Bootstrap Studies of the ρ Meson and the "Generalized Potential" of the Modified Cheng Representation*

W. J. ABBE
Physics Department, University of Michigan, Ann Arbor, Michigan†
and
Physics Department, University of Georgia, Athens, Georgia‡

AND

P. KAUS
Physics Department, University of California, Riverside, California

AND

P. NATH AND Y. N. SRIVASTAVA

Physics Department, Northeastern University, Boston, Massachusetts

(Received 7 October 1966)

A bootstrap calculation for the width of the ρ resonance is performed using a simplified version of a Reggeized bootstrap theory proposed recently by the authors. A phenomenological Pomeranchuk input trajectory has been assumed. The $I=2$ channel is eliminated from the appropriate crossing relation and no statements about this channel are necessary. The 2-π continuum states are assumed to be dominated by the ρ for the $I=1$ and by the f^0 for the $I=0$. The input ρ trajectory is parametrized to produce the ρ resonance at the observed mass. The ρ width is then determined by the maximum satisfaction of the crossing relations. The calculation yields a "best" width of 125 MeV for the ρ. The problem concerning the simultaneous bootstrap of the ρ mass and width is briefly discussed, and a systematic procedure for obtaining the "generalized potential" of the modified Cheng representation is given.

I. INTRODUCTION

MOST calculations performed so far to bootstrap the ρ ($\approx$750 MeV) resonance have assumed that the force needed to produce the ρ in the direct channel is furnished mainly by the exchange of the (elementary) ρ in the crossed channels.[1–3] These calculations have mostly been performed in the N/D formalism where one solves for the $I=1$, $J=1$ π-π amplitude using an $L=1$ partial-wave projection of the input force. Since the asymptotic behavior (in energy) of the elementary ρ ($J=1$) exchange force blows up as $s^{J=1}$, a cutoff on the integrals in N/D equations is necessary which is adjusted to force the resonance at the experimental mass. Such calculations have invariably obtained a width of the ρ which is much too large. For example, a single-channel calculation using elastic unitarity gives a width of $\approx$600 MeV as compared with an input width of $\approx$100 MeV.[3] It has been generally believed that this discrepency between the input and the output widths was presumably due to the neglect of other channels; in particular, the $\pi\omega$ and $K\bar{K}$ are strongly coupled to the π-π channel. However, while it is generally true that the closed inelastic channels do in fact work to reduce the width of a resonance,[4,5] it is improbable that such an effect would reduce the width by a numerical factor of 5 or 6. In fact, multichannel[4] as well as single-channel calculations using inelastic unitarity[6] show that the reduction in width is about a factor of 2 or less.

Let us now take the point of view that the failure of the previous dynamical calculations to obtain the ρ self-consistently is perhaps due to the lack of uniformity in the way the ρ is treated in the direct channel (as dynamic) *versus* the crossed channel (as elementary). In the true spirit of the bootstrap hypothesis, one must treat the ρ as dynamic not only in the direct but in the crossed channel as well. This is tantamount to saying that one must determine self-consistently not only the ρ, but the entire Regge trajectory on which the ρ happens to lie. The force to produce the ρ in the direct channel comes, therefore, not only from the exchange of the ρ, but the exchange of the ρ-trajectory itself. As is well known, the Regge input force has a nice asymptotic behavior and no subtractions or cutoffs are necessary to obtain convergence. Until recently, the most serious attempt to develop a theory for bootstrapping entire Regge trajectories has been the strip approximation.[7] This theory has some serious drawbacks, however,

* Part of this work was completed at the University of California, Riverside, California, where it was supported in part by the U. S. Atomic Energy Commission.
† Address until September, 1967.
‡ Permanent address.
[1] F. Zachariasen, Phys. Rev. Letters 7, E268 (1961). Also see lectures given at the Pacific International Summer School in Physics, 1965, Honolulu, Hawaii (unpublished).
[2] D. Wong, Phys. Rev. 126, 1220 (1962).
[3] M. Bander and G. Shaw, Phys. Rev. 135, B267 (1964); also in this connection see J. D. Amand and G. C. Joshi, Nuovo Cimento 38, 1588 (1965).
[4] J. Fulco, G. Shaw, and D. Wong, Phys. Rev. 137, B1242 (1965).
[5] P. Nath and Y. N. Srivastava, Phys. Rev. 138, B404 (1965).
[6] P. W. Coulter and G. Shaw, Phys. Rev. 138, B1273 (1965).
[7] G. F. Chew, Phys. Rev. 129, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* 135, B208 (1965).

one of them being that it introduces an arbitrary parameter called the strip width.

Recently, the authors proposed a theory of Reggeized bootstraps which is completely free of any undetermined arbitrary parameters.[8] The theory was based on a representation of scattering amplitudes expressed in terms of Regge trajectories alone,[9] in contrast to the earlier proposals which explicitly involved both the Regge parameters: namely, the trajectories as well as the residues.[7,10] Furthermore, the representation used in I had the remarkable property that the individual contribution from each Regge pole to the scattering amplitude explicitly reflected the correct analytic behavior expected of the exact amplitude. Partial as well as total scattering amplitudes calculated through this representation in successive approximations when one, two, or more high-level trajectories were retained gave such a fast convergence in potential theory that amplitudes computed retaining only the trajectories which reach the physical l-plane almost reproduced the exact known results.[8,9] This is an encouraging feature of the theory proposed in I, since the critical test of any bootstrap program based on Regge trajectories must of necessity be determined by how fast the theory converges in terms of the number of trajectories.

In the present paper we use a simplified version of the program developed in I to bootstrap the ρ. Since our Regge pole representation for the total scattering amplitude $f(s,t)$ constructed in I is valid for $s>4\mu_\pi^2$ and all t, we can impose crossing only at the unphysical (s,t) values $s>4\mu_\pi^2$ and $t>4\mu_\pi^2$. Now the partial-wave expansion for $f(s,t)$ converges only for $t<(2\mu_\pi)^2$, since $2\mu_\pi$ is the lowest mass that can be exchanged in the crossed channel and satisfaction of crossing requires, therefore, that we calculate $f(s,t)$, etc., by Regge continuation. To simplify the numerical problem as much as possible, we would like to impose crossing at real values of (s,t). On the other hand, the background integral in $f(s,t)$, for example, converges on the cut in t (i.e., $t>4\mu_\pi^2$) only in the limiting sense as we approach the real t axis from above in the complex t plane, being completely oscillatory on the cut itself.[8] This way of calculating the background integral on the cut (i.e., in the limiting sense) is computationally a somewhat difficult problem, and we explore, as a first attempt, an approximate calculation which is computationally more manageable. In the present calculation we have achieved this by artificially pushing the cut in t starting at the 2π continuum to the mass of the ρ ($\approx 5.43\mu_\pi$) for $I=1$ and to the mass of the f^0 ($\approx 8.95\mu_\pi$) for the $I=0$. Partial-wave expansions can now be used for $t\lesssim\mu_\rho^2\approx 30\mu_\pi^2$. We found that this choice allows a reasonable rate of convergence of the series for $t\lesssim 16\mu_\pi^2$.

Instead of actually attempting to simultaneously self-consistently calculate both the $I=0$ and the $I=1$ channels of the π-π system, as would be done in a complete calculation, we here rather addressed ourselves to the question: Can one assume an $I=0$ (Pomeranchuk) trajectory, obtained from experiment, and see a "best" satisfaction of the crossing relations in the region of the actual experimental ρ parameters?

Our input consists of the $I=0$ (Pomeranchuk) trajectory in a form used by Ahmadzadeh and Sakmar.[11] The ρ trajectory is assumed to be real analytic with only a right-hand cut in the ν_s plane. No statements (or approximations) regarding the $I=2$ channel are necessary, since it can be eliminated from the appropriate crossing relations.[8] In this paper, we have attempted a partial answer to the bootstrap problem; namely, we adjust the ρ-trajectory parameters to obtain a resonance at the experimental mass of the ρ and then determine the width of the ρ resonance by adjusting the other parameters until a maximum satisfaction of crossing occurs. Our result in brief is that we obtain a "best" width of the ρ by this procedure which is ≈ 125 MeV.

Section II is devoted to a review of the basic equations of our formalism and a general formulation of the problem. In Sec. III we discuss the details of the calculation and the results. In an Appendix, an iterative procedure for determining the "generalized potential" for the modified Cheng representation is presented.

II. FORMULATION OF THE PROBLEM

The π-π elastic partial-wave S matrix can be represented[8,9] in terms of Regge trajectories $\alpha_n(\nu_s)$ as follows for $\nu_s>0$:

$$\ln S^I(l,\nu_s)=\sum_n\left\{\int_{\alpha_n^I(\nu_s)}^{\alpha_n^{*I}(\nu_s)}\frac{\exp[(l'-l)\xi_I(\nu_s)]}{l'-l}dl'-i\frac{h_I^2}{\nu_s^{p_I}}\times\frac{\exp[-(l-c_I+n)\xi_I(\nu_s)]}{l-c_I+n}\times\int_{\mu_I}^{\infty}\sigma_I(\mu')P_{n-1}\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu'\right\}+\frac{ih_I^2}{\nu_s^{p_I}}\times\int_{\mu_I}^{\infty}\sigma_I(\mu')Q_l\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu',\quad(2.1)$$

[8] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **141**, 1513 (1966), hereafter referred to as I. Preliminary results of the work described in the text were presented at the April, 1966 meeting of the American Physical Society, Washington, D. C. [Bull. Am. Phys. Soc. **11**, 381 (1966)].

[9] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **140**, B1595 (1965); W. J. Abbe and G. A. Gary, Bull. Am. Phys. Soc. **11**, 901 (1966).

[10] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964).

[11] A. Ahmadzadeh and I. A. Sakmar, Phys. Letters **5**, 145 (1963). It now appears quite likely that the f^0 does not lie on the P trajectory: B. R. Desai and P. G. O. Freund, Phys. Rev. Letters **16**, 622 (1966); B. R. Desai (to be published). In this case the parameters of the P would be somewhat different, but this should not affect this calculation too much.

where I is the isotopic spin index and ν_s and ξ are defined by

$$\nu_s=\tfrac{1}{4}s-\mu_\pi^2, \tag{2.2}$$

$$\cosh\xi_I(\nu_s)=1+\frac{(2\mu_I)^2}{2\nu_s}, \tag{2.3}$$

where μ_I is the lowest mass exchanged in the crossed channel, and the parameters h_I, p_I, c_I, and the "generalized potential" $\sigma_I(\mu')$ (see Appendix) are to be determined self-consistently from the crossing relations. The partial-wave amplitude is defined by

$$f_l^I(\nu_s)=\frac{S^I(l,\nu_s)-1}{2i\rho(\nu_s)}, \tag{2.4}$$

where

$$\rho(\nu_s)=\left(\frac{\nu_s}{\nu_s+\mu_\pi^2}\right)^{1/2} \tag{2.5}$$

and has the behavior for large l as[8]

$$f_l^I(\nu_s)\underset{l\to\infty}{\longrightarrow}\frac{e^{-l\xi^I(\nu_s)}}{\sqrt{l}},\quad \mathrm{Re}l>-\tfrac{1}{2}, \tag{2.6}$$

$$\cosh\xi_I(\nu_s)=1+\frac{\mu_I^2}{2\nu_s}.$$

Let us now consider the partial-wave expansion for the total invariant amplitude $f(\nu_s,t)$, where t is the invariant momentum transfer in the c.m. system

$$f^I(\nu_s,t)=\sum_{l=0}^{\infty}(2l+1)f_l^I(\nu_s)P_l\left(1+\frac{t}{2\nu_s}\right)[1+(-)^{I+l}]. \tag{2.7}$$

Since

$$P_l(z)\underset{l\to\infty}{\longrightarrow}\frac{e^{l\eta}}{\sqrt{l}}, \tag{2.8}$$

where

$$\cosh\eta=z=1+\frac{t}{2\nu_s}, \tag{2.9}$$

the series (2.7) will converge for $t<\mu_I^2$. In π-π scattering $\mu_I=2\mu_\pi$, and the representation (2.7) is then valid only for $s>4\mu_\pi^2$ and $t<4\mu_\pi^2$. Similarly, we can obtain $f(\nu_t,s)$ valid for $t>4\mu_\pi^2$ and $s<4\mu_\pi^2$ so that we have to perform a Watson-Sommerfeld transformation in (2.7) to obtain a common region of the (s,t) plane where crossing can be imposed. This is, however, a time-consuming operation computationally as explained in Sec. I, and we shall instead perform an approximate calculation which simplifies the problem considerably. Our approximation consists in pushing the cuts of $f(\nu_s,t)$ in t which start at the 2-π continuum states to the mass of the ρ for the $I=1$ channel and to the mass of the f^0 for the $I=0$ channel (assuming of course that the Pomeranchuk particle is indeed the f^0). This enables us to use the partial-wave expansions for $t\lesssim\mu_\rho^2\approx30\mu_\pi^2$. In practice, a reasonable rate of convergence of the series is obtained for $t\lesssim16\mu_\pi^2$.

This is then the essential procedure in our calculation: We set the "generalized potential" (see Appendix) $\sigma_I(\mu')=\delta(\mu'-\mu_I)$ and we have set $\mu=\mu_\rho\approx5.83\mu_\pi$ for $I=1$ and $\mu=\mu_{f^0}\approx8.95\mu_\pi$ for $I=0$ and calculated both sides of the crossing relation:

$$f^0(\nu_s,t)-f^0(\nu_t,s)=-2[f^1(\nu_s,t)-f^1(\nu_t,s)]. \tag{2.10}$$

It should be noted that (2.10) is exact; the $I=2$ channel has simply been algebraically eliminated from the crossing relations.

III. DISCUSSION OF THE CALCULATION AND RESULTS

For the $I=0$ (Pomeranchuk) trajectory we have used the one determined phenomenologically by Ahmadzadeh and Sakmar[11]:

$$\mathrm{Im}\alpha^{I=0}(\nu_s)=\frac{3.79\nu_s^{1.533}}{259.7+(\nu_s-55.24)^2}, \tag{3.1}$$

$$\mathrm{Re}\alpha^{I=0}(\nu_s)=1+\left(\frac{\nu_s+1}{\pi}\right)P\int_0^\infty\frac{\mathrm{Im}\alpha^{I=0}(\nu')}{(\nu'-\nu)(\nu'+1)}d\nu', \tag{3.2}$$

and we have set $\mu_\pi=1$. This trajectory was determined so that the following three experimental facts are fitted:

(i) $\alpha^{I=0}(\nu_s=-1)=1$.

(ii) $\mathrm{Re}\alpha^{I=0}(s=\mu_{f^0}^2)=2;\quad \mu_{f^0}=1250$ MeV. (3.3)

(iii) $\Gamma_{f^0}\approx\left[\frac{\mathrm{Im}\alpha(s)}{s^{1/2}d\,\mathrm{Re}\alpha(s)/ds}\right]_{s=\mu_{f^0}^2}\approx200$ MeV.

In addition, the trajectory was assumed to go to -1 at large ν_s. Since the above trajectory given by (3.1) and (3.2) was fit for values of ν_s near threshold, one naturally expects it to be valid only in such a region, and it should be noted that in the calculation described below we use it in the small region near threshold $(0<\nu_s\lesssim3)$ where it is expected to hold.

For the $I=1$ channel, we have used the trajectory:

$$\alpha^{I=1}(K)=\alpha_0+(\alpha_\infty-\alpha_0)\left(\frac{K}{K+K_0}\right)^2\left[1-\frac{B}{(K+K_0)^{2p}}\right]+\left(\frac{A}{\alpha_0-\frac{1}{2}}\right)\left[\left(\frac{K}{K+K_0}\right)^{2\alpha_0+1}-\left(\frac{K}{K+K_0}\right)^2\right]\left(\frac{1}{K+K_0}\right), \tag{3.4}$$

where $k=iK\equiv(\nu_s)^{1/2}$ and $A>0$, $B>0$, $0<p\leqslant\frac{1}{2}$, and $\mathrm{Re}K_0>0$. The trajectory (3.4) has the following properties:

(i) Real analytic with only a right-hand cut in the ν_s plane.

(ii) $\text{Im}\alpha(\nu_s)>0$ for $\nu_s>0$ and $=0$ for $\nu_s<0$.

(iii) $d\text{Re}\alpha(\nu_s)/d\nu_s>0$ as $\nu_s\to 0$ and approaches α_∞ from above.

(iv) The threshold behavior is correct up to order $\nu_s^{\alpha_0+1/2}$ or ν^2, whichever is appropriate, and all $-\frac{1}{2}<\alpha_0<1$. It also has the correct logarithmic behavior for $\alpha_0=\frac{1}{2}$.

(v) For $\text{Re}K_0>0$, there are no singularities in the upper half k plane or physical ν_s plane.

(vi) $\text{Im}\alpha(\nu_s)\to 1/\nu_s^p$ for $0<p\leqslant\frac{1}{2}$.

Since the most important parameters in (3.4) are A, K_0, and α_0, we assume $p=\frac{1}{2}$, $B=0$, and $\alpha_\infty=-1$, so that for a given α_0 we have a two-parameter trajectory with K_0 and A which determine the width and position of the ρ resonance. Also since

$$\alpha^{I=1}(\nu_s)\underset{\nu_s\to\infty}{\to}-1+i\frac{2(\alpha_0+1)}{\sqrt{\nu_s}}K_0+\frac{3K_0^2(\alpha_0+1)+2AK_0}{\nu_s}+\cdots,\quad(3.5)$$

we may identify h^2 as[8]

$$h_{I=1}^2=4(\alpha_0+1)K_0\rangle 0,\quad(3.6)$$

since

$$K_0>0,\quad \alpha_0>-1.$$

If we denote the left-hand side of (2.10) by $A^0(s,t)$ and the right-hand side by $A^1(s,t)$, then the deviation from exact satisfaction of crossing symmetry can conveniently be expressed by $\sin\epsilon(s,t)$,[12] where

$$\sin\epsilon(s,t)=\frac{|A^0(s,t)-A^1(s,t)|}{\sqrt{2}\{[A^0(s,t)]^2+[A^1(s,t)]^2\}^{1/2}}.\quad(3.7)$$

The procedure is to calculate $A^0(s,t)$ and $A^1(s,t)$ for various s-t pairs, compute $\sin\epsilon(s,t)$ from (3.7) for each pair, and then average. Since both sides of (2.10) are complex, we do this for both the real and imaginary parts and calculate $\langle\langle\sin\epsilon\rangle\rangle$ where

$$\langle\langle\sin\epsilon\rangle\rangle=\tfrac{1}{2}[\langle\sin\epsilon_R\rangle+\langle\sin\epsilon_I\rangle].\quad(3.8)$$

The procedure of the calculation is now as follows: First, we vary the parameters of the $I=1$ trajectory until a resonance at the experimental mass of the ρ is obtained. This trajectory, along with the $I=0$ trajectory is then used in (2.1) to calculate the partial-wave S matrix for $I=1$ and $I=0$. The total amplitudes are computed through the partial-wave expansion (2.7) and finally $\langle\langle\sin\epsilon(s,t)\rangle\rangle$ is calculated for a set of (s,t) pairs. With a fixed input (experimental) mass of the ρ, the input widths are systematically changed to search for the point where crossing relation (2.10) is best satisfied on the average: namely, when a minimum is found in $\langle\langle\sin\epsilon(s,t)\rangle\rangle$.

[12] See, for example, R. E. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964).

TABLE I. Values of the figure of merit $\langle\langle\sin\epsilon\rangle\rangle$ as a function of the ρ width Γ in MeV.

Γ (MeV)	150	125	100	75	50
$\langle\langle\sin\epsilon\rangle\rangle$	0.82	0.67	0.72	0.74	0.74

For each value of the input ρ width, we evaluate $\sin\epsilon(s,t)$ by calculating the crossing relation at the following eleven points of the s-t plane: $s=15$, $t=5$, $5\frac{1}{2}$, 6, $6\frac{1}{2}$, $\cdots$, $9\frac{1}{2}$, 10. The choice of points was dictated by a desire to stay away from the line $s=t$ where (2.10) is an identity, and by the rate of convergence of the partial-wave series (2.7).

The results are shown in Table I where the input consists of the experimental ρ mass (≈ 750 MeV) and $\alpha_0^{I=1}=\frac{1}{2}$. We see from the table that $\langle\langle\sin\epsilon\rangle\rangle$ has its minimum value of 0.67 when the width $\Gamma=125$ MeV (the width was varied in 25-MeV steps) and may be compared to the experimental value of 106 MeV; this is certainly an improvement over the results of N/D calculations which, when the mass is fixed at the experimental value, find a width too large by as much as a factor of about 6.[12a]

The resulting ρ trajectory corresponding to the best values of its parameters, namely,

$$\Gamma_\rho=125\text{ MeV},$$
$$M_\rho=750\text{ MeV},$$

is then

$$\alpha^{I=1}(\nu_s)=\tfrac{1}{2}+\left(\frac{-i\sqrt{\nu_s}}{K_0-i\sqrt{\nu_s}}\right)^2\times\left[-\frac{3}{2}+2A\frac{\ln[-i\sqrt{\nu_s}/(K_0-i\sqrt{\nu_s})]}{K_0-i\sqrt{\nu_s}}\right],$$

with the values of A and K_0 given by

$$A=0.409,\quad K_0=4.455,$$

and $-i=\exp(-i\pi/2)$. In the low-energy limit, which was actually used in the calculation, this trajectory reduces to

$$\text{Re}\alpha^{I=1}(\nu_s)\underset{\nu_s\to 0}{\to}\frac{1}{2}+\left(\frac{\sqrt{\nu_s}}{K_0}\right)^2\left[\frac{3}{2}-\frac{2A}{K_0}\ln\left[\frac{(\sqrt{\nu_s})}{K_0}\right]\right],$$

$$\text{Im}\alpha^{I=1}(\nu_s)\underset{\nu_s\to 0}{\to}\frac{\pi A}{K_0}\left(\frac{\sqrt{\nu_s}}{K_0}\right)^2.$$

If the crossing relations were exactly satisfied, then $\langle\langle\sin\epsilon\rangle\rangle$ would have to be zero, so the question arises that although we see the crossing relations best satisfied

[12a] Under the approximations of the present calculation, it would not be meaningful to vary the width Γ too far from the experimental value, and hence the rather narrow range indicated in the table. More detailed calculations with wider variations in Γ are presently under way.

(at least in the narrow region in which we looked) for $\Gamma_\rho=125$ MeV, why is $\langle\langle\sin\epsilon\rangle\rangle(\approx 0.67)$ so large in this case? A glance at Figs. 1 and 2 immediately shows where the trouble lies, and as one might have guessed, the source of trouble is in approximating the 2-π continuum states by a pole at the ρ for $I=1$ and at the f^0 for the $I=0$.

In Figs. 1 and 2 we have plotted the real and imaginary parts of the invariant amplitudes $f^I(s,t)$ in the unphysical region. For example, the curve labeled (1) in Fig. 1 is $\mathrm{Re}f^{I=1}(t,\ s=15)$ plotted as a function of t for s fixed at 15; the curve labeled (2) is $\mathrm{Re}f^{I=1}(t=15,\ s)$ plotted as a function of s for t fixed at 15, and similarly for (3) and (4), while the curves for $\mathrm{Im}f^I(s,t)$ are shown in Fig. 2. If crossing were exactly satisfied, then for a particular (s,t) pair we should have

$$[2(1)-2(2)]+[(3)-(4)]=0. \qquad (3.9)$$

Now from Fig. 1 we see that (3–9) is quite well satisfied; indeed, the average value of $\sin\epsilon$ for this case is

$$\langle\sin\epsilon_R\rangle=0.440$$

as indicated on the figure. However, when we look at Fig. 2, we see the crossing relations for $\mathrm{Im}f^I(s,t)$ are poorly satisfied in this case since

$$\langle\sin\epsilon_I\rangle=0.896.$$

As discussed in the Appendix, the representation of the total invariant amplitude (2.7) may be written, when the modification is made with a "superposition of Yukawas" with weight function $\sigma(\mu')$:

$$f^I(\nu_s,t)=\sum_{l=0}^{\infty}(2l+1)[f_l^I(\nu_s)-f_l^{IB}(\nu_s)]P_l\left(1+\frac{t}{2\nu_s}\right)$$
$$\times[1+(-)^{I+l}]+\tfrac{1}{2}h_I^2(\nu_s+1)^{1/2}\left[P\int_{\mu_I^2}^{\infty}\frac{\sigma_I(\sqrt{t'})}{t'-t-i\delta}\frac{dt'}{\sqrt{t'}}\right.$$
$$\left.+(-)^I\int_{\mu_I^2}^{\infty}\frac{\sigma_I(\sqrt{t'})}{t'+t+4\nu_s}\frac{dt'}{\sqrt{t'}}\right], \qquad (3.10)$$

where we have abbreviated

$$f_l^{IB}(\nu_s)=h_I^2\frac{(\nu_s+1)^{1/2}}{2\nu_s}\int_{\mu_I}^{\infty}\sigma_I(\mu')Q_l\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu', \qquad (3.11)$$

with μ_I the lowest mass exchanged in the crossed channel, and c_I has been set to 0 and p_I to $\frac{1}{2}$ so certain partial-wave sums are more easily performed. Now in the above calculation the weight function was set equal to a delta function:

$$\sigma_I(\mu')=\delta(\mu'-\mu_0^I) \qquad (3.12)$$

and this is tantamount to replacing the t cut by a pole, since then the first integral of (3.10) simply gives the Born pole, and now the results of Fig. 2 for $\mathrm{Im}f^I(s,t)$ are understandable.

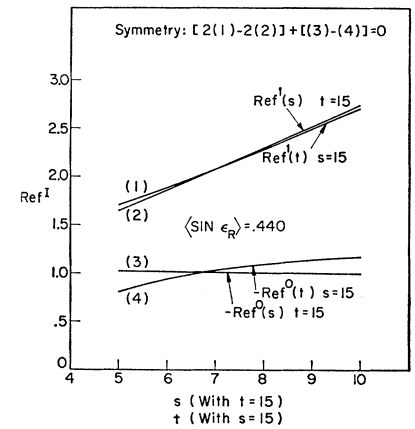

FIG. 1. The real part of the invariant amplitude as a function of t for s fixed and s for t fixed, as discussed in the text, for both $I=0$ and $I=1$.

For example, we see that $\mathrm{Im}f^0(t)$ with s fixed at 15 goes to a constant at $t=4$, whereas $\mathrm{Im}f^0(s)$ as a function of s with t fixed at 15 is forced to zero at threshold by the representation, since the Born term does not contribute to the imaginary part. Since a similar behavior prevails for the imaginary part of the $I=1$ amplitudes, the crossing relations are poorly satisfied in this case.

Let us now suppose that the weight function $\sigma(\mu')$ is not given simply by the delta function (3.12); then we can see qualitatively that the above situation must improve. Specifically, suppose the weight function has the form (see Appendix):

$$\sigma_I(\sqrt{t})\sim\mathrm{Im}_tf^I(t),\quad s\text{ fixed}, \qquad (3.13)$$

and from unitarity in channel t it follows that[13]

$$\sigma_I(\sqrt{t})>0. \qquad (3.14)$$

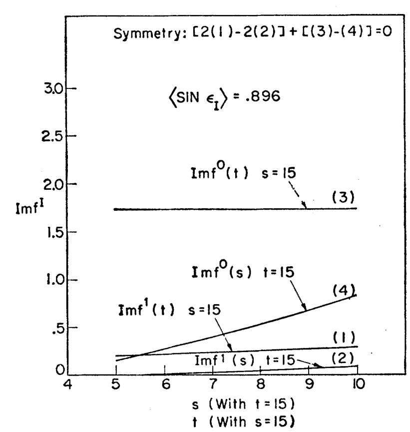

FIG. 2. The imaginary part of the amplitudes as in Fig. 1.

[13] Here we have tacitly assumed that $h>0$; of course, if this is not true, we cannot hope to bind anything.

In this case, the imaginary part of (3.10), when s and t are both above 4, is

$$\mathrm{Im}_{s,t}f^I(s,t)=\sum(2l+1)\mathrm{Im}f_l^I(\nu_s)P_l\left(1+\frac{t}{2\nu_s}\right)$$

$$\times[1+(-)^{I+l}]+\pi h_I^2\frac{(\nu_s+1)^{1/2}}{2}\frac{\sigma_I(\sqrt{t})}{\sqrt{t}}, \quad (3.15)$$

where $\sigma_I(\sqrt{t})$ satisfies (3.14) and from (3.13) goes to zero as t goes to its threshold value.

We can now see that (3.15) will qualitatively solve the difficulty in Fig. 2, since now, as $t\rightarrow 4$, s fixed $\sigma_I(\sqrt{t})\rightarrow 0$ and the curve (3) still goes to a constant, but now increasing as t increases because of $\sigma_I(\sqrt{t})$. However, when we switch the variables we have $s\rightarrow 4$ and t fixed, and while $\mathrm{Im}f^I(\nu_s)\rightarrow 0$, $\sigma_I(\sqrt{t})$ is now constant, so it goes to a constant rather than to zero as before. Furthermore, as s increases (t still fixed) the curve will increase because of both $f_l^I(\nu_s)$ and $(\nu_s+1)^{1/2}$, and so we see that qualitatively the above results will be improved by replacing $\sigma(\mu')$ by more than a simple delta function.

A systematic iterative procedure for determining $\sigma(\mu')$ is presented in the Appendix, and calculations along these lines are presently in progress.

IV. CONCLUSIONS

In the present paper we have attempted a very crude answer to a complicated question: Can one assume the existence of a phenomenological Pomeranchuk trajectory (obtained from experiment) and predict a maximum satisfaction of the crossing relations in physical regions of the ρ parameters, without making any statements or approximations regarding the $I=2$ channel? In view of the somewhat encouraging results obtained for the ρ width, we may now feel a little less hesitant to attempt the more difficult calculation wherein both the position and width of the resonance are determined self-consistently, or, in other words, a many-parameter bootstrap calculation. For the complete ρ bootstrap it is perhaps essential to revert to the full formalism of the theory as developed in I and the Appendix of this paper, and attempt to satisfy crossing in the s-t plane by calculating total invariant amplitudes via the Regge term and the background integral. This is so because the two-way self-consistency problem for the mass M_ρ of the ρ and its width Γ_ρ demands a very sensitive condition, namely, that the figure of merit curves close in the $\Gamma_\rho-M_\rho$ plane, and preliminary calculations show[14] that this is unlikely to happen in a calculation under the present approximations. Finally, a generalized potential for the modified Cheng representation has been defined, and a systematic iterative procedure for obtaining it has been given.

[14] W. J. Abbe, Ph.D. thesis, University of California, Riverside, California, Chap. 8 (unpublished).

ACKNOWLEDGMENT

The calculations presented here were performed at the Computing Center, University of California, Riverside, California on an IBM 1620II.

APPENDIX

The purpose of this appendix is to define the "generalized potential" of the modified Cheng representation, and present a systematic iterative procedure for determing it, at any stage of approximate satisfaction of the crossing relations.

The modified Cheng representation,[8,9] in its most general form,[15] is derived from a generalization of the asymptotic behavior of potential theory trajectories in the $\nu_s=k^2$ plane; that is,

$$\alpha_n(\nu_s)\underset{\nu_s\rightarrow\infty}{\longrightarrow}-n+c_n+\frac{ih^2}{2\nu_s^{p_n}}\times\int_\mu^\infty\sigma(\mu')P_{n-1}\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu', \quad (A1)$$

where $n=1, 2, 3, \cdots$, and the potential is of the form

$$V(r)=\int_\mu^\infty\sigma(\mu')\frac{e^{-\mu' r}}{r^q}d\mu'. \quad (A2)$$

In what follows, we shall take $c_n=0$, $p_n=\frac{1}{2}$, and $q=1$, since this choice makes certain sums easy to perform analytically, but the procedure may be generalized. Equations (A1) and (A2) then lead to the following representation for the single channel S-matrix element[16]

$$\ln S(l,\nu_s)=\sum_n\left\{\int_{\alpha_n(\nu_s)}^{\alpha_n^*(\nu_s)}\frac{\exp[(l'-l)\xi(\nu_s)]}{l'-l}dl' -\frac{ih^2}{\sqrt{\nu_s}}\frac{\exp[-(l+n)\xi(\nu_s)]}{l+n}\int_\mu^\infty\sigma(\mu')P_{n-1}\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu'\right\}$$

$$+\frac{ih^2}{\sqrt{\nu_s}}\int_\mu^\infty\sigma(\mu')Q_l\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu', \quad (A3)$$

where $\nu_s=k^2=\frac{1}{4}s-\mu_\pi^2$, $\cosh\xi(\nu_s)=1+(2\mu)^2/2\nu_s$. It is therefore natural to define the $\sigma(\mu')$ in (A3) as the "generalized potential" for the modified Cheng representation; in general, of course, it will depend on the energy ν_s.

In potential theory there is no procedure for determining $\sigma(\mu')$, except of course the trivial one of adjusting it to fit the experimental data, and as is well known, there will in general be many different $\sigma(\mu')$ that will predict the *same* experimental results. The nice thing

[15] See Chap. 6 of Ref. 14.

[16] A procedure for extending the modified Cheng representation to multichannel processes has been developed: W. J. Abbe, P. Nath, and Y. N. Srivastava, Nuovo Cimento (to be published).

about the relativistic strong-interaction problem is that the crossing relations, when coupled with unitarity and assumptions about analyticity, may actually determine the $\sigma(\mu')$ uniquely. One iterative procedure for determining it follows:

Suppressing the isospin index, the total amplitude for the π-π case is

$$f(\nu_s,t)=\sum_{l=0}^{\infty}(2l+1)f_l(\nu_s)P_l\left(1+\frac{t}{2\nu_s}\right), \quad \text{(A4)}$$

where, as usual,

$$f_l(\nu_s)=\left[\frac{S(l,\nu_s)-1}{2i\sqrt{\nu_s}}\right](\nu_s+1)^{1/2}. \quad \text{(A5)}$$

In the large ν_s limit, the first two terms of (A3) cancel and we have

$$f_l^B(\nu_s)\equiv(\nu_s+1)^{1/2}\frac{h^2}{2\nu_s}\int_\mu^\infty \sigma(\mu')Q_l\left(1+\frac{\mu'^2}{2\nu_s}\right)d\mu'. \quad \text{(A6)}$$

If we add and subtract (A6) in (A4), we obtain

$$f(\nu_s,t)=\sum_{l=0}^{\infty}(2l+1)[f_l(\nu_s)-f_l^B(\nu_s)]P_l\left(1+\frac{t}{2\nu_s}\right)$$
$$+(\nu_s+1)^{1/2}\frac{h^2}{2}P\int_{\mu^2}^{\infty}\frac{\sigma(\sqrt{t'})}{t'-t-i\delta}\frac{dt'}{\sqrt{t'}}, \quad \text{(A7)}$$

and now the sum in (A7) converges for $t<(2\mu)^2$, since

$$f_l(\nu_s)-f_l^B(\nu_s)\underset{l\to\infty}{\longrightarrow}\exp[-l\xi(\nu_s)].$$

From (A7) we obtain the well known fact that $\pi[\sigma(\sqrt{t})/\sqrt{t}](h^2/2)(\nu_s+1)^{1/2}$ is just the discontinuity of $f(\nu_s,t)$ across the t axis, which can be obtained from (A7) in the region $\mu^2<s$, $t<(2\mu)^2$ or, for $\mu=2\mu_\pi$, $4\mu_\pi^2<s$, $t<16\mu_\pi^2$. [This is also a good region because our representation (A3) is a candidate for being exact there, as far as inelastic channels are concerned.] However, in general for all s, $t>4\mu_\pi^2$, one must use the full Regge analytically continued amplitude.

One could, of course, simply put a parametrized form for $\sigma(\mu')$ and attempt to determine the parameters by forcing (A7) to satisfy the crossing relations (if indeed this is possible). However, this seems to us like a cumbersome and inefficient procedure, since one would like to take full advantage of any stage of approximate satisfaction of crossing relations as a guide toward proceeding to a more accurate stage, and simply parametrizing $\sigma(\mu')$ anew each time is of little avail in this connection.

Secondly, since $\sigma(\mu')$ is always required to be positive[13] at least up to the first inelastic threshold, as demanded by unitarity in channel t, we would like to have this incorporated automatically, rather than having to check it each time.

Third, we require that $\sigma(\mu')$ have something to do with the amplitude in channel t, and moreover, if we can achieve this directly, the number of parameters will be greatly reduced, since then no more will appear than appear in the representation (A3) via the trajectories. The following procedure then suggests itself (superscripts in parentheses correspond to stages of iteration):

We start the zeroth iterate by setting

$$\sigma^{(0)}(\mu')=\delta(\mu'-\mu). \quad \text{(A8)}$$

This choice then corresponds roughly to selecting a simple Yukawa potential in channel t. The total amplitude from (A7) is then

$$f^{(0)}(\nu_s,t)=\sum_{l=0}^{\infty}(2l+1)[f_l^{(0)}(\nu_s)-f_l^{B(0)}(\nu_s)]P_l\left(1+\frac{t}{2\nu_s}\right)$$
$$+(\nu_s+1)^{1/2}\frac{h^2}{\mu^2-t}. \quad \text{(A9)}$$

This is the level of iteration at which the calculation described in the text was performed, with the further approximation that $\mu=\mu_\rho$ for $I=1$ and μ_{f0} for $I=0$. After determining the approximate value of $f^{(0)}(\nu_s,t)$, the next stage of iteration is then found by simply taking the discontinuity of $f^{(0)}(\nu_t,s)$ across the t axis:

$$\sigma^{(1)}(\sqrt{t})=\frac{2\sqrt{t}}{\pi h^2\sqrt{(\nu_s+1)}}\times\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}f_l^{(0)}(\nu_t)P_l\left(1+\frac{s}{2\nu_t}\right)$$
$$\vdots \quad \text{(A10)}$$
$$\sigma^{(N)}(\sqrt{t})=\frac{2\sqrt{t}}{\pi h^2\sqrt{(\nu_s+1)}}\times\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}f_l^{(N-1)}(\nu_t)P_l\left(1+\frac{s}{2\nu_t}\right)$$

and so on. This iteration procedure then satisfies our general requirements for the ith iterate:

(a) $\sigma^{(i)}(\sqrt{t})>0$ (unitarity in channel t since the modified Cheng representation is automatically unitary).

(b) $\sigma^{(i)}(\sqrt{t})$ is directly determined by cross-channel amplitudes, only at a previous stage of iteration, and is of course energy dependent.

(c) It is assumed that $\mu=2\mu_\pi$, so that the only parametrization necessary is that of the trajectories $\alpha_n(\nu_s)$. Full use is then made of any stage of approximate satisfaction of the crossing relations.

(d) When only one trajectory is retained ($n=1$), it is not difficult to show that for $p_n>\frac{1}{2}$, both integrals in (A3) involving $\sigma(\mu')$ converge at early stages of iteration. We have not been able to prove it rigorously for higher trajectories and higher stages of iteration, but since $\sigma^{(N)}(\mu')$ is obviously still a delta-type function, peaked around the physical particles in channel t, this presumably will cause no difficulty. Furthermore, since the main reason for making the modification was to achieve rapid convergence with only the top trajectory, it suffices to show that the integrals converge with $n=1$ only.

Questions of convergence of this iteration procedure can only be settled numerically, and are presently under investigation, the preliminary results of a "first stage" having been discussed in the text. Finally, it should be stressed that the above procedure, using the partial-wave expansion, is valid only for $s>4\mu_\pi^2$ and $t<16\mu_\pi^2$ (a nice region as far as inelastic channels are concerned), and must be replaced by the full Watson-Sommerfeld continuation (background integral plus Regge term) in larger regions of the s-t plane.

PHYSICAL REVIEW VOLUME 154, NUMBER 5 25 FEBRUARY 1967

Optical-Model Parameters and Cross Sections for the ρ-Meson-Nucleon Interaction According to a Modification of the Coherent Droplet Model

C. Iddings and L. Marshall
*University of Colorado, Boulder, Colorado**

(Received 13 May 1966; revised manuscript received 28 November 1966)

A modified droplet optical-model method is developed for evaluating cross sections for short-lived ($\sim10^{-23}$ sec) particles produced in two-body reactions. Using elastic-scattering and ρ-production differential cross sections for 1.7-BeV/c π^-p interactions in this model, we estimate the total ρ-nucleon cross section as 12 mb.

Recent experimental results on two-body cross sections at high energies are almost universally characterized by sharp forward peaking and high inelasticity.[1] Motivated by such results, theoretical interpretations in terms of optical models have been proposed by several authors.[2–6] These models are proving useful in correlating data from various reactions and in giving a rough interpretation of the dynamics of the interaction.

We notice that sharp forward peaking occurs even at relatively low energies; for example in 1.7-BeV/c π^-p interactions both the elastic scattering[7] and ρ-production[8] reactions display marked diffraction patterns and therefore analysis using an optical model seems justifiable.

In this article, we shall analyze some results on ρ production in 1.7-BeV/c π^-p interactions[7,8] according to a simple extension of a model due to Byers and Yang.[5] Our conclusion is that the *elastic* ρ^-p interaction has an effective cross section of the form

$$\{d\sigma/dt\}_{\text{elastic}}=S_\rho e^{-\gamma_\rho t},\quad S_\rho=(8.0\pm2)\text{ mb}/(\text{BeV}/c)^2,\quad \gamma_\rho=(2.5\pm0.4)\ (\text{BeV}/c)^{-2},\tag{1}$$

where $t\geqslant0$ is the square of the momentum transfer between incoming and outgoing nucleons. This cross section is to be compared with that for π^-p elastic

* Research supported by the U. S. Atomic Energy Commission Under Contract No. AT-(11-1) 1537.

[1] See, e.g., S. J. Lindenbaum, in *Nucleon Structure Conference*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, Stanford, California, 1964) and Aachen-Berlin-CERN Collaboration, CERN Report No. 65-23, 1965 (unpublished); Phys. Letters **19**, 608 (1965); D. R. O. Morrison, CERN Report No. 66-14, 1966 (unpublished).

[2] R. Serber, Phys. Rev. Letters **10**, 357 (1963).

[3] (a) J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965); K. Gottfried and J. D. Jackson, Nuovo Cimento **34**, 735 (1965). (b) J. D. Jackson, J. T. Donohue, K. Gottfried, R. Keyser, and B. E. Svensson, Phys. Rev. **139**, B428 (1965); for values $0.90\leqslant C_\rho\leqslant1.00$, $\frac{3}{4}\gamma_\pi\leqslant\gamma_\rho\leqslant\gamma_\pi=0.12$ quoted in this reference, the corresponding total ρ-nucleon cross section at 1.6 BeV/c is $74\leqslant\sigma(\rho^-p)_{\text{tot}}\leqslant107$ mb. For a production reaction $\pi p\to Xp$, the absorption model uses the parameters C_{in}, γ_{in}, C_{out}, and γ_{out} corresponding to our parameters A_π, (x_π/k_π^2), A_ρ, and (x/k_ρ^2), so that the total cross sections corresponding to our expression (8) become

$$\sigma_{\text{tot}}(\text{out})=2\pi\lambda\!\!\!^{-}{}_{\text{out}}^2(C_{\text{out}}/\gamma_{\text{out}});\quad \sigma_{\text{tot}}(\text{in})=2\pi\lambda\!\!\!^{-}{}_{\text{in}}^2(C_{\text{in}}/\gamma_{\text{in}}).$$

The Jackson prescription allowing no free parameters requires $C_{\text{out}}=1$ and $\gamma_{\text{out}}=\frac{3}{4}\gamma_{\text{in}}$. It follows that a consequence of this prescription in the absorption model is

$$\frac{\sigma_{\text{tot}}(\pi p)}{\sigma_{\text{tot}}(Xp)}=\frac{3}{4}\frac{\lambda\!\!\!^{-}{}_\pi^2}{\lambda\!\!\!^{-}{}_X^2}\leqslant\frac{3}{4},$$

since $C_{\text{in}}\leqslant1$ and $\lambda\!\!\!^{-}{}_\pi^2\leqslant\lambda\!\!\!^{-}{}_X^2$, corresponding to $M_\pi\geqslant M_X$. This result seems nonphysical. However, Jackson *et al.* note that their model's predictions are not very sensitive to precise values of C_{out} and γ_{out}. Thus within the framework of the absorption model a broad range of values of $\sigma_{\text{tot}}(Xp)$ seems possible, in particular values which do not fulfill the relationship noted above. We quote their comment on the one-pion-exchange absorption model: "It is not permissible to treat C and γ as independently variable parameters in reactions mediated by pion exchange. . . . Nevertheless we shall treat C and γ as independent parameters throughout, since this provides us with the only simple way of incorporating final-state interactions which may be stronger and more long-range than initial-state interactions."

[4] L. Durand, III, and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

[5] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1965).

[6] (a) S. Drell and J. Trefil, Phys. Rev. Letters **16**, 552 (1966); **16**, 832 (1966); (b) M. Ross and L. Stodolsky, Phys. Rev. **150**, 1172 (1966).

[7] D. Allen, G. Fisher, G. Godden, J. Kopelman, L. Marshall, and R. Sears, Phys. Letters **21**, 468 (1966).

[8] D. Allen, G. Fisher, G. Godden, J. Kopelman, L. Marshall, and R. Sears, Phys. Rev. Letters **17**, 53 (1966).