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# Separable Expansions of the Two-Body $T$ Matrix for Local Potentials and Their Use in the Faddeev Equation*

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An expansion of the off-shell two-body $T$ matrix is introduced in the form of a sum of terms separable in the initial- and the final-momentum variables. The convergence of this expansion is tested against exact solutions of several local potential problems. The substitution of the expansion into the Faddeev equations yields a set of coupled integral equations in one variable. As an example, the binding energy of three identical bosons is calculated using an $S$-wave Yukawa potential for the two-body interaction. It is found that a stationary state exists for the potential strength $G \geq 1.4$, whereas a two-body bound state requires $G \geq 1.8$. An off-shell effective-range formula is introduced for problems in which the shape of the two-body potential is not well known. It is shown that the difference between the solutions of the Yukawa potential and the exponential potential with the same scattering length and effective range is comparable to the deviation of the off-shell effective-range formula from either solution.

## I. INTRODUCTION

RECENTLY, a number of authors[1,2] have presented various methods for the calculation of the off-shell two-body $T$ matrix and have suggested possibilities of applying them to the three-body problem. In the present paper we show that the direct inversion of the Lippmann-Schwinger equation is a practical way of solving the two-body $T$ matrix. After examining the analytic structure of the solution, we propose an expansion of the off-shell $T$ matrix in the form of a sum of terms, each being separable in the initial- and final-momentum variables. The substitution of this expansion into the three-body Faddeev equation[3–5] yields an integral equation in only one variable and is soluble by ordinary numerical methods. This method allows us to solve the Faddeev equation without recourse to the usual assumption of separable potentials.[4,5]

The study of the Lippmann-Schwinger equation also leads to a two-parameter effective-range-type formula for the off-shell two-body $T$ matrix. This formula may be useful for problems where information on the two-body interaction is limited to a few phenomenological parameters, such as those of the low-energy nuclear reactions.

In Sec. II we review the derivation of the Lippmann-Schwinger equation starting with the Schrödinger equation. Then we replace the momentum variables by discrete indices and convert the integral equation into a matrix equation. The off-shell $T$ matrix is then evaluated by a matrix-inversion operation. In Sec. III we propose an expansion of the off-shell $T$ matrix and compare the exact solution of the Yukawa-potential problem with the expansion. In Sec. IV we introduce a two-parameter off-shell effective-range formula. This formula is compared to the exact solutions of the Yukawa and the exponential potentials, all with the same scattering length and effective range. Finally, in Sec. V, we reduce the Faddeev equation using the expansion of the two-body $T$ matrix given in Sec. III and solve for the three-body ground state energy with $s$-wave two-body Yukawa interactions. The convergence of the expansion is tested by the stability of the numerical solutions. The extension to include higher orbital interactions is straightforward, but no calculation is done in this paper.

## II. THE TWO-BODY $T$-MATRIX

We start with the two-body Schrödinger equation for a given energy $k^2$ and a fixed angular momentum $l$:

$$H|k^2\rangle = k^2|k^2\rangle. \quad (1)$$

(The $l$ index is suppressed.) The corresponding free-particle equation can be written as

$$H_0|k^2\rangle_0 = k^2|k^2\rangle_0, \quad (2)$$

where

$$H_0 = H - V. \quad (3)$$

From these equations one obtains

$$|k^2\rangle_{\epsilon\to0^+} = |k^2\rangle_0 - (H_0 - k^2 - i\epsilon)^{-1}V|k^2\rangle. \quad (4)$$

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[1] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965); K. L. Kowalski, *ibid.* **15**, 798 (1965).

[2] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 105 (1966).

[3] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[4] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[5] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139**, B1085 (1965). The same results for the partial-wave reduction are also obtained by T. Osborn and H. P. Noyes, Phys. Rev. Letters **17**, 215 (1966).

This is easily verified by applying $(H_0-k^2-i\epsilon)$ to both sides of (4). The $-i\epsilon$ gives the outgoing wave boundary condition.

Now we multiply (4) on the left by $-{}_0\langle k'|V$ and insert a complete set of free-particle state vectors on both sides of $(H_0-k^2-i\epsilon)^{-1}$;

$$-{}_0\langle k'^2|V|k^2\rangle=-{}_0\langle k'^2|V|k^2\rangle_0+\sum_{k''^2,k'''^2}{}_0\langle k'^2|V|k''^2\rangle_0\,{}_0\langle k''^2|(H_0-k^2-i\epsilon)^{-1}|k'''^2\rangle_0\times{}_0\langle k'''^2|V|k^2\rangle. \quad (5)$$

Since the $T$ matrix is defined by

$$_0\langle k'^2|T|k^2\rangle_0\equiv-{}_0\langle k'^2|V|k^2\rangle, \quad (6)$$

Eq. (5) can also be written as

$$_0\langle k'^2|T|k^2\rangle_0=-{}_0\langle k'^2|V|k^2\rangle_0-\sum_{k''^2}{}_0\langle k'^2|V|k''^2\rangle_0\times{}_0\langle k''^2|T|k^2\rangle_0/(k''^2-k^2-i\epsilon). \quad (7)$$

This is the Lippmann-Schwinger equation.

As we shall see later, the Faddeev equation requires a simple generalization of (7), namely,

$$T(k'^2,k^2,s)=-V(k'^2,k^2)-\frac{1}{\pi}\int_0^\infty dk''^2\frac{k''V(k'^2,k''^2)T(k''^2,k^2;s)}{k''^2-s-i\epsilon}. \quad (8)$$

Clearly, Eq. (8) can be reduced to (7) if we set

$$_0\langle k'^2|T|k^2\rangle_0=T(k'^2,k^2;k^2). \quad (9)$$

In writing Eq. (8) we have chosen the normalization

$$T(k^2,k^2;k^2)=(e^{i\delta_l}\sin\delta_l)/k, \quad (10)$$

where $\delta_l$ is the conventional phase shift. It can be shown (for example, by iteration) that $T(k'^2,k^2;s)$ is symmetric in $k'^2$ and $k^2$ for a fixed value of $s$ provided that $V(k'^2,k^2)$ is symmetric.

Now were replace the momentum variables $k'^2$, $k^2$, and $k''^2$ by a discrete spectrum and introduce the matrices $T(s)$ and $V$ with elements

$$T_{ij}(s)=T(k_i^2,k_j^2;s), \quad (11)$$
$$V_{ij}=V(k_i^2,k_j^2).$$

In matrix notation, Eq. (8) becomes

$$T(s)=-V-VR(s)T(s), \quad (12)$$

where $R(s)$ is a diagonal matrix whose elements are

$$R_{ij}(s)=\delta_{ij}\cdot\frac{1}{\pi}\int_{(k_i^2)_-}^{(k_i^2)_+}ds'\frac{\sqrt{s}}{s'-s-i\epsilon}. \quad (13)$$

Here $(k_i^2)_-$ and $(k_i^2)_+$ are lower and upper boundaries of that segment of the momentum variable which is replaced by the discrete value $k_i^2$. It is important to evaluate the integral in (13) analytically because of the singularity from the denominator.

As it stands, the integral in (13) for the highest value of $i$ formally diverges if the range in $k^2$ is infinite. However, Eq. (12) shows that the corresponding value of $R_{ii}$ is always multiplied by $V_{mi}$ with the same $i$. Therefore, there is no divergence as long as $V$ vanishes sufficiently rapidly at high momentum. This is indeed the case for ordinary local potentials.

Since $V$ and $R(s)$ are known matrices, Eq. (12) is readily solved:

$$T(s)=-[I+VR(s)]^{-1}V. \quad (14)$$

For ordinary potentials such as the Yukawa or the exponential, Eq. (14) gives the off-shell $T$ matrix quite accurately by using only 10 to 20 points for the momentum spectrum.

## III. EXPANSION OF THE OFF-SHELL $T$ MATRIX

Changing back to the momentum variables we can write the solution of Eq. (14) as

$$T(k'^2,k^2;s)=F(k'^2,k^2;s)/D(s), \quad (15)$$

where $D(s)$ is the determinant of the matrix $[I+VR(s)]$. Clearly, it does not depend on the momentum variables $k'^2$ or $k^2$. On the other hand, all the poles of $T(k'^2,k^2;s)$ in the $s$ variable for positive values of $k'^2$ and $k^2$ must appear as zeros of $D(s)$. This is because $F(k'^2,k^2;s)$ has no singularity in $s$ (positive $k'^2,k^2$) except for a branch point at $s=0$ with a branch cut extending to $+\infty$.

As a function of $k'^2$ and $k^2$, $F$ also has no singularity in the region $k^2$, $k'^2$ positive and $s$ negative because the potential is analytic in $k'^2$ and $k^2$. In view of this simple analytic structure, we make an expansion in the form

$$F(k'^2,k^2;s)=\sum_{n,m=0}C_{nm}(s)\left(\frac{k'^2}{k'^2+\mu^2}\right)^n\left(\frac{k^2}{k^2+\mu^2}\right)^m\times\frac{1}{(k'^2+\mu^2)(k^2+\mu^2)}. \quad (16)$$

For a given $s$ and a fixed number of terms in (16), the parameters $C_{nm}$ and $\mu^2$ can be chosen to optimize the fit to $F(k'^2,k^2;s)$ for all positive values of $k'^2$ and $k^2$. The value of $\mu$ is roughly the inverse of the force range, and the extra factor $[(k'^2+\mu^2)(k^2+\mu^2)]^{-1}$ provides a convergent behavior for $k'^2$ or $k^2$ much greater than $\mu^2$.

As long as the potential $V(k'^2,k^2)$ is symmetric under the interchange of $k'^2$ and $k^2$, the $T$ matrix is also a symmetric function of $k'^2$ and $k^2$. Therefore,

$$C_{nm}(s)=C_{mn}(s). \quad (17)$$

In practice it is convenient to divide the expansion into terms with $m=n$ and terms with $m\neq n$. For the examples of the Yukawa potential and the exponential

TABLE I. Comparison of the expansion of the off-shell $T$ matrix with the exact solution for a unit-range Yukawa potential with $G=1.4$.

| No. of terms $(m,n)$ | $s=-0.111$ $k^2=0.111$ $k'^2=0.111$ | $s=-0.111$ $k^2=0.111$ $k'^2=3.45$ | $s=-0.111$ $k^2=3.45$ $k'^2=3.45$ | $s=-3.45$ $k^2=0.111$ $k'^2=0.111$ | $s=-3.45$ $k^2=0.111$ $k'^2=3.45$ | $s=-3.45$ $k^2=3.45$ $k'^2=3.45$ | $s=-32.11$ $k^2=0.111$ $k'^2=0.111$ | $s=-32.11$ $k^2=0.111$ $k'^2=3.45$ | $s=-32.11$ $k^2=3.45$ $k'^2=3.45$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 term (0,0) | 2.34 | 0.846 | 0.306 | 1.37 | 0.495 | 0.179 | 1.10 | 0.397 | 0.144 |
| 2 terms (0,0), (1,1) | 2.35 | 0.895 | 0.505 | 1.20 | 0.462 | 0.273 | 1.10 | 0.413 | 0.206 |
| 5 terms (0,0), (1,1), (2.2) (0,1), (1,0) | 2.34 | 0.856 | 0.562 | 1.32 | 0.400 | 0.340 | 1.19 | 0.328 | 0.280 |
| Exact solution | 2.32 | 0.878 | 0.578 | 1.31 | 0.411 | 0.346 | 1.18 | 0.333 | 0.289 |

potential, it is found that terms with $m=n$ are more important than those with $m\neq n$. A typical example is shown in Table I for a unit-range Yukawa potential with coupling strength $G=1.4$. As one can see, a five-term fit with $(m,n)$ equal to (0,0), (1,1), (2,2), (0,1), (1,0) is very close to the exact solution for a wide range of $k'^2$, $k^2$, and $s$.

## IV. OFF-SHELL EFFECTIVE-RANGE FORMULA

For physical problems in which the two-body interaction is not known very well, it is desirable to find a direct parametrization of the $T$ matrix. For the on-shell $T$ matrix, the ordinary effective-range formula is known to be a useful representation in the low-energy region. A simple derivation of this formula can be obtained through the examination of analytic structures.[6] In this section we propose an analogous formula for the off-shell $T$ matrix, also based on analytic structure.

As was shown in Sec. III, the function $F(k'^2,k^2;s)$ has no singularity in the region $k'^2$, $k^2$ positive, and $s$ negative. On the other hand, $F$ will, in general, have singularities for negative $k'^2$ or $k^2$. Since the determinant $D(s)$ already contains all the zeros corresponding to bound-state poles of the $T$ matrix and has a branch cut for positive $s$, the simplest approximation for $F$ is an $s$-independent function with a pole in $k'^2$ and $k^2$. We consider the two-parameter formula:

$$F(k'^2,k^2;s)=\Gamma/(k'^2+k^2+\mu^2)\,, \tag{18}$$

and

$$\begin{aligned} D(s)&=1-\frac{1}{\pi}\int_0^\infty ds'\frac{\Gamma\sqrt{s}}{(2s'+\mu^2)(s'-s-i\epsilon)}\\ &=1-\Gamma\left[\frac{i\sqrt{s}+\mu/\sqrt{2}}{2s+\mu^2}\right]. \end{aligned} \tag{19}$$

Aside from the analytic properties stated above, the $T$ matrix obtained by $F/D$ reduces to the ordinary effective-range formula when one takes the on-shell limit $k'^2=k^2=s$. That is,

$$\begin{aligned} T(s,s;s)&=\left(\frac{\Gamma}{2s+\mu^2}\right)\left[1-\Gamma\left(\frac{i\sqrt{s}+\mu/\sqrt{2}}{2s+\mu^2}\right)\right]^{-1}\\ &=\left[\left(\frac{\mu^2}{\Gamma}-\frac{\mu}{\sqrt{2}}\right)+\left(\frac{2}{\Gamma}\right)s-i\sqrt{s}\right]^{-1}. \end{aligned} \tag{20}$$

The scattering length and the effective range are identified as

$$-1/a=(\mu^2/\Gamma-\mu/\sqrt{2}),\quad \tfrac{1}{2}r=2/\Gamma. \tag{21}$$

Another property of our representation for $F$ is that the on-shell limit is pure real for positive $s$. This is also required by the exact solution of the Lippmann-Schwinger equation. From Eqs. (13) and (14), it is easily seen that the matrix $[I+VR(s)]$ is pure real except for the $i$th column with $s$ lying between $(k_i^2)_+$ and $(k_i^2)_-$. Therefore, aside from the factor $D^{-1}(s)$, the inverse matrix $[I+VR(s)]^{-1}$, given by the co-factors, is pure real along the $i$th row. Hence the on-shell $F$ function $F(s,s;s)$ is also real.

Finally, $F(k'^2,k^2;s)$ can be expanded in the form (16) as follows:

$$\begin{aligned} F(k'^2,k^2;s)&=\Gamma/(k'^2+k^2+\mu^2)\\ &=\frac{\Gamma\mu^2}{(k'^2+\mu^2)(k^2+\mu^2)[1-(k'^2/(k'^2+\mu^2))(k^2/(k^2+\mu^2))]}\\ &=\sum_{m=0}^{\infty}\Gamma\mu^2\left(\frac{k'^2}{k'^2+\mu^2}\right)^m\left(\frac{k^2}{k^2+\mu^2}\right)^m\frac{1}{(k'^2+\mu^2)(k^2+\mu^2)}. \end{aligned} \tag{22}$$

Clearly, only equal-power terms enter into this expansion. Since we have found that equal-power terms are more important than terms with $m\neq n$ in the expansion of the exact $T$ matrix, this can be considered as a further support for our off-shell effective-range formula given by (18) and (19).

[6] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters 3, 191 (1959).

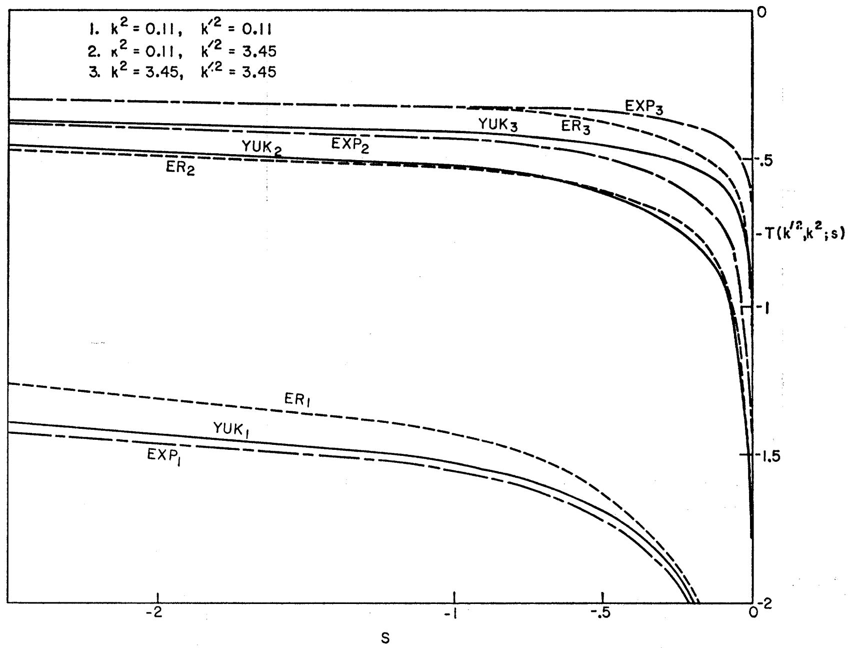


FIG. 1. Comparison of the off-shell effective-range formula with exact solutions of Yukawa and exponential potentials, all with $a=-6.99$ and $r=2.44$.

For a practical test of our formula, we take a Yukawa potential and an exponential potential both having the same scattering length and effective range. Specifically

$$V_{\text{Yuk}}(r)=1.4e^{-r}/r \Rightarrow a=-6.99,\quad r=2.44, \tag{23}$$

$$V_{\text{exp}}(r)=1.18\times1.66e^{-1.66r} \Rightarrow a=-6.99,\quad r=2.44.$$

We use the units $M=\hbar=1$.

With these potentials we calculate the exact $T$ matrix by solving the Lippmann-Schwinger equation as described in Sec. II. Then we evaluate the off-shell effective-range formula using (18) and (19) with $\Gamma$ and $\mu$ determined by (21). The results are plotted in Fig. 1. As one can see, the difference between the Yukawa solution and the exponential solution is comparable to the deviation of the effective-range formula from either of them. Therefore, we conclude that our simple effective-range formula is a practical representation of the off-shell $T$ matrix for problems where the shape of the two-body potential is not well determined. Clearly, our formula can also be generalized to include a sum of terms of the form (18) and the corresponding modification of (19). The number of terms depends on the amount of information available for the two-body $T$ matrix.

## V. THE THREE-BODY PROBLEM

We now consider the interaction of three identical neutral spinless bosons and limit ourselves to the $S$-wave interaction between any two particles. Using the notations of Ahmadzadeh and Tjon,[5] the Faddeev equation can be reduced to the form

$$\Psi_s(p,q)=\phi(p,q)+\frac{1}{\pi q}\int_0^\infty dp_2^2\int_0^\infty dq_2^2\int_{-1}^{+1}d\cos\theta_{22}\times\delta(q^2-q_1^2)T(p^2,p_1^2,s-q^2)\times\left(\frac{p_2q_2}{p_2^2+q_2^2-s}\right)\Psi_s(p_2,q_2), \tag{24}$$

where

$$q_1^2=\tfrac{1}{4}(3p_2^2+q_2^2-2\sqrt{3}p_2q_2\cos\theta_{22}),\qquad p_1^2=\tfrac{1}{4}(p_2^2+3q_2^2+2\sqrt{3}p_2q_2\cos\theta_{22}). \tag{25}$$

In terms of $\cos\theta_{22}$, the $\delta$ function becomes

$$\delta(q^2-q_1^2)=\begin{cases}\dfrac{2}{\sqrt{3}p_2q_2}\delta\left[\cos\theta_{22}+\left(\dfrac{4q^2-3p_2^2-q_2^2}{2\sqrt{3}p_2q_2}\right)\right], & \tfrac{1}{3}(2q-q_2)^2<p_2^2<\tfrac{1}{3}(2q+q_2)^2,\\ 0, & \text{otherwise.}\end{cases} \tag{26}$$

After performing the $\cos\theta_{22}$ integration using the $\delta$ function, Eq. (24) reduces to

$$\Psi_s(p,q)=\phi(p,q)+\frac{2}{\sqrt{3}\pi q}\int_0^\infty dq_2^2\int_A^B dp_2^2T(p^2,p_1^2;s-q^2)\times\left(\frac{1}{p_2^2+q_2^2-s}\right)\Psi_s(p_2,q_2), \tag{27}$$

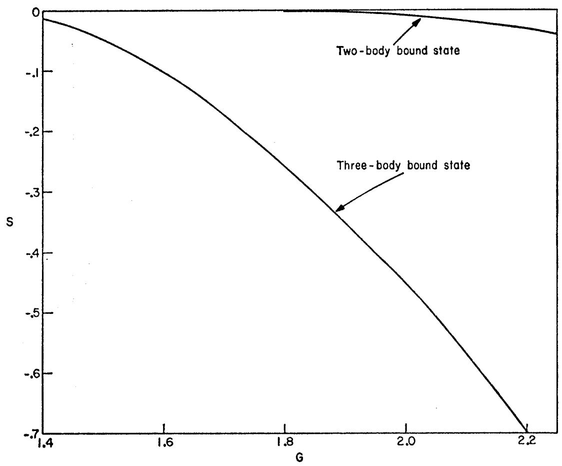


FIG. 2. Three-body bound-state energy and two-body bound-state energy versus the Yukawa coupling strength $G$.

where

$$A=\tfrac{1}{3}(2q+q_2)^2,\quad B=\tfrac{1}{3}(2q-q_2)^2, \tag{28}$$

and $p_1^2$ is now given by

$$p_1^2=p_2^2+q_2^2-q^2. \tag{29}$$

As it stands, (27) is a Fredholm equation in two variables. If one divides the $q^2$ and $p^2$ variables into $I$ and $J$ discrete points, respectively, then the kernel of the equation becomes a square matrix of rank$I\times J$. A direct inversion of the matrix equation is impractical, not only because of the size of the matrix but also because the range of $p_2^2$ depends on $q$ and $q_2$. On the other hand, if we make use of the expansion (16) for $T(p^2,p_1^2;s-q^2)$, then we can reduce the integral equation to a single-variable equation as shown below.

With the expansion (16), Eq. (27) can be written as

$$\Psi_s(p,q)=\phi(p,q)+\frac{2}{\sqrt{3}\pi q}\int_0^\infty dq_2^2\int_A^B dp_2^2 \times\sum_{n,m}^{N,M}\left(\frac{C_{nm}(s-q^2)}{D(s-q^2)}\right)\left(\frac{p^2}{p^2+\mu^2}\right)^n\left(\frac{p_1^2}{p_1^2+\mu^2}\right)^m \times\frac{1}{(p^2+\mu^2)(p_1^2+\mu^2)}\frac{1}{(p_2^2+q_2^2-s)}\Psi_s(p_2,q_2). \tag{30}$$

Since the $p$ dependence of the integral is given entirely by functions of the form $p^{2n}/(p^2+\mu^2)^{n+1}$, we can write

$$\Psi_s(p,q)=\phi(p,q)+\sum_{n=0}^{N}\chi_n(q,s)\left(\frac{p^2}{p^2+\mu^2}\right)^n\frac{1}{(p^2+\mu^2)}. \tag{31}$$

Substituting (31) into (30), we obtain

$$\chi_n(q,s)=\phi_n(q)+\int_0^\infty dq_2^2\sum_{r=0}^{N}K_{nr}(q,q_2;s)\chi_r(q,s), \tag{32}$$

where

$$K_{nr}(q,q_2;s)=\frac{2}{\sqrt{3}\pi q}\sum_{m=0}^{M}\frac{C_{nm}(s-q^2)}{D(s-q^2)} \times\int_A^B dp_2^2\left(\frac{p_1^2}{p_1^2+\mu^2}\right)^m\left(\frac{p_2^2}{p_2^2+\mu^2}\right)^r \times\frac{1}{(p_1^2+\mu^2)(p_2^2+\mu^2)(p_2^2+q_2^2-s)}, \tag{33}$$

$$\phi_n(q)=\frac{2}{\sqrt{3}\pi q}\sum_{m=0}^{M}\frac{C_{nm}(s-q^2)}{D(s-q^2)}\int_0^\infty dq_2^2\int_A^B dp_2^2 \times\left(\frac{p_1^2}{p_1^2+\mu^2}\right)^m\frac{\phi(p_2,q_2)}{(p_1^2+\mu^2)(p_2^2+q_2^2-s)}. \tag{34}$$

Now, (32) is a set of coupled single-variable equations and is readily solved by numerical methods provided $(N+1)$ is not too large. As we have seen in Table I, an excellent fit of the two-body $T$ matrix can be obtained with $(N+1)$ equal to three. If we divide the $q^2$ variable into $I$ discrete points, then Eq. (32) becomes a matrix equation where the kernel is a $(NI+I)\times(NI+I)$ matrix:

$$\eta_i(s)=\xi_i+\sum_{j=1}^{NI+I}\kappa_{ij}(s)\eta_j(s), \tag{35}$$

where

$$\eta_{nI+m}(s)=\chi_n(q_m),\quad \xi_{nI+m}=\phi_n(q_m),$$

and

$$\kappa_{nI+m,rI+m'}(s)=(\Delta q_2^2)_{m'}K_{nr}(q_m,q_{m'},s). \tag{36}$$

In matrix notation the solution of (35) is simply the vector given by

$$\eta(s)=[I-\kappa(s)]^{-1}\xi. \tag{37}$$

Since the bound states of the three-body system corresponds to zeros of the determinant of $[I-\kappa(s)]$, it is not necessary to evaluate the inhomogeneous term $\xi$ if only the energy level is desired. In practice, it is sufficient to take $I$ between 10 and 20. Therefore, the dimension of $\kappa$ is under 60 by 60. Whether we are calculating $\eta$ or simply evaluating the determinant of $(I-\kappa)$, the operation takes just a few seconds on a computer.

As an illustration of the application of the expansion method, we take the two-body interaction to be a Yukawa potential of unit range. For an attractive potential strength $G\geq 1.4$, we find that the three-body system has a bound state. The value $G=1.4$ is considerably lower than the minimum-coupling strength for the formation of a two-body bound state $(G=1.8)$.[7] In Fig. 2 we plot the binding energy of the three-body system versus $G$. As a reference we also plot the two-body binding energy for $G>1.8$.

[7] M. Luming, Phys. Rev. 136, B1120 (1964).

Aside from testing the accuracy of the expansion against the exact two-body $T$ matrix as shown in Table I, we also test the sensitivity of the three-body solution to the expansion. This is done in two ways. First, we compare the required coupling strength $G$ with the three-term expansion against that with the five-term expansion for a fixed binding energy. The difference is within a few percent. For the second test we take a simple function for the inhomogeneous term $\phi(p,q)$ and solve for $\Psi(p,q)$ using a five-term expansion for the two-body $T$ matrix. Having obtained $\Psi(p,q)$, we substitute it into the right-hand side of the original Faddeev equation (27) with the kernel given now by the exact $T$ matrix. This gives a function $\bar{\Psi}(p,q)$ which is generally within 2% of the solution obtained by the five-term expansion.

The above results indicate that the Faddeev equation, along with the expansion method, is a practical tool for calculating the three-body wave function as well as the energy levels. For the two-body interaction in the higher orbital states, the expansion of Ref. 5 can be applied along with our treatment of the off-shell $T$ matrix. Finally, we remark that the scattering of a two-body bound state by a third particle can also be handled in essentially the same way as above except that the kernel of the integral equation will be complex and the numerical solution will involve the inversion of a complex matrix.

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# Example of an "Inelastic" Bound State*

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An example is given of a bound state which occurs in a channel with a repulsive Born approximation. The bound state occurs because of the attraction provided at low energy by three-particle intermediate states.

## I. INTRODUCTION

FOR some time, physicists have speculated that inelastic channels in particle collisions might give rise to resonances or bound states. Such a mechanism is well known in nuclear physics,[1] and it has been applied to particle physics by several authors, including, for instance, Cook and Lee.[2] These authors performed a matrix $N/D$ calculation to see if the higher nucleon resonances might be driven by the opening of the $\rho N$ channel. The interest in such a mechanism stems from the fact that particle exchange on the left is not always attractive in the channels where resonances are known to occur. Put in modern terminology, there is sometimes a breakdown in naive bootstrap philosophy, which assumes that elastic unitarity and the crossing matrix are sufficient principles for the prediction of resonances. The inelasticity mechanism is invoked as a cure for the breakdown of bootstrap theory.

Unfortunately, all past attempts to assess the effects of inelasticity have been marred by an enormous number of approximations and simplifications. It has never been clear whether it was the inelasticity or the approximations which produced the resonances. In the present paper, we wish to correct this situation by presenting a model calculation in which the dynamics are carefully evaluated, without important approximations other than the exclusion of states involving more than three particles. Specifically, our scattering amplitudes will have the hallowed properties of analyticity, crossing symmetry, and unitarity

We study a reaction in which the single-particle exchange poles provide a repulsive force. Corresponding to this, when we construct a scattering amplitude satisfying crossing and elastic unitarity (one-meson approximation), no bound state appears. This is in agreement with naive bootstrap theory. However, when we construct a scattering amplitude satisfying crossing and two- and three-particle unitarity (two-meson approximation), a bound state appears when the coupling is sufficiently strong. The bound state can only be a result of inelasticity because we have the elastic calculation for comparison. In addition, the development of a *bound* state indicates that inelasticity can affect the low energy properties of scattering amplitudes. Contrary to popular belief, inelastic effects are not limited to energies where three-particle phase space is large.

## II. THE MODEL

The model we study is the charged scalar static model. This model has a spin-zero source which can exist in either positive ($p$) or neutral ($n$) charge states,

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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chaps. VIII–X.
[2] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962); **127**, 297 (1962).