Lorentz Commutators of Internal Dynamical Variables

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A definition of internal dynamical variables which includes the unitary-spin generators is used to determine the pure Lorentz commutators of these variables in the presence of nonconserving interactions. A geometrical interpretation of the nonvanishing Lorentz commutators is presented in terms of the hyperplane formalism, previously discussed by the author.

1. INTRODUCTION

THE converse of the McGlinn theorem,[1] that translationally invariant dynamical variables which are not conserved do not commute with the Lorentz generators, can be applied to such internal dynamical variables as isotopic spin, hypercharge, or indeed any of the $SU(3)$ generators.[2] The conservation of every one of these dynamical variables is violated by at least one of the prominent interactions among the photon, leptons, and hadrons. This raises two interesting questions concerning the commutators of these variables with the Lorentz generators. First, what is the interpretation of the nonvanishing commutators for variables which, in a casual frame of mind, one tends to think of as Lorentz invariants? Second, what, precisely, is the (operator) value of the commutator?

Clearly the answer to the second question must depend on the dynamics since if the internal variables are conserved they do commute with the Lorentz generators. In the general case, however, one might still seek an expression for the Lorentz commutator in terms of the commutator of the variable with the Hamiltonian. Such an expression is derived in the next section.

In the third section of this article the interpretation of nonvanishing Lorentz commutators is provided via the hyperplane formalism.[3] It is shown there that a precise meaning can be given to the notion that internal dynamical variables are Lorentz invariants even in the presence of nonconserving interactions.

The derivation in Sec. 2 is carried out in the conventional formalism to facilitate the reader's comprehension. The physical interpretation seems much clearer in the hyperplane formalism, however, and it was in that formalism that the author first obtained the result.

2. LORENTZ COMMUTATORS OF INTERNAL DYNAMICAL VARIABLES

For this discussion an internal dynamical variable $A(t)$ will be defined by the conditions of translational invariance,

$$[\mathbf{P},A]=0\,, \tag{2.1}$$

rotational invariance,

$$[\mathbf{J},A]=0\,, \tag{2.2}$$

and momentum independence. The last condition means that the eigenvalue spectrum of A is independent of the total momentum eigenvalue; i.e., for any eigenvalue A' of A the states

$$|\mathbf{P},A',\alpha\rangle \tag{2.3}$$

exist for all values of the P_i' from $-\infty$ to $+\infty$.

It may be objected that (2.2) does not allow for variables analogous to the *relative* position or momentum of two subsystems that may enter into the structure of a particle. Such variables are somewhat less *internal* than those which are uninfluenced by rotations and the present definition includes the currently interesting unitary-spin generators.[4] The generalization of the following arguments to internal *vector* variables is straightforward, in any case.

From the states (2.3) one can construct the states

$$|\mathbf{R}',A',\alpha\rangle \equiv (2\pi\hbar)^{-3/2}\int d^3P' \exp\left(\frac{i}{\hbar}\mathbf{P}'\cdot\mathbf{R}'\right)|\mathbf{P}',A',\alpha\rangle\,, \tag{2.4}$$

which are simultaneous eigenstates of A and the Newton-Wigner position operator $\mathbf{R}$.[5] It follows that

$$[\mathbf{R},A]=0\,. \tag{2.5}$$

If the spin vector $\mathbf{S}$ is now defined by

$$\mathbf{J}\equiv\mathbf{R}\times\mathbf{P}+\mathbf{S}\,, \tag{2.6}$$

then the Lorentz generator $\mathbf{N}$ is well known to be given by[6]

$$\mathbf{N}=H\!:\!\mathbf{R}+[(\mathbf{P}\times\mathbf{S})/\mathbf{P}^2](H-Mc)-ct\mathbf{P}\,, \tag{2.7}$$

where the colon indicates a symmetrized product and

$$Mc\equiv|(H^2-\mathbf{P}^2)^{1/2}|\,. \tag{2.8}$$

[1] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964). See also the papers reprinted in *Symmetry Groups in Nuclear and Particle Physics* [edited by F. J. Dyson (W. A. Benjamin, Inc., New York, 1966), pp. 246–268] which discuss refinements and generalizations of the McGlinn theorem.
[2] See Acknowledgment at the end of this article.
[3] G. N. Fleming, J. Math. Phys. **7**, 1959 (1966).
[4] It does not, of course, include the $SU(6)$ generators. The corresponding analysis for $SU(6)$ is presently under study.
[5] T. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[6] L. L. Foldy, Phys. Rev. **122**, 275 (1961); F. Coester, Helv. Phys. Acta **38**, 7 (1965).

From (2.1), (2.2), (2.5), and (2.6) it follows that

$$[\mathbf{S},A]=0 \qquad (2.9)$$

and then, finally, from (2.7)

$$[\mathbf{N},A]=\mathbf{R}:[H,A]+[(\mathbf{P}\times\mathbf{S})/\mathbf{P}^2][H-Mc,A]. \qquad (2.10)$$

The commutator of A with Mc can, of course, be expressed in terms of $[H,A]$ but only as an infinite series since from (2.8)

$$Mc=H(I-\tfrac{1}{2}\mathbf{P}^2/H^2-\tfrac{1}{8}(\mathbf{P}^2/H^2)^2-\cdots). \qquad (2.11)$$

If the dynamics $[H,A]$ are given then the Lorentz commutators $[\mathbf{N},A]$ are uniquely determined by (2.10) and (2.11).

3. INTERPRETATION OF THE NONZERO LORENTZ COMMUTATOR

If the time-dependent variable $A(t)$ is not a Lorentz invariant, then what is its transformation rule? That a fundamental difficulty is involved in the very phrasing of the question is easily demonstrated. A transformation rule, inherently kinematical in nature and having no dependence on specific dynamical assumptions, can exist for $A(t)$ only if there exists an operator $B(t)$ such that

$$A'(t')=B(t). \qquad (3.1)$$

If t' is given, however, then t is inherently ambiguous. The value of t depends on spatial coordinates $\mathbf{x}'$ as well as t' while $A'(t')$ has no dependence on $\mathbf{x}'$ and, in fact, may not refer to points of space at all. Clearly (3.1) can hold only if A' and B are independent of their temporal variables, i.e., only if they are conserved.

At this point it is natural to conclude that only dynamical variables which depend on spatial coordinates as well as the time can play a fundamental role in the theory since such variables can have simple transformation rules. This is a position frequently held by advocates of local quantum field theory. An equally natural and less restrictive alternative exists, however, and it does not seem unreasonable that it should be considered seriously at this late date. Thus, instead of *imposing the restriction* that only field-like variables $A(\mathbf{x},t)$, can be the basic variables of the theory one may *invoke the generalization* of regarding dynamical variables on arbitrary space-like hyperplanes on an equal footing. Instead of asking what happens *at a given time* one may ask what happens *on a given hyperplane*.[7]

Many years ago Tomanaga and later Schwinger[8] pointed out the conceptual advantages of focussing one's attention on the physical situation on an arbitrary curvilinear space-like surface, an invariant geometrical construct. The present notion is not simply a special case of this older idea. Within the framework of special relativity it is only through the use of local field variables that one can define variables on *arbitrary* space-like surfaces, of which hyperplanes are a special case.[9] Quite independently of field-theoretic considerations, however, one can define dynamical variables on hyperplanes via the active interpretation of Poincaré transformations applied to variables defined at a given time. This latter approach has the further advantage of separating the purely kinematical problem of how a dynamical variable on a given hyperplane appears in different reference frames from the purely dynamical problem of how the dynamical variable changes from one hyperplane to another.[10]

From this point of view the intuitive notion that an internal dynamical variable is Lorentz-invariant can be realized in the requirement that the internal variables on a given hyperplane as measured from two inertial frames are the same, or

$$A'(\eta',\tau')=A(\eta,\tau), \qquad (3.2)$$

where the hyperplane parameters η_μ and τ define the hyperplane (in the corresponding reference frame) as the set of space-time points satisfying the linear equation[11]

$$\eta_\mu x^\mu=\tau;\quad \eta_\mu\eta^\mu=1. \qquad (3.3)$$

In fact (3.2) is not an arbitrary choice but is dictated by a consideration of the appearance in the two frames of the same measurement of the internal variable A on the fixed hyperplane.

The nonvanishing commutator of A with the Lorentz generators is now seen as a dynamical fact due to the dependence of A on the *orientation* of the hyperplane in space-time, i.e., dependence of A on η_μ. This dynamical dependence is related to the dependence of A on the *location* of the hyperplane, i.e., dependence of A on τ, which is expressed by the commutator of A with the Hamiltonian H.[12] It is precisely this relation between the dependence of A on the orientation and location of the hyperplane that is expressed by (2.10).

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[7] See Ref. 3, Sec. 4.

[8] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **1**, 27 (1946); J. Schwinger, Phys. Rev. **82**, 914 (1951).

[9] See Ref. 3, p. 1963.

[10] See Ref. 3, Sec. 6.

[11] See Ref. 3, p. 1965.

[12] Strictly speaking, for observables on noninstantaneous hyperplanes, a linear transformation of the conventional Poincaré generators must be performed to obtain the generators of *pure* reorientations or relocations (orthogonal timelike displacements) of the hyperplane. Nevertheless, the resulting *hyperplane generators* are very analogous to $\mathbf{N}$ and H, respectively, and the preceding statements are exact if the initial hyperplane is instantaneous. See Ref. 3 for a complete discussion of this point.