We further assume that g is so large that $g^2\gamma > \omega_0$, where ω_0 is still the energy above which the asymptotic forms of Δ_- and u^2 may be used. This energy is independent of g^2, so clearly g^2 can be chosen large enough to put $g^2\gamma > \omega_0$. Then in Eq. (A2) we may use

$$|\Delta_-(\omega_1)|^2 = \frac{g^4\mu^2\beta^2}{\Delta_+^2(\infty)[(1-g^2\gamma/\omega)^2 + g^4\eta^2/16\pi^2\omega^{2n}]}. \quad \text{(A3)}$$

We further assume that g is sufficiently large that

$$\eta^2/16\pi^2\gamma^2 < g^{4(n-1)}. \quad \text{(A4)}$$

Then

$$(1-g^2\gamma/\omega_1)^2 + g^4\eta^2/16\pi^2\omega_1^{2n} \leqslant 1, \quad (\omega_1 \geqslant g^2\gamma) \quad \text{(A5)}$$

and

$$|\Delta_-(\omega_1)|^2 \geqslant g^4\mu^2\beta^2/\Delta_+^2(\infty), \quad (\omega_1 \geqslant g^2\gamma). \quad \text{(A6)}$$

Since $|\Delta_+(\omega)|^2$ has a nonzero lower bound κ which is independent of g,

$$I_1 \geqslant \frac{4^n g^4\mu^2\beta^2\kappa^2\eta^2}{\Delta_+^2(\infty)\omega^{n-1}} \int_{g^2\gamma}^{\omega/2} \frac{d\omega}{\omega_1^{n+1}}$$
$$= \lambda g^{2(2-n)}\omega^{-n+1}, \quad (\omega > 3g^2\gamma > \omega_0), \quad \text{(A7)}$$

where λ is independent of g and ω. The limit $3g^2\gamma$ is chosen so that the integral in Eq. (A7) may be bounded from below by an expression proportional to $(g^2\gamma)^{-n}$.

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Determination of the Nucleon-Nucleon Elastic-Scattering Matrix. V. New Results at 25 MeV

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Analyses of N-N experiments at 25 MeV have been hampered by a lack of complete scattering data, especially for the scattering states with isotopic spin $T=0$. In particular, there is an ambiguity in the single-energy $T=0$ solutions at 25 MeV. This ambiguity, which we discuss here in detail, is partly resolved by the addition of new (n,p) data. Some new (p,p) data have also been added. The resulting phases more closely resemble the values expected from potential models—with which they are compared. The new selection of data permits a determination of the pion-nucleon coupling constant ($g_\pi^2 = 14.3 \pm 1.3$), whereas the older selections did not. An investigation of the parabolic approximation for each of the phases indicates the extent to which one can believe the uncertainties as given by an error-matrix calculation. The energy-dependent analyses in this energy region have been improved by having the S phases extrapolate to the scattering length and effective-range expansions at low energies. The resulting phases give excellent fits to the data at 10 MeV as well as at 25 MeV. Experiments that would further improve the analysis at 25 MeV are suggested. The present results are in some disagreement with a recently released Dubna analysis at 23 MeV.

I. INTRODUCTION

In previous papers in this series[1–4] we have published the results of energy-dependent and energy-independent phase-shift analyses in the energy range from 25 to about 350 MeV. Both (p,p) and (n,p) data were analyzed, and the isotopic spin $T=0$ and $T=1$ amplitudes were determined. However, whereas the (p,p) experiments are reasonably complete and give reliable values for the $T=1$ phase shifts, the (n,p) experiments are patently incomplete, and the $T=0$ scattering matrix obtained from our analyses must be considered with this fact in mind. For an incomplete data set, multiple phase solutions may exist. Even for a correct type of solution, the phases may be somewhat inaccurate, and the phase-shift uncertainties as given by an error matrix calculation may be grossly inaccurate. In particular, the least-squares sum (χ^2) hypersurface in the neighborhood of the solution minimum, that is for variations of a few standard deviations for each parameter, may not be parabolic. These statements are well illustrated in the present analysis of nucleon-nucleon data near 25 MeV.

In our previous analyses at 25 MeV,[3,4] we obtained $T=1$ and $T=0$ scattering matrices. However, we were unable to obtain a value for the pion-nucleon coupling constant g_π^2, and some of the $T=0$ phases, ϵ_1 in particular, had obviously misleading values and/or errors. Even for an energy as low as 25 MeV, triple-scattering parameters are needed for an accurate phase-shift analysis. These were incomplete for the (p,p) system and nonexistent for the (n,p) system.

Recently, additional experiments have been completed near 25 MeV that modify our previous results

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and in fact bring them into closer agreement with theoretical predictions based on potential models. In the course of doing the phase-shift re-analysis, we discovered other sets of $T=0$ solutions at 25 MeV (using the original data set[3,4]) that also give reasonable values for χ^2. Incorporating the new data changes eliminated most but not all of the ambiguity in the energy-independent analysis. It also permitted us to determine a reliable value for g_π^2. Adding the requirement that the S phases approach the scattering length and effective range at low energies gave an energy-dependent result that is essentially unambiguous and that gives an excellent fit to precision (p,p) cross section measurements near 10 MeV, with no searching on the phase-shift energy-dependent parameters (which were obtained from 25–350-MeV data fits) being required.

In Sec. II we discuss the changes in the data selection. Section III gives the energy-independent phase-shift results and the value we obtained for g_π^2. In Sec. IV we compare the phases to the values given by various potential models. Section V is a discussion of the use of error matrix techniques, and it includes an evaluation of the "parabolicity" of the $T=0$ and $T=1$ scattering matrices. In Sec. VI we compare the present results with a similar analysis carried out by the Dubna group. Section VII gives new energy-dependent results. Section VIII is a discussion of N-N data that would help to resolve existing ambiguities. Our conclusions are given in Sec. IX. As in the preceding papers in this series, we assume charge independence for the $T=1$ phases. Any deviations from charge independence would be small compared to uncertainties caused by the incompleteness of the data selection.

II. DATA SELECTION

The data used in our analyses[5–17] are itemized in Table I. A few comments on how this data set differs from that used in our previous analyses[3,4] are in order.

Three new data points were added to the (p,p) data set. They are

$$C_{nn}(25.7\text{ MeV}, 90°),^5\quad A_{xx}(25.7\text{ MeV}, 90°),^5$$

and

$$C_{nn}(27\text{ MeV}, 90°).^6$$

One datum,[7] $R(27.6\text{ MeV}, 39°)$, was removed because it was felt to be incompatible with the newly included data. Our general procedure in selecting data has been to accept all of the measurements in a given energy band, excluding only data which are for one reason or another suspect or which are redundant, and to carry out the phase-shift analysis. Then the detailed contributions to χ^2 are examined, and any isolated data points that contribute more than 4 to the χ^2 sum are arbitrarily deleted. Thus, our criterion for data selection is based on the internal consistency of the data. If a number of points in one set of data seem to be inconsistent with the rest of the data selection, that particular set may be rejected. We would like to have a clearcut statistical argument for judging data points, but historically the errors that have been uncovered in nucleon-nucleon scattering data are such a blend of systematic and statistical errors that statistical arguments are not very meaningful. As a practical matter, the values and errors obtained for the phase shifts are essentially the same whether errant points are kept or rejected. The principal effect of the errant points is to raise the value of the χ^2 sum.

Six new data points were added to the (n,p) set. They are

$$C_{NN}(23.0\text{ MeV}, 140°),^8\quad C_{NN}(23.0\text{ MeV}, 174°),^8$$
$$P(23.0\text{ MeV}, 140°),^8\quad \sigma^{\text{tot}}(23.51\text{ MeV}),^9$$
$$\sigma^{\text{tot}}(25.31\text{ MeV}),^9\quad\text{and}\quad \sigma^{\text{tot}}(27.29\text{ MeV}).^9$$

Previous Livermore analyses[1–4] did not use (n,p) total cross sections. These are now included as normalized data, where the normalization error as given in Table I is derived from the percentage dispersion in the energy of the incident neutron beam. The three large-angle (n,p) differential cross section points[15] at 27.5 MeV were given a separate normalization constant from the small-angle points. The normalization constant for the large-angle points was not constrained; the fit to the total (n,p) cross sections listed above served to insure correct overall normalization for the (n,p) differential cross section. The "decoupling" of these large-angle points gives a much closer agreement between the value of $^1P_1$ as experimentally determined and the one-pion-exchange value. The basis for this "decoupling" is of course experimental.

The tabulated values in the column called "Normalization constant" in Table I are the results given by the error matrix as obtained for solution C (next section).

Some D measurements have been made for the (n,p)

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TABLE I. Data used in phase-shift analyses at 25 MeV.

Energy (MeV)	Type of data	c.m. angle (deg)	Datum	Experimental error	Normalization error	Normalization constant[a]	Ref.
25.63	$\sigma(\theta)$ (p,p)	10.07	109.60	2.925	0.0093	0.984 ± 0.006	10
		12.08	56.31	0.918			
		14.09	33.20	0.295			
		16.11	23.76	0.176			
		18.12	19.90	0.147			
		19.13	18.70	0.139			
		20.13	17.98	0.133			
		22.15	17.33	0.128			
		24.16	17.09	0.127			
		25.16	17.16	0.127			
		26.17	17.17	0.127			
		28.18	17.30	0.128			
		30.19	17.43	0.129			
		32.31	17.68	0.131			
		34.22	17.80	0.132			
		36.23	17.93	0.133			
		40.25	18.20	0.135			
		44.27	18.33	0.136			
		50.30	18.52	0.137			
		60.34	18.56	0.138			
		70.37	18.65	0.138			
		80.38	18.60	0.138			
		89.61	18.59	0.138			
25.62	$\sigma(\theta)$ (p,p)	90.0	18.30	0.110			11
27.4	$P(\theta)$ (p,p)	45	0.003	0.004			12
30.0	$P(\theta)$ (p,p)	45	-0.0004	0.003	0.04	1.004 ± 0.04	13
27.6	$R(\theta)$ (p,p)	23.2	-0.324	0.054	0.03	1.006 ± 0.079	7, 14
		54.6	-0.243	0.032			
27.6	$A(\theta)$ (p,p)	23.2	0.012	0.030	0.03	1.003 ± 0.03	14
		39.0	0.037	0.025			
		54.6	0.090	0.022			
25.7	$C_{nn}(\theta)$ (p,p)	90.0	-0.72	0.02			5
25.7	$A_{xx}(\theta)$ (p,p)	90.0	-0.92	0.02			5
27.0	$C_{nn}(\theta)$ (p,p)	90.0	-0.689	0.07			6
27.5	$\sigma(\theta)$ (n,p)	7	28.5	1.3	0.03	0.989 ± 0.026	15
		14	28.5	1.5			
		21	29.6	1.8			
		31	28.3	1.6			
		41	27.5	1.9			
		51	27.3	2.0			
		62	26.5	2.1			
		72	27.0	2.1			
27.5	$\sigma(\theta)$ (n,p)	159	25.3	1.6	10[c]	1.09 ± 0.05	15
		166	26.8	1.6			
		173	29.9	1.9			
22.5	$\sigma(\theta)$ (n,p)	65	33.3	1.9	1[c]	0.99 ± 0.026	16
		75	32.8	1.5			
		85	32.7	1.2			
		95	33.2	1.2			
		105	32.2	1.2			
		115	32.4	1.3			
		125	34.7	1.4			
		135	35.6	1.5			
		145	37.0	1.7			
		155	34.0	1.9			
		165	34.3	2.6			
		175	35.9	3.7			
23.1	$P(\theta)$ (n,p)	50	0.049	0.014	0.017	1.000 ± 0.017	17
		70	0.053	0.010			
		90	0.052	0.007			
		110	0.031	0.007			
		130	0.025	0.009			
		150	-0.004	0.009			
23	$P(\theta)$ (n,p)	140	0.011	0.005			8
23	$C_{NN}(\theta)$ (n,p)	140	0.074	0.024			8
		174	-0.014	0.011			
23.51	σ^{tot} (n,p)		397.7	16.0	0.024[b]	1.001 ± 0.021	9
25.31	σ^{tot} (n,p)		362.9	13.5	0.025[b]	1.004 ± 0.022	9
27.29	σ^{tot} (n,p)		335.4	11.5	0.026[b]	1.002 ± 0.023	9

[a] These are the normalization uncertainties from solution "C," as given by the error matrix.
[b] Normalization errors on total cross sections are derived from the energy dispersion in the incident neutron beam.
[c] Floated freely.

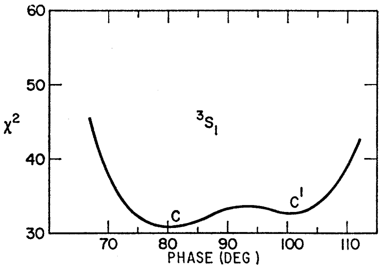

FIG. 1. Parameter study of $^3S_1$, showing solutions C and C'. The $^3S_1$ phase shift is varied systematically, and the other phases are allowed to readjust to minimize χ^2 at each value. The phases are Stapp nuclear bar phase shifts in degrees.

system at 23 MeV.[18] However, these measurements have little effect on our analysis, and the experimenters regard them as preliminary data.[19] Thus we have not included them here.

III. ENERGY-INDEPENDENT SOLUTION AMBIGUITIES

In previous Livermore analyses[3,4] of the then-available (n,p) data at 25 MeV, little attention was given to the detection of ambiguities of the solutions among the $T=0$ states (see Sec. II of Ref. 3). However, the 25-MeV solution as published[3,4] is not compatible with the recently available (n,p) C_{nn} measurements of the Los Alamos group.[8] Upon closer examination, it was determined that there are four solutions which are competitive in χ^2 and which are essentially of different character. The $T=0$ states corresponding to the four solutions, which we have denoted as A, B, A', and B', are given in Table II. The $T=1$ states are rather rigorously fixed by the (p,p) data and change little among the four solutions. The four solutions can be thought of as twofold ambiguities in the $^3S_1$ and ϵ_1 phases.

The data revisions which went into our present analysis removed the ambiguity in ϵ_1 and resulted in the two solutions which we have denoted as C and C', and which are given in Table III. These two solutions result from an ambiguity in $^3S_1$, as seen in Fig. 1, where we depict a parameter study on $^3S_1$. It is suspected that further ambiguities may exist in the $T=0$ D states

TABLE II. T (isospin) $=0$ states and χ^2 for four solutions,[a] using the *old* data of Refs. 3 and 4.

	A[b]	B	A'	B'
$^1P_1$	0.26	1.04	2.69	1.46
$^3S_1$	77.8	77.6	106.5	104.1
ϵ_1	7.01	−5.92	7.58	−8.43
$^3D_1$	−2.86	−2.98	−3.47	−3.64
χ^2	29.62	31.3	29.6	32.9

[a] All phase shifts in this paper are Stapp nuclear bar phase shifts in degrees.
[b] This is essentially the solution published in Refs. 3 and 4.

[18] R. B. Perkins and J. E. Simmons, in *Proceedings of the International Conference on Nuclear Physics, Paris, 1964* (Editions du Centre National de la Recherche Scientifique, Paris, 1965), Vol. 2, p. 164.

[19] J. E. Simmons (private communication).

TABLE III. Phase shifts[a] and χ^2 for solutions C and C', using the revised data of Table I.

	Solution C	Solution C'	OPEC values ($g_\pi^2=14$)	Energy derivative (deg/MeV)
χ^2	31.04	32.92		
$^1S_0$	48.68 ±0.28	48.62±0.28	47.88[b]	−0.435
$^3P_0$	7.83 ±0.59	7.73±0.56	11.5	0.21
$^3P_1$	−4.80 ±0.28	−4.92±0.29	−6.93	−0.156
$^3P_2$	2.38 ±0.12	2.46±0.13	0.45	0.131
$^1D_2$	0.87 ±0.086	0.86±0.088	0.57	0.041
ϵ_2	−1.15 ±0.20	−1.12±0.20	−0.83	−0.043
$^3F_2$	0.117±0.14	0.12±0.14	0.11	0.0074
$^1P_1$	−2.06 ±2.66	−2.59±2.08	−7.04	−0.018
$^3S_1$	79.48 ±4.61	100.8 ±4.79	82.2[c]	−0.995
ϵ_1	−1.58 ±2.35	−0.67±1.6	∼2[d]	−0.0178
$^3D_1$	−3.53 ±4.46	2.12±2.85	−2.23	−0.172
$^3D_2$	4.91 ±2.25	−4.28±1.86	3.38	0.264
$^3D_3$	0.04 ±2.10	0.48±1.41	−0.23	0.0138

[a] See footnote a of Table II.
[b] $^1S_0$ from eff range for $a=-7.815$ F, $r_0=2.795$ F (Ref. 26).
[c] $^3S_1$ from eff range for $a=5.4$ F, $r_0=1.73$ F (Ref. 27).
[d] ϵ_1 from a number of low-energy models which give the correct S-D admixture in the deuteron (see Sec. IV).

which, because of the lack of (n,p) data, were not searched in previous Livermore analyses,[3,4] but which were fixed at their one-pion-exchange-contribution (OPEC) values. Since the D states which we determine (solution C) are reasonable in their resemblance to OPEC, and since they are determined with rather broad errors, we decided to defer any further searching until such time as more precise (n,p) data become available. A parameter study of g_π^2 taken around solution C yielded the value $g_\pi^2=14.3\pm1.3$. The surprising aspect of this determination is not so much the value obtained, which may be partly accidental, but rather that we were able to secure any determination at all from the data. Previous attempts[3,4] which employed earlier data all resulted in indeterminate or wholly fictitious values for g_π^2.

IV. COMPARISON WITH POTENTIAL MODELS

Three models were studied for their compatibility against our present choice of data. They are the Hamada-Johnston (H-J) potential model,[20] the Yale potential model,[21] and the Bryan-Scott (B-S) one-boson-exchange potential model.[22] The phase shifts used were those as published at 25 MeV for the Yale and for the B-S models, and as published at 20 MeV for the H-J model. In all cases the local phase-shift energy dependence, for comparison purposes, was assumed to be linear, $(\partial\delta/\partial T)$, as derived from our energy-dependent analysis[4] (values are given in Table III). The results of these studies are listed in Table IV, where we have tabulated χ^2 *after adjustment of normalization parameters to achieve a best fit.* We have further decomposed χ^2 into

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[21] M. H. Hull, K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **128**, 830 (1962).

[22] R. A. Bryan and B. L. Scott (to be published).

the contribution from (p,p) data, χ_{pp}^2, and the contribution from (n,p) data, χ_{np}^2. Also tabulated in Table IV are the Yale and H-J models with the phases ${}^1S_0$, ${}^3S_1$, and ϵ_1 adjusted to obtain a best fit. In this way we hope to determine how much of the mismatch is an S-wave effect. (The B-S model does not attempt to fit the S phases.) The Yale and B-S models give quite good quantitative fits to the data. Despite the rather large values of χ^2 for the Hamada-Johnston model, we found it to fit everything well except the 25.63-MeV (p,p) differential cross section; the mismatch to these data persisted even after searching ${}^1S_0$.

TABLE IV. Compatibility of present data with predictions of various potential models.

Model	χ^2	χ_{pp}^2	χ_{np}^2
Solution C of present analysis	31.0	19.4	11.6
Yale potential[21]	87	53.6	23.4
Hamada-Johnston[20]	111.8	83.8	28
Bryan-Scott[22] (free ${}^1S_0$, ${}^3S_1$, ϵ_1)	57.4	40.7	16.7
Yale potential[21] (free ${}^1S_0$, ${}^3S_1$, ϵ_1)	61.3	46.2	15.1
Hamada-Johnston[20] (free ${}^1S_0$, ${}^3S_1$, ϵ_1)	97.2	81.7	15.5

V. VALIDITY OF THE ERROR ANALYSIS

We have suggested in another publication[4] that for the purpose of determining model parameters one could represent the data at a single energy by the reduced correlation matrix (reduced second-derivative matrix) derived from a phase-shift analysis at that energy. The reduced correlation matrix is derived from the full correlation matrix, which describes the variations in χ^2 about the minimum solution position as a function both of phase shifts (denoted as vector δ) and of data normalization parameters (denoted as a vector α). That is, to second order in variations of α and δ about the minimum value $(\chi_0^2,\delta_0,\alpha_0)$, χ^2 may be approximated as

$$\chi^2 \simeq \chi_0^2 + \tfrac{1}{2}(\Delta\delta^T,\Delta\alpha^T)\begin{bmatrix} A & B \\ \tilde{B} & C \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta\alpha \end{bmatrix}, \qquad (1)$$

where

δ_0 = phase shifts at minimum,

α_0 = normalization parameters at minimum,

$\chi_0^2 = \chi^2(\delta_0,\alpha_0)$,

$\Delta\delta = \delta - \delta_0$,

$\Delta\alpha = \alpha - \alpha_0$,

$$A_{jk} = \left.\frac{\partial^2\chi^2}{\partial\delta_j\partial\delta_k}\right|_{(\delta_0,\alpha_0)},$$

$$B_{jn} = \left.\frac{\partial^2\chi^2}{\partial\delta_j\partial\alpha_n}\right|_{(\delta_0,\alpha_0)},$$

$$C_{mn} = \left.\frac{\partial^2\chi^2}{\partial\alpha_m\partial\alpha_n}\right|_{(\delta_0,\alpha_0)},$$

$(\ )^T$ = transposed vector,

$\tilde{B}$ = transposed matrix.

Solving for the value $\Delta\alpha_{\min}$ which minimizes this expression for χ^2 for a particular choice $\Delta\delta$, we obtain

$$\Delta\alpha_{\min} = -C^{-1}\tilde{B}\Delta\delta \qquad (2)$$

and

$$\chi_r^2(\Delta\delta) = \chi^2(\Delta\alpha_{\min},\Delta\delta) = \chi_0^2 + \tfrac{1}{2}\Delta\delta^T(A - BC^{-1}\tilde{B})\Delta\delta. \qquad (2a)$$

We define the reduced correlation matrix A_r by the equation

$$A_r = A - BC^{-1}\tilde{B}. \qquad (3)$$

Equation (2a) describes a reduced hypersurface $\chi_r^2(\Delta\delta)$ in which the data normalization parameters do not appear explicitly, but in which they are always varied implicitly to achieve minimum χ^2. The conventional error associated with any quantity, $\rho(\delta)$, *including the phase shifts themselves*, which may be calculated in terms of the phase shifts is simply

$$\Delta\rho = \rho(\delta) - \rho(\delta_0) = (2\beta^T A_r^{-1}\beta)^{1/2}, \qquad (4)$$

where

$$\beta_j = \partial\rho/\partial\delta_j|_{\delta_0}. \qquad (5)$$

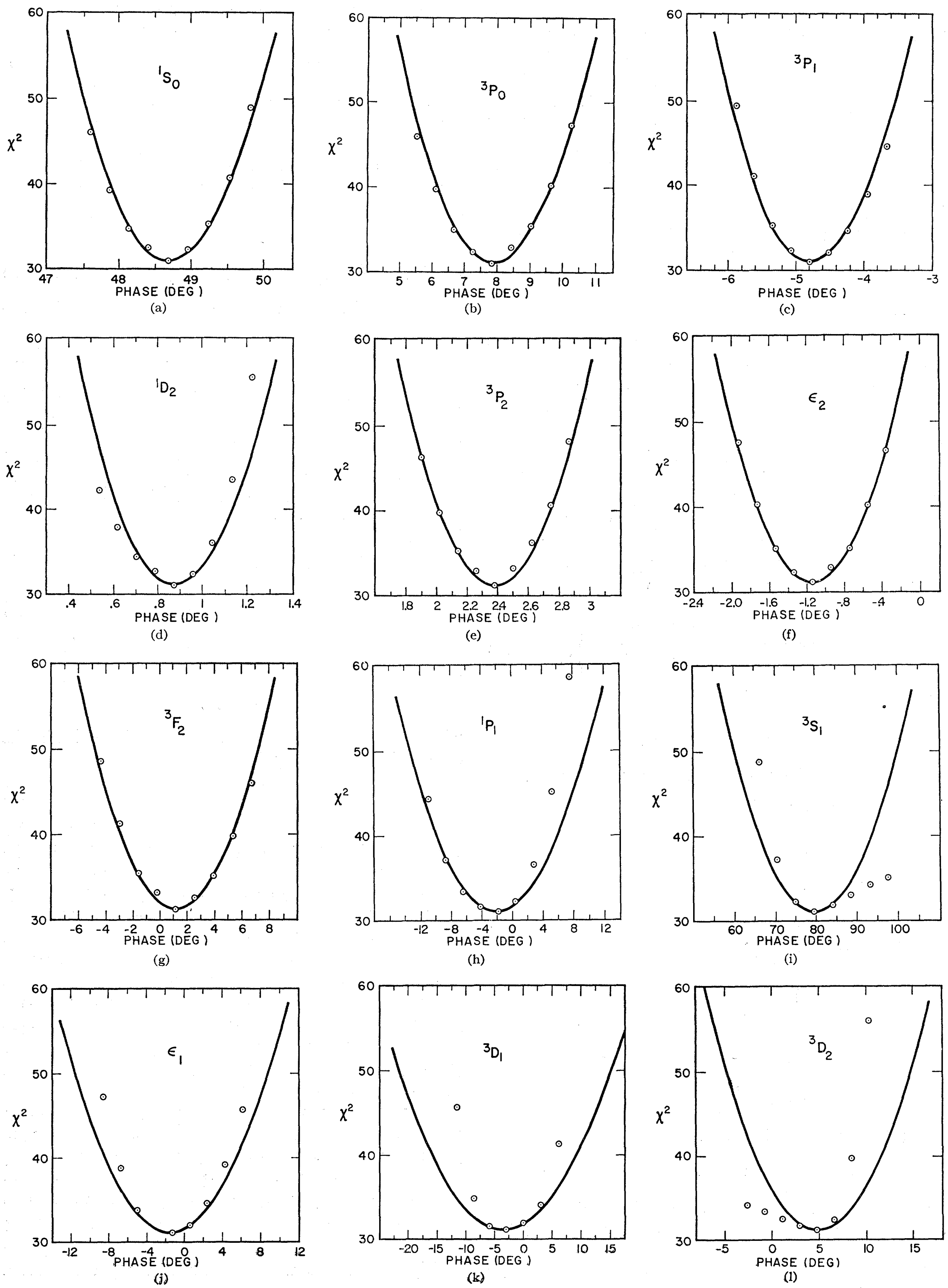
If the observable, $\rho(\delta)$ is one component, say δ_k, of the phase-shift vector, then

$$\beta_j = \delta_{jk},$$

with δ_{jk} being the Kronecker delta, and we obtain the conventional error matrix result:

$$\Delta\delta_j = [2(A_r^{-1})_{jj}]^{1/2}. \qquad (6)$$

There is no doubt that the reduced matrix given by Eq. (3) can be used *locally* (for small $\Delta\delta$) to describe the actual χ^2 hypersurface as calculated from the data. The question then is: over what range of $\Delta\delta$ is this approximation valid? We cannot, of course, answer this question entirely, since it would involve a reasonably complete mapping of a 13-dimensional hypersurface (there being 13 adjustable phase shifts in our analysis). We did, instead, attempt to achieve some insight into this question by comparing the results of "parameter studies" on all 13 searched phases against the predictions of the error matrix. It can be shown mathematically that the error in a phase shift as obtained through Eq. (6) is just the width (corresponding to a χ^2 increase of 1 from the minimum value) of a parabola obtained by a step-by-step variation of the phase shift in question, while allowing all the remaining parameters to be searched at each step. This procedure is here called a "parameter study," and such studies were carried out for all 13 phases around solution C. The results are given in Fig. 2. We conclude from these results that over a



$^1S_0$
$^3P_0$
$^3P_1$
$^1D_2$
$^3P_2$
ϵ_2
$^3F_2$
$^1P_1$
$^3S_1$
ϵ_1
$^3D_1$
$^3D_2$
χ^2
PHASE (DEG)
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)

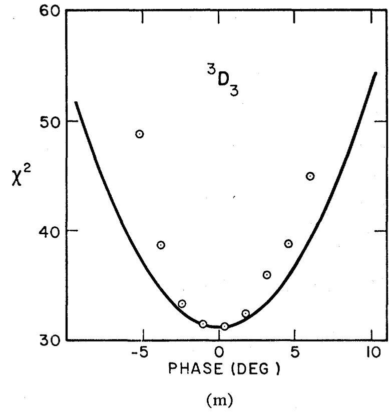


FIG. 2. Comparison of parameter studies around solution C with the predictions of a parabolic approximation. The points are derived from parameter studies, as in Fig. 1, and the solid curves are predictions of the parabolic approximation. The $T=1$ phases closely follow the parabolic approximation, whereas the $T=0$ phases do not.

range of at least three or four standard deviations in the phase shift, the parabolic approximation is quite good for the $T=1$ states, but it is questionable at best for the $T=0$ states. This result undoubtedly originates from the comparative abundance and quality of the (p, p) data and the paucity of (n, p) data at 25 MeV.

When the parameter study does not yield a parabola over a range of at least one standard deviation (χ^2 increase of one), as in the case of $^3D_2$, then the error as given by Eq. (6) can be very misleading. The error limits may in fact not be symmetric. The (n,p) data at 25 MeV are simply not sufficient to accurately determine $^3D_2$.

VI. COMPARISON WITH THE DUBNA SOLUTION

Workers at Dubna[23] have recently completed a phase shift analysis at 25 MeV, using approximately the same data selection as in our Table I. However, where we found *at least* two (roughly equivalent) solutions C and C' (Table III), the Dubna preprint[23] listed only one solution. The $T=1$ phases of all solutions are about the same. The ambiguity we found occurs in the $T=0$ phases. The Dubna solution has $T=0$ phases that correspond partly to solution C' ($^3S_1$) and partly to solution C (the D waves). The Dubna solution is shown in Table V.

Private correspondence with the Dubna group[24] established the fact that they had shifted the 25.63-MeV (p,p) differential cross section data (Table I) to 23.1 MeV by using interpolations based on the measured values of $\sigma(90°)$. However, this changes the normalization of the data but does not allow for any change in the shape of angular distribution. The (p,p) differential cross section is changing fairly rapidly in this energy region, and we found that a change in the shape of the cross section does in fact occur. This is evidenced by the fact that the Dubna solution when matched appropriately to our data selection gives a χ^2 sum of 124 for the six smallest-angle (p,p) differential cross section points. When we use their phases with their method of shifting the cross section, the discrepancy in χ^2 disappears. Thus the two computer codes are in agreement, and the phase shift differences are due to differences in handling the data, and not to coding errors. Also, the Dubna workers apparently do not assign an over-all normalization parameter to the differential cross sections, so that a change in the choice of normalization can have an appreciable effect on the phase shifts. The accurate measurement of (p,p) $\sigma(90°)$ at 25.62 MeV (Table I) was not available to the Dubna workers at the time they carried out their phase shift analysis.[24]

TABLE V. Recent Dubna solutions (Ref. 23) at 23.1 MeV.

	$l_{max}=2$	$l_{max}=3$
f^2	(0.08)	(0.08)
χ^2	67.4	54.47
$^1S_0$	50.01 ± 0.20	50.54 ± 0.26
$^3P_0$	9.19 ± 0.35	7.81 ± 0.71
$^3P_1$	-5.13 ± 0.17	-4.69 ± 0.43
$^3P_2$	2.78 ± 0.09	2.38 ± 0.37
$^1D_2$	0.81 ± 0.03	1.08 ± 0.07
ϵ_2		-1.18 ± 0.25
$^3F_2$		-0.20 ± 0.33
$^3F_3$		0.43 ± 0.63
$^3F_4$		-0.14 ± 0.17
$^1P_1$	0.69 ± 1.05	-0.19 ± 1.36
$^3S_1$	97.49 ± 4.09	96.41 ± 4.74
ϵ_1	-4.86 ± 1.39	-4.38 ± 1.22
$^3D_1$	-2.97 ± 1.29	-3.04 ± 1.93
$^3D_2$	3.42 ± 4.03	2.56 ± 5.14
$^3D_3$	-0.03 ± 1.16	-0.47 ± 1.04
$^1F_3$		0.26 ± 0.65

When we used the $l_{max}=3$ Dubna solution of Table V as a starting point and searched against our data selection, the solution went over essentially into solution C of Table III. In solution C the $^3F_3$, $^3F_4$, and $^1F_3$ are assigned to OPEC values, since the data are not sufficiently accurate or complete to determine these phases.

When the Dubna workers searched on the pion-nucleon coupling constant, they obtained $g_\pi^2\sim32\pm5$ for $l_{max}=2$, and $g_\pi^2\sim58\pm32$ for $l_{max}=3$. These results are at variance with the value $g_\pi^2=14.3\pm1.3$ that we obtain for solution C of Table III.

To summarize, we find that shifting differential cross-section data in energy must be done with allowances for changes in shape as well as in normalization. (It can be done on a computer, using the results of energy-dependent analyses.) Also, correct assignment of over-all normalization parameters is important, both for a proper evaluation of the error matrix and for a determination of possible solution ambiguities. We find a real ambiguity in the $T=0$ phases at 25 MeV. The differences between the phases of solution C in Table III and the solution of Table V are directly related to the manner in which the differential cross section data are treated.

[23] S. I. Bilenkaya, Z. Janout, Yu. M. Kazarimov, and F. Lehar, Yadernaya Fiz. 4, 892 (1966) [English transl.: Soviet J. Nucl. Phys. (to be published)].

[24] We would like to thank Dr. Lehar for an interesting private communication.

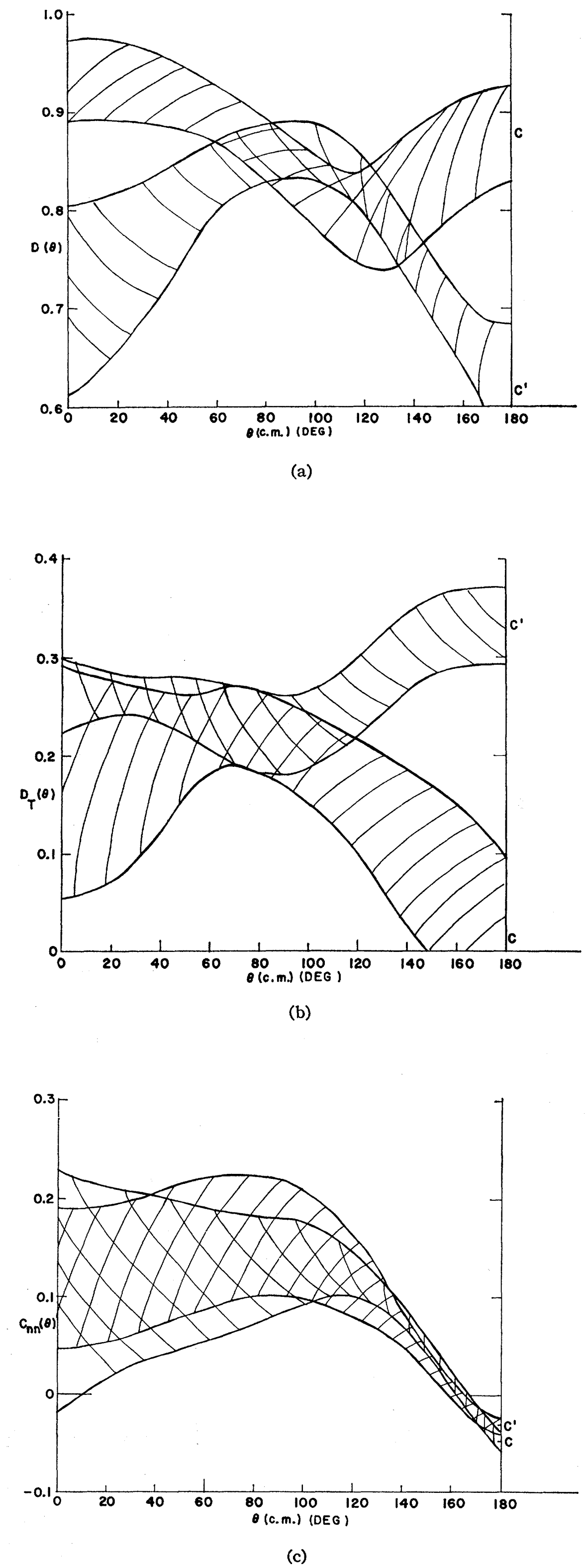


FIG. 3. Predictions from solutions C and C' for (n,p) observables D, D_T, and C_{nn}. The first two measurements can be used to distinguish between C and C', but the C_{nn} measurement cannot.

VII. ENERGY-DEPENDENT ANALYSIS

In paper IV of this series, we listed our energy-dependent results from 25–400 MeV. The present modifications in the data at 25 MeV affect the low-energy end of our solution. Also, the fitting of $^1S_0$ and $^3S_1$ to scattering length and effective range formulas at the low-energy end improve this solution. The result is that the solution now gives an excellent quantitative fit to (p,p) differential cross-section data[25] at 10 MeV ($\langle\chi^2\rangle_{\rm av}\sim 1$) without any adjustment in the parameters, even though these parameters were all obtained (except for the scattering lengths and effective ranges) from data at 25 MeV and above.

One further improvement in the energy-dependent analysis was to establish a "corridor-of-errors" for each of the phase shifts. The limits of this corridor at each energy represent the values that the phase shift can be shifted up or down and keep the χ^2 increase to one, *with the other phases allowed to vary in accommodation with the shift.* These results will be published in a forthcoming paper.

VIII. EXPERIMENTS NEEDED TO RESOLVE THE 25-MeV AMBIGUITY

There is a considerable theoretical justification for favoring solution C over solution C', largely through the close correspondence between the phase shifts of solution C and the corresponding low-energy predictions[26,27] (see Table III). It would be desirable, however, to base such a choice on the information from the data. We have examined the predictions of the laboratory observables for both of the solutions C and C', to establish which measurements are best capable of effecting a separation (in χ^2) between them. We determined that, of the (n,p) observables, depolarization D or depolarization transfer D_T, at either large or small scattering angles, if measured to within an accuracy of about 0.1, can distinguish between C and C' (see Fig. 3).

Experimental efforts are currently underway[19] to determine C_{nn} at 90° and A_{zz} at 175° for the (n,p) system. Fairly accurate measurements of these quantities could cause large changes in the $T=0$ states for either solution, C or C'. For this reason, and because they will most certainly decrease the errors on the $T=0$ states, they will be important experiments. It is doubtful, however, that they will be useful in a separation of solutions C and C'. This is illustrated in Fig. 3, where predictions for C_{nn}, D, and D_T from solutions C and C' are plotted. The D and D_T measurements distinguish between C and C', whereas the C_{nn} measurements do not.

It is interesting to inquire how the phase shifts will change if a new measurement for some observable, $\rho^{\rm exp}\pm\epsilon^{\rm exp}$, is added to the present set of data. This

[25] L. H. Johnston and D. E. Young, Phys. Rev. 90, 989 (1959).
[26] H. P. Noyes, Phys. Rev. Letters 12, 171 (1964).
[27] H. P. Noyes, Nucl. Phys. 74, 508 (1965).

TABLE VI. Characteristic changes in $T=0$ phases from the values given by solution C, for various possible (n,p) experiments. $\rho(\delta_0)$ is the observable predicted by solution C. η is defined in Eq. (9) of the text. Δ^1P_1, etc., are the quantities defined in Eq. (11) of the text. These quantities are used in Eqs. (8) and (7) to give the phase shift variations corresponding to different values for the observables.

	(n,p) experiment at 25 MeV						
	$C_{nn}(90°)$	$C_{nn}(175°)$	$A_{zz}(175°)$	$D(5°)$	$D(175°)$	$D_T(5°)$	$D_T(175°)$
$\rho(\delta_0)$	0.13	−0.02	0.17	0.93	0.87	0.16	0.02
η	0.048	0.01	0.18	0.04	0.05	0.11	0.09
Δ^1P_1	−2.3	0.19	2.6	−2.3	−1.35	−2.5	−2.2
Δ^3S_1	0.75	0.98	−2.7	1.69	0.7	2.9	3.5
ϵ_1	2.1	0.3	−2.31	2.27	1.6	2.3	2.0
Δ^3D_1	1.78	0.86	−3.16	3.69	3.9	3.6	3.7
Δ^3D_2	−0.86	−0.4	−1.68	−1.8	−1.7	−1.8	−2.0
Δ^3D_3	0.85	0.43	−1.48	1.8	1.9	1.7	1.7

prediction can be made by using the reduced correlation matrix A_r described in Sec. V. The phase changes are given by

$$\Delta\delta_\rho=(\Delta\chi)\Delta\delta_\rho{}^\circ, \tag{7}$$

where

$\Delta\delta_\rho$ = change in the phase shifts resulting from the addition of measurement $\rho^{\exp}\pm\epsilon^{\exp}$,

$$\Delta\chi\equiv\frac{\rho(\delta_0)-\rho^{\exp}}{[(\epsilon^{\exp})^2+\eta^2]^{1/2}}, \tag{8}$$

[note that $(\Delta\chi)^2$ is the resulting change in χ^2],

$\rho(\delta_0)$ = value of the observable predicted by the solution,

A_r = reduced correllation matrix for solution C,

$\eta^2\equiv 2\beta^T A_r{}^{-1}\beta$ = "theoretical" error squared, coming from the uncertainties in the other observables already included. (9)

$$\beta_l\equiv\partial\rho/\partial\delta_l|_{\delta_0}=\text{gradient of }\rho\text{ at }\delta_0, \tag{10}$$

$\Delta\delta_\rho{}^0\equiv=(2/\eta)A_r{}^{-1}\beta$ = the "characteristic" change associated with the measurement of the observable ρ. (11)

To illustrate the amount by which compatible (small increases in χ^2) measurements can affect the $T=0$ phases for solution C, we have tabulated in Table VI the characteristic changes, $\Delta\delta_\rho{}^0$, corresponding to a number of possible (n,p) measurements. We can readily see from Table VI that our present determination for $T=0$ states is subject to large changes resulting from future (n,p) experiments. We have not tabulated the changes in $T=1$ states for the various (n,p) experiments because they change only slightly from the values given by solution C.

IX. CONCLUSIONS

The present analysis represents an improvement over our previous results[3] at 25 MeV, both with regard to the $T=0$ phases and with regard to the determination of $g_\pi{}^2$. We have made a preliminary re-analysis at 50 MeV which suggests that a better data selection will remove the anomalous behavior of the ϵ_1 phase shift. We plan to publish these results, together with the concomitant changes produced in the energy-dependent analysis, in forthcoming papers.

ACKNOWLEDGMENTS

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