0^\infty dk\frac{\sin kr}{k^2-k'^2}\right.$$

$$\left.-E\frac{d}{dr}\int_0^\infty dk\frac{\cos kr}{k^2-k'^2}\right]=\frac{1}{2\pi^2 r}\left[-\left(k'+\frac{m^2}{2k'}\right)[\sin k'r\,\text{Ci}(k'r)-\cos k'r\,\text{Si}(k'r)]+\tfrac{1}{2}\pi E\cos k'r++\frac{1}{r}\right].$$

When combined with the residue terms, this yields

$$\frac{1}{2\pi^2 r}\left[-\left(k'+\frac{m^2}{2k'}\right)[\sin k'r\,\text{Ci}(k'r)-\cos k'r\,\text{Si}(k'r)]+\tfrac{1}{2}\pi E\cos k'r+\frac{1}{r}\pm i\pi E\sin k'r\right].$$

$k'+m^2/2k'$ are the first terms in the expansion of $(m^2+k'^2)^{1/2}=E$. This suggests that a better approximation when $m\neq 0$ is given by

$$G_{m\sim 0}^{(\pm)}(r)\equiv\frac{E}{2\pi^2 r}\left[\cos k'r\,\text{Si}(k'r)-\sin k'r\,\text{Ci}(k'r)+\tfrac{1}{2}\pi\cos k'r+\frac{1}{Er}\pm i\pi\sin k'r\right].\quad(20)$$

The Green's function simplifies as $r\to\infty$:

$$G_{m\sim 0}^{(\pm)}(r)\underset{r\to\infty}{\sim}\frac{2E}{4\pi}\frac{e^{\pm ik'r}}{r}.\quad(21)$$

The factor k' in $G_{m=0}^{(\pm)}(r)$ has been replaced by E in $G_{m\sim 0}^{(\pm)}(r)$. Note that this form goes over into the correct nonrelativistic form $(2m/4\pi)e^{\pm ik'r}/r$.[8]

It is noted here that the zero-mass approximation did not simplify the partial-wave Green's function sufficiently to compute the integral, Eq. (15), in closed form except for $l=0$ (see Appendix III).

V. COMPARISON BETWEEN THE B-T EQUATION AND SERBER'S K-G EQUATION

The squared B-T equation in operator form is

$$(-\mathbf{k}^2+\mathbf{k}'^2)|\psi\rangle=[(m^2+\mathbf{k}^2)^{1/2}V(|\mathbf{x}|)+V(|\mathbf{x}|)\times(m^2+\mathbf{k}^2)^{1/2}+V^2(|\mathbf{x}|)]|\psi\rangle.\quad(3)$$

Except for the sign of the V^2 term, this "reduces" to the K-G equation of Serber[9]

$$(-\mathbf{k}^2+\mathbf{k}'^2)|\psi\rangle=(2EV-V^2)|\psi\rangle,$$

if the operators $(m^2+\mathbf{k}^2)^{1/2}$ are replaced by the energy eigenvalue E.

[8] The factor $2m$ is included in the Green's function instead of in U.

[9] R. Serber, Phys. Rev. Letters **10**, 357 (1963); Rev. Mod. Phys. **36**, 649 (1964).

But a comparison between the squared B-T equation

$$(\nabla^2+\mathbf{k}'^2)\psi(\mathbf{x})=\int d^3x''\,[K(|\mathbf{x}-\mathbf{x}''|)V(|\mathbf{x}''|)+V(|\mathbf{x}|)K(|\mathbf{x}-\mathbf{x}''|)]\psi(\mathbf{x}'')+V^2(|\mathbf{x}|)\psi(\mathbf{x})\quad(4)$$

and Seber's K-G equation

$$(\nabla^2+\mathbf{k}'^2)\psi(\mathbf{x})=[2EV(|\mathbf{x}|)-V^2(|\mathbf{x}|)]\psi(\mathbf{x}),$$

or a comparison between the radial squared B-T equation

$$\left(\frac{d^2}{dr^2}+k'^2-\frac{l(l+1)}{r^2}\right)u_l(r)=\int_0^\infty dr'\,\kappa(r,r';l)u_l(r')\times[V(r')+V(r)]+V^2(r)u_l(r)\quad(10)$$

and Serber's radial K-G equation

$$\left(\frac{d^2}{dr^2}+k'^2-\frac{l(l+1)}{r^2}\right)u_l(r)=[2EV(r)-V^2(r)]u_l(r)$$

shows that the B-T equation is actually different from Serber's K-G equation.

The Born approximation applied to Eq. (3) yields a scattering amplitude

$$f_{\text{B-T}}^{\text{Born}}(\mathbf{k}',\mathbf{k}'')=-\frac{1}{4\pi}\langle\mathbf{k}''|(m^2+\mathbf{k}^2)^{1/2}V(|\mathbf{x}|)+V(|\mathbf{x}|)(m^2+\mathbf{k}^2)^{1/2}+V^2(|\mathbf{x}|)|\mathbf{k}'\rangle$$

$$=-\frac{1}{4\pi}[\langle\mathbf{k}''|V(|\mathbf{x}|)|\mathbf{k}'\rangle(m^2+\mathbf{k}''^2)^{1/2}+\langle\mathbf{k}''|V(|\mathbf{x}|)|\mathbf{k}'\rangle(m^2+\mathbf{k}'^2)^{1/2}+\langle\mathbf{k}''|V^2(|\mathbf{x}|)|\mathbf{k}'\rangle],$$

and, since $|\mathbf{k}'|=|\mathbf{k}''|$, and $(m^2+\mathbf{k}'^2)^{1/2}=(m^2+\mathbf{k}''^2)^{1/2}=E$,

$$f_{\text{B-T}}^{\text{Born}}(\mathbf{k}',\mathbf{k}'')=-(1/4\pi)[2E\langle\mathbf{k}''|V(|\mathbf{x}|)|\mathbf{k}'\rangle+\langle\mathbf{k}''|V^2(|\mathbf{x}|)|\mathbf{k}'\rangle].$$

For Serber's K-G equation, the Born approximation yields

$$f_{\text{K-G}}{}^{\text{Born}}(\mathbf{k}',\mathbf{k}'') = -(1/4\pi)\langle\mathbf{k}''|2EV(|\mathbf{x}|)-V^2(|\mathbf{x}|)|\mathbf{k}'\rangle$$
$$= -(1/4\pi)[2E\langle\mathbf{k}''|V(|\mathbf{x}|)|\mathbf{k}'\rangle - \langle\mathbf{k}''|V^2(|\mathbf{x}|)|\mathbf{k}'\rangle].$$

Thus despite the differences between the B-T and Serber's K-G equations, in the first Born approximation they yield the same result except for the sign of the V^2 term. The matrix elements for Serber's potential are

$$\langle\mathbf{k}''|\frac{ie^{-\Lambda|\mathbf{x}|}}{|\mathbf{x}|}|\mathbf{k}'\rangle = -\frac{2i}{(2\pi)^2}\frac{1}{\Lambda^2+|\mathbf{k}'-\mathbf{k}''|^2},$$

$$\langle\mathbf{k}''|\frac{e^{-2\Lambda|\mathbf{x}|}}{|\mathbf{x}|^2}|\mathbf{k}'\rangle = \frac{2}{(2\pi)^2}\frac{\tan^{-1}(|\mathbf{k}'-\mathbf{k}''|/2\Lambda)}{|\mathbf{k}'-\mathbf{k}''|}.$$

It is easily seen that already in the second Born approximation $f_{\text{K-G}}$ and $f_{\text{B-T}}$ differ not only because of the sign of the V^2 term, but also because the operator $(m^2+k^2)^{1/2}$ can no longer be replaced everywhere by E. In particular the second Born term in $f_{\text{B-T}}$ consists of eight terms, four of which are identical with those in $f_{\text{K-G}}$ and four additional terms which do not occur in $f_{\text{K-G}}$.

If the V^2 term is neglected in Serber's K-G equation then it becomes

$$(\nabla^2+k'^2)\psi(\mathbf{x}) = 2EV\psi(\mathbf{x}),$$

and the exact Green's function, appropriate if the $2E$ is included in G instead of in the potential, is

$$G_{\text{K-G}}{}^{(\pm)}(r) = \frac{2E}{4\pi}\frac{e^{\pm ik'r}}{r}.$$

This coincides with the asymptotic form of the Green's function, Eq. (21), obtained from the nonsquared B-T equation.

The first Born approximation for the scattering amplitude obtained from the unsquared B-T equation is easily got from Eq. (19) with k' replaced by E [see Eq. (20)]:

$$f_{m\sim 0}{}^{(+)\text{Born}}(\mathbf{k}',\mathbf{k}'') = -\frac{2E}{4\pi}\langle\mathbf{k}''|V(|\mathbf{x}'|)|\mathbf{k}'\rangle.$$

This expression for $f_{m\sim 0}{}^{(+)\text{Born}}$ is identical with the first terms in $f_{\text{B-T}}{}^{\text{Born}}$ and in $f_{\text{K-G}}{}^{\text{Born}}$. Because of the factor E, this term is expected to dominate at very high energies.

The approximation of Eq. (20) by Eq. (21) actually depends on the magnitude of $k'r$, and so becomes valid at any impact parameter r for sufficiently large k'. However, kr' = constant corresponds roughly to l = constant. Therefore, an approximation corresponding to Eq. (21) is never expected to apply to Eq. (15) for the first few partial waves. For small-angle scattering—peripheral collisions—the first partial waves play a minor role at high k', so Eq. (21) is, in the high-energy limit, expected to be adequate. But for large-angle scattering—deeply penetrating collisions—it would seem necessary, even in the high-energy limit to use the full Green's function, Eq. (20).

Since the momentum-transfer distribution computed from the B-T equation is not expected to differ in the low-momentum-transfer region from the predictions obtained from Serber's Klein-Gordon equation, and since the latter leads to the result that[10]

$$\text{Re}f(0°)/\text{Im}f(0°) \approx -0.01,$$

it is unlikely that Serber's potential used in the B-T equation could produce a real part sufficiently large that

$$\text{Re}f(0°)/\text{Im}f(0°) \approx -0.3,$$

as required by experiment,[11,12] unless the B-T equation leads to very large real parts in the low-l phase shifts. Also, in order to fit the high-momentum-transfer data, such large real parts would have to interfere strongly at high angles; in fact, the cross section at high-momentum transfers calculated at 30.0 GeV/c is already too large owing mainly to the contributions of the real parts.

VI. PROSPECTS

A determination of the effect of the (unsquared) B-T Green's function $G_{m\sim 0}$, Eq. (20), on large-angle scattering, perhaps for Serber's potential, is indicated. Following the implications of Ref. 10, this may require a partial-wave analysis, and because of the complexity either of the B-T Green's function $g_{m\sim 0}$ and g or of the B-T kernels K and κ, such a computation would require significantly more analysis than the procedure given, for example, in Ref. 10.

The arbitrariness of the potential in B-T theory allows models which involve momentum-dependent potentials. As noted in Appendix I, this might be used to represent Lorentz contraction of the protons. This picture introduces a new distinction between longitudinal and transverse directions, which might conceivably have some relation with the empirical formula

$$d\sigma/d\omega = Ae^{-p_\perp/p_0},$$

where $p_\perp = p\sin\theta$,[13] mentioned in Sec. II of Ref. 10.

Finally it is suggested with great reservation that the consequences of replacing the usual Green's functions of field theory with the B-T Green's function might be of interest.

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[10] D. Avison, preceding paper, Phys. Rev., **155**, 1570 (1967).

[11] L. Kirillova, L. Khristov, V. Nikitin, M. Shafronova, L. Strunov, V. Sviridov, Z. Korbel, L. Rob, P. Markov, Kh. Tchernev, T. Todorov, and A. Zlateva, Phys. Letters **13**, 93 (1964).

[12] G. Belletini, G. Cocconi, A. N. Diddens, E. Lillethun, J. Pahl, J. P. Scanlon, J. Walters, A. M. Wetherell, and P. Zanella, Phys. Letters **14**, 164 (1965).

[13] J. Orear, Phys. Rev. Letters **12**, B1263 (1964).

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APPENDIX I: THE FORM OF THE POTENTIAL

In Serber's model, the very-high-energy scattering of two protons is reduced to a discussion of the center-of-mass scattering of a single particle by a complex spherically symmetric potential. However, in the center-of-mass system the two particles may be pictured as severely Lorentz-contracted spheres. The overlap of such spheroids would still be squashed. The spatial extent of the potential is expected to represent this overlap function. Thus the potential is not necessarily expected to possess spherical symmetry, but only cylindrical symmetry.

Serber's optical integral calculation in fact injects a kind of total squashing: The phase function $\chi(\rho)$, calculated from an integral through the potential, is used as a source distribution on a plane ("phase plate") perpendicular to the scattering axis. From this two-dimensional distribution the differential cross section is computed. The partial-wave calculation of course introduces no squashing, and to the extent that the optical-model and the partial-wave calculations agree, the effect of squashing would be judged unimportant.

The potential which occurs in the B-T theory is required to be a rotationally invariant function of $\mathbf{x}$ and $\mathbf{k}$, the coordinate and momentum variables for the relative motion. It is noted that this requirement does not exclude powers of $\cos\theta=\mathbf{k}\cdot\mathbf{x}/|\mathbf{k}||\mathbf{x}|$, which could be used to represent asphericity. (This should really be replaced by an appropriate Hermitian operator.) The cylindricity about $\mathbf{k}$ is easy to picture as literal squashing of the potential if $\mathbf{k}$ is nearly constant (small-angle, peripheral collisions); the rotation of the axis of cylindrical symmetry as $\mathbf{k}$ changes is in fact an extra refinement in representing a Lorentz contraction. In spite of this motivation to introduce $\cos\theta$ dependence, this has not been done in order to keep things simple, and also to keep the Serber potential intact so as to consider in isolation the effect of replacing Serber's K-G formula by the B-T formula.

It should be noted that the v of the B-T model is Hermitian whereas Serber's potential is not. This reflects the fact that the B-T model deals with scattering without change in particle number, whereas Serber's model represents the effect in the elastic channel of many inelastic channels. Nevertheless, given that the inelasticity is to be represented by an imaginary potential, the B-T formula is taken to indicate where a potential function is to be inserted.

Note also that the non-Hermitian character of the potential introduces no difficulty in setting up the B-T scattering equations. (See also Sec. IV A.)

APPENDIX II: TWO OPERATOR EQUATIONS

Equations (7) and (9) of the text are formally derived here. They are applied for $g=g_l=R_l$, for $g=g_l=VR_l$, and for $g=g_l=\{1/[(m^2+\mathbf{k}^2)^{1/2}-E]\}VR_l$.

$$F(\mathbf{k}^2)[\sum_{l,m} Y_l^m(\theta,\phi)g_l]=\sum_{l,m} Y_l^m(\theta,\phi)F(k_l^2)[g_l], \quad (7)$$

where

$$k_l=\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{d}\right)+\frac{l(l+1)}{r^2}\right]^{1/2}.$$

Derivation. Expand

$$F(\mathbf{k}^2)=\sum_{q=0}^{\infty} a_q(\mathbf{k}^2)^q\,;$$

then,

$$F(\mathbf{k}^2)[\sum_{l,m} Y_l^m(\theta,\phi)g_l]=\sum_q\sum_{l,m} a_q(\mathbf{k}^2)^q[Y_l^m(\theta,\phi)g_l].$$

But

$$\mathbf{k}^2Y_l^m(\theta,\phi)g_l(r)=-\nabla^2Y_l^m(\theta,\phi)g_l(r)=Y_l^m(\theta,\phi)k_l^2g_l(r),$$

and

$$\begin{aligned}(\mathbf{k}^2)^2Y_l^m(\theta,\phi)g_l(r)&=(-\nabla^2)(-\nabla^2)Y_l^m(\theta,\phi)g_l(r)\\&=-\nabla^2Y_l^m(\theta,\phi)k_l^2g_l(r)\\&=Y_l^m(\theta,\phi)(k^2)^2g_l(r),\end{aligned}$$

since

$$\begin{aligned}-\nabla^2k_l^2Y_l^m(\theta,\phi)g_l(r)&=-k_l^2\nabla^2Y_l^m(\theta,\phi)g_l(r)\\&=Y_l^m(\theta,\phi)(k_l^2)^2g_l(r).\end{aligned}$$

By induction it follows that

$$\begin{aligned}(\mathbf{k}^2)^qY_l^m(\theta,\phi)g_l(r)&=-(\nabla^2)^qY_l^m(\theta,\phi)g_l(r)\\&=Y_l^m(\theta,\phi)(k_l^2)^qg_l(r)\end{aligned}$$

from which Eq. (7) follows by multiplying a_q and summing.

$$F(k_l^2)[g]|_r=\int_0^\infty dr'\,g(r')\times\left[\frac{2}{\pi}r'^2\int_0^\infty dk\,F(k^2)j_l(kr)j_l(kr')\right]. \quad (9)$$

Derivation. The action of $F(k_l^2)$ may be replaced by an integral operator: Expand

$$g(r)=\int_0^\infty dk\,\gamma_l(r)j_l(kr),$$

then

$$F(k_l^2)[g]|_r=\int_0^\infty dk\,\gamma_l(r)F(k^2)j_l(kr),$$

where k is the eigenvalue of k_l. To find $\gamma_l(r)$, the selfreciprocal property of the Bessel transform[5] is used;

$$g(r)=\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty dk\,\gamma_l(k)\frac{J_{l+\frac{1}{2}}(kr)}{(kr)^{1/2}},$$

$$\left(\frac{2}{\pi}\right)^{1/2}rg(r)=\int_0^\infty dk\left(\frac{\gamma_l(k)}{k}\right)J_{l+\frac{1}{2}}(kr)(kr)^{1/2},$$

$$\frac{\gamma_l(k)}{k}=\int_0^\infty dr'\left[\left(\frac{2}{\pi}\right)^{1/2}r'g(r')\right]J_{l+\frac{1}{2}}(kr')(kr')^{1/2}$$

$$=\int_0^\infty dr'\left[\left(\frac{2}{\pi}\right)^{1/2}r'g(r')\right]\times\left[\left(\frac{2}{\pi}\right)^{1/2}(kr')^{1/2}j_l(kr')\right](kr')^{1/2},$$

Finally,

$$\gamma_l=\frac{2k^2}{\pi}\int_0^\infty dr'\,r'^2g(r')j_l(kr').$$

From this,

$$F(k_l^2)[g]|_r=\int_0^\infty dk\frac{2k^2}{\pi}\int_0^\infty dr'\,r'^2g(r')j_l(kr')F(k^2)j_l(kr)$$

which, upon interchanging the order of integration becomes Eq. (9).

APPENDIX III: THE S-WAVE PHASE SHIFT

On specializing Eqs. (15) and (16), to the s wave and letting $m\to 0$, these equations become

$$g(r,r';0)=\frac{2}{\pi}\int_0^\infty dk\frac{\sin kr\,\sin kr'}{k-k'}\tag{23}$$

and

$$u_0(r)=-\int_0^\infty dr'\,g(r,r';0)V(r')u_0(r')+k'^{-1}\sin k'r.\tag{24}$$

To evaluate Eq. (23):

$$g(r,r';0)=\frac{1}{\pi}\int_0^\infty dk\frac{\cos kr_-}{k-k'}-\frac{1}{\pi}\int_0^\infty dk\frac{\cos kr_+}{k-k'},$$

where

$$r_-=r-r'\quad\text{and}\quad r_+=r+r'.$$

Leaving, for the moment, the question of contour open, we have

$$\int_0^\infty dk\frac{\cos kr_\pm}{k-k'}=\text{P.V.}\int_0^\infty dk\frac{\cos kr_\pm}{k-k'}+(\pm,0)\frac{1}{2}\oint dk\frac{\cos kr_\pm}{k-k'},$$

$$\frac{1}{2}\oint dk\frac{\cos kr_\pm}{k-k'}=i\pi\cos k'r_\pm.$$

For $r_\pm>0$,

$$\text{P.V.}\int_0^\infty dk\frac{\cos kr_\pm}{k-k'}=-\{\cos k'r_\pm\,\text{Ci}(k'r_\pm)+(\sin k'r_\pm)\times[\tfrac{1}{2}\pi+\text{Si}(k'r_\pm)]\},$$

see Ref. 5; so that

$$g(r,r';0)|_{r>r'}=-(1/\pi)[\cos k'r_-\,\text{Ci}(k'r_-)+(\sin k'r_-)(\tfrac{1}{2}\pi+\text{Si}(k'r_-))]+(1/\pi)[\cos k'r_+\,\text{Ci}(k'r_+)+(\sin k'r_+)(\tfrac{1}{2}\pi+\text{Si}(k'r_+))]+(\pm,0)\times i[\cos k'r_+-\cos k'r_-].$$

Because of the symmetry between r and r' in Eq. (23),

$$g(r,r';0)|_{r<r'}=g(r',r;0)|_{r>r}.$$

The asymptotic expressions for $g(r,r';0)$ are

$$g(r,r';0)\underset{r\to\infty}{\sim}2\sin k'r'[\cos k'r+(\pm,0)(-i)\sin k'r],$$

and

$$g(r,r';0)\underset{r\to 0}{\sim}0,$$

independent of the choice of $(\pm,0)$.

Inserting Eq. (23) into Eq. (24) yields the radial wave equation valid for large r:

$$u_0(r)\underset{r\to\infty}{\sim}-2[\cos k'r+(\pm,0)(-i)\sin k'r]\times\int_0^\infty dr'(\sin k'r')V(r')u_0(r')+k'^{-1}\sin k'r.$$

The boundary condition on $u_0(r)$ is

$$u_0(r)\underset{r\to 0}{\sim}0.$$

The normalization of $u_0(r)$ is implicitly fixed by the $k'^{-1}\sin k'r$ term. With this normalization and using the "0" alternative from $(\pm,0)$, the phase shift δ_0 is defined by

$$u_0(r)\underset{r\to\infty}{\sim}\frac{1}{k'}[\sin k'r+\tan\delta_0\cos k'r].$$

This alternative corresponds to the contour $\frac{1}{2}(C_++C_-)$ and not to the C_+ contour (see Fig. 1) appropriate to a scattering problem; nevertheless, the phase shift δ, defined as half the relative phase of the asymptotic outgoing and incoming spherical waves is insensitive to this issue. The s-wave phase shift is therefore given in terms of the radial wave function by

$$\tan\delta_0=-2k'^2\int_0^\infty dr'\,r'j_0(k'r')V(r')u_0(r').$$

It is noted that this is the same as the corresponding expression for the Schrödinger equation except for the factor of[8] $2k$ and of course the presence of the B-T radial wave function u_0 instead of the Schrödinger wave function. The first Born approximation is of course

$$\tan\delta_0 = -2k'^2\int_0^\infty dr'\, r'[j_0(k'r')]^2 V(r').$$

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Decay Modes $\eta \to \pi^+\pi^-\pi^0\gamma$ and $\eta \to \pi^0\gamma\gamma$*

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We calculate rates and photon-energy spectra for the decay modes $\eta \to \pi^+\pi^-\pi^0\gamma$ and $\eta \to \pi^0\gamma\gamma$, with the aid of a new model. The relevance of the decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$ to the possibility of C violation in electromagnetic interactions is discussed.

I. INTRODUCTION

IN this article, we propose a mechanism which can account for a partial decay rate $\eta \to \pi^0\gamma\gamma$ comparable in magnitude to the other major η decays. We also make a detailed analysis of the decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$, which has not been treated previously and for which no experimental data are yet available. The ratio of these two decays is predicted by the model. The γ-ray energy spectra, whose knowledge is helpful in the experimental detection of these decays, are also presented.

A recent experiment of DiGiugno *et al.*[1] indicates that (37.5 ± 3.6) percent of the neutral decay products of η consist of the decay mode $\eta \to \pi^0\gamma\gamma$. They also obtain for the ratio to the $\eta \to \gamma\gamma$ decay, $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to 2\gamma)] = 0.9\pm0.1$. In apparent disagreement, Wahlig *et al.*[2] report $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)] < 0.5$. Nevertheless, two other experiments seem to confirm the abundant occurrence of the $\eta \to \pi^0\gamma\gamma$ decay. Strugalski *et al.*[3] obtain $R[(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)] = 0.86\pm0.40$, while Grunhaus[4] gives (27 ± 9) percent for the percentage of $\pi^0\gamma\gamma$ among the neutral decay products of eta.

A copious rate for $\eta \to \pi^0\gamma\gamma$ causes some theoretical embarrassment. The ratio of the other two detected radiative decays of eta, namely $\eta \to \pi\pi\gamma$ and $\eta \to \gamma\gamma$ has been successfully accounted for[5,6] by using a basic trilinear vector-vector-pseudoscalar-meson interaction, followed by transitions $\rho \to 2\pi$ and (vector meson) $\to \gamma$. This rho-dominance model also fits very well[7] the photon-energy spectrum in the decay $\eta \to \pi^+\pi^-\gamma$. If this is assumed to be the mechanism for all η radiative decays, then $\eta \to \pi^0\gamma\gamma$ is expected to be very small. Roughly, we estimate

$$(\eta \to \pi^0\gamma\gamma)/(\eta \to \pi^+\pi^-\gamma) \simeq (\rho^0 \to \pi^0\gamma)/(\rho^0 \to \pi^+\pi^-)$$

by using an effective $\eta\rho\gamma$ vertex, which gives $<1\%$ for this ratio. Alles, Baracca, and Ramos[8] have calculated $(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma)$ with this model including all possible vector-meson intermediate states and obtain $(\eta \to \pi^0\gamma\gamma)/(\eta \to \gamma\gamma) = 1.06\times10^{-3}$. When considering also η-X mixing they show that this number cannot be significantly improved without badly damaging the $(\eta \to \pi^+\pi^-\gamma)/(\eta \to \gamma\gamma)$ ratio.

II. FORMULATION OF MODEL AND CALCULATIONS

As the trilinear meson interactions VVP and VPP (V is the vector-meson nonet, P the pseudoscalar-meson octet) fail to account for $\eta \to \pi\gamma\gamma$ by a factor of 10^3, it is reasonable to expect that improvements like form factors, etc. will not change this factor significantly. We suggest therefore that the large rate for $\eta \to \pi^0\gamma\gamma$ is related to quadrilinear meson interactions. A well-known example of this kind is the $\lambda(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2$ term of the interaction Lagrangian. It is natural to enlarge this

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[1] C. DiGiugno, R. Querzoli, G. Troise, F. Vanoli, M. Giorgi, P. Schiavon, and V. Silvestrini, Phys. Rev. Letters **16**, 767 (1966).

[2] M. A. Wahlig, E. Shibata, and I. Manelli, Phys. Rev. Letters **17**, 221 (1966).

[3] Strugalski *et al.*, in Proceedings of the 13th International Conference on High-Energy Physics, Berkeley, California, 1966 (to be published).

[4] J. Grunhaus, Ph.D. thesis, Columbia University, New York, 1966 (unpublished), and (private communication).

[5] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[6] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); **16**, 424 (1966).

[7] F. S. Crawford, Jr., and LeRoy R. Price, Phys. Rev. Letters **16**, 333 (1966).

[8] W. Alles, A. Baracca, and A. T. Ramos, Nuovo Cimento **45**, A272 (1966).