Number of Subtractions in Partial-Wave Dispersion Relations*

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Under the usual assumption of unitarity and analyticity for the partial-wave amplitude $f_l(s)$, it is proved that the dispersion relation for $f_l(s)$ requires no more than one subtraction for any angular momentum l, provided that $|f_l(s)|\leq \exp[C(\ln|s|)^{2-\epsilon}]$, $\epsilon>0$, holds for $|s|\to\infty$, and that the number of times that the sign of the discontinuity $\mathrm{Im}\,f_l(s+i0)$ changes in the interval $(s,0)$ does not increase more rapidly than $C'(\ln|s|)^{1-\epsilon}$ as $s\to-\infty$ along the negative real axis.

I. INTRODUCTION

THE partial-wave dispersion relation is one of the most frequently used tools in the dynamical study of strongly interacting particles. However, its theoretical foundation has been in a rather unsatisfactory state. First of all, there is still no rigorous derivation based on the axiomatic field theory of the analyticity domain, usually assumed for partial-wave amplitudes, although such a domain has been obtained in perturbation theory in some cases.[1] This being the case, it is perhaps not surprising that little attention has been paid until recently to the question of the number of subtractions necessary for writing down a meaningful dispersion relation. The first discussion of this problem was given in our preliminary paper under some restrictive assumptions.[2] The same problem has since been discussed by Jin and Kang,[3] Woolcock,[4] and by Contogouris and Martin[5] under various assumptions. The purpose of this paper is to present a detailed treatment of this problem under more general assumptions than those of Ref. 2.

For simplicity we shall restrict our consideration to the elastic scattering of spinless particles of equal mass. We denote by s the square of the total energy of the incident particles in the center-of-mass system. We assume that the partial-wave amplitude $f_l(s)$ has the following properties:

(a) It is regular in the cut s plane with two cuts $(-\infty,s_0)$ and $(4\mu^2,+\infty)$, real in the interval $(s_0,4\mu^2)$, and continuous on the cuts.[6] It has no essential singularity at any finite point on the cuts.
(b) It has the threshold behavior $f_l(s)\equiv(s-4\mu^2)^l F_l(s)$, where $F_l(s)$ has a finite limit as $s\to 4\mu^2$.[7]

(c) On the right-hand cut, $f_l(s)$ satisfies the unitarity condition

$$0\leq[(s-4\mu^2)/s]|f_l(s)|^2\leq[(s-4\mu^2)/s]^{1/2}\,\mathrm{Im}f_l(s)\leq 1.$$

To complete the usual set of assumptions we have to add some restriction on the asymptotic behavior of $f_l(s)$ for large s. A quick survey of these assumptions indicate, however, that they are probably not sufficient for determining the number of subtractions. At the same time it seems reasonable to guess that the necessary additional information will be related to the number of zeros of $f_l(s)$ or the number of times (denoted by ν) that the discontinuity $\mathrm{Im}f_l(s+i0)$ changes its sign on the left-hand cut. In fact it was found[2] that if ν is finite, the dispersion relation for $f_l(s)$ requires no more than one subtraction for any angular momentum l. In this paper we are mainly interested in finding out the extent to which the assumption of finite ν can be relaxed without losing the above result.

As was noted in Ref. 2, there seems to be a strong correlation between the behavior of the discontinuity $\mathrm{Im}f_l(s+i0)$ on the left-hand cut and the restriction on the asymptotic behavior of $f_l(s)$ required for limiting the number of subtractions. For instance, when ν is finite, it is sufficient to impose the very weak assumption that $f_l(s)$ satisfies

$$|f_l(s)|<\exp(C|s|^{1-\delta}),\quad \delta>0,$$

as $|s|\to\infty$. If we want to allow for $\mathrm{Im}f_l(s+i0)$ the possibility of infinite ν, on the other hand, we will have to make a much stronger assumption on the asymptotic behavior. We have not been able to determine how these properties are related to each other in general; however, we have at least found that the problem of subtraction can be given a definite answer under these assumptions:

(d) For sufficiently large s, $f_l(s)$ satisfies

$$|f_l(s)|<\exp[C(\ln|s|)^{2-\epsilon}]\quad\text{for any}\quad\epsilon>0.$$

(e) The number of times that the discontinuity $\mathrm{Im}f_l(s+i0)$ changes its sign in the interval $(s,0)$ does not exceed $C'(\ln|s|)^{1-\epsilon}$ as s goes to $-\infty$ along the negative real s axis.

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[1] J. G. Taylor, Nuovo Cimento **22**, 92 (1961); N. Nakanishi, Phys. Rev. **126**, 1225 (1962).
[2] T. Kinoshita, Phys. Rev. Letters **16**, 869 (1966).
[3] Y. S. Jin and K. Krang, Phys. Rev. **152**, 1227 (1966).
[4] W. S. Woolcock, Phys. Rev. **153**, 1449 (1966).
[5] A. P. Contogouris and A. Martin, Nuovo Cimento (to be published).
[6] In general the discontinuity $\mathrm{Im}f_l(s+i0)$ will be a tempered distribution. Thus it will be necessary to regularize it over a small interval of values of s. We assume that this averaging is already done and $\mathrm{Im}f_l(s+i0)$ is continuous on the real s axis.
[7] G. Roepstorff and J. L. Uretsky, Phys. Rev. **152**, 1213 (1966); Y. S. Jin, *ibid.* (to be published). For a critical comment of these papers see A. Martin, Nuovo Cimento (to be published).

We note that we have not been able to weaken the assumption (e) even if we replaced the assumption (d) by the stronger condition

$$|f_l(s)|<|s|^N \quad \text{for} \quad |s|\to\infty\,,$$

where N is a fixed positive number.

This is as far as we have been able to go. We hesitate to consider our result as very satisfactory, not only because the assumption (e) looks rather artificial and is probably too strong, but also because everything was obtained by mathematical manipulation alone and we have gained very little insight into the physical significance of the oscillation of $\mathrm{Im} f_l(s+i0)$ on the left-hand cut.

Although the case of finite ν was already discussed in Ref. 2, we shall repeat the argument in Sec. II in order to improve some details. The case where the assumptions (d) and (e) are made is treated in Sec. III. Lemmas on Herglotz functions used in this paper are discussed in Appendix A. An inequality on the minimum modulus of entire functions of order 0 is derived in Appendix B.

II. FINITE ν CASE

As was discussed in Ref. 2, our approach to the problem of subtraction is to relate $f_l(s)$ to a Herglotz function using the technique introduced by Jin and Martin.[8] Suppose $f_l(s)$ has p real zeros $r_1, r_2, \cdots, r_p$ in the interval $(s_0, 4\mu^2)$ and $2q$ complex zeros $c_1, c_1^*, \cdots, c_q, c_q^*$ in the cut s plane. Suppose also that $\mathrm{Im} f_l(s+i0)$ changes its sign at $s_1, s_2, \cdots, s_\nu$ on the left-hand cut, where ν is assumed to be finite in this section. Our first step is to construct the function

$$g(s)=\left[\prod_{i=1}^{p}(s-r_i)\prod_{j=1}^{q}(s-c_j)(s-c_j^*)\right]^{-1}F_l(s)\,, \quad (1)$$

which has no zero in the cut s plane. Unitarity condition (c) allows us to choose the phase of $g(s+i0)$ to be between 0 and π in the interval $(4\mu^2, +\infty)$. Then the phase of $g(s)$ is either 0 or π in the interval $(s_0, 4\mu^2)$, according to the assumption (b). Since $\mathrm{Im} f_l(s+i0)$ changes its sign at $s_1, s_2, \cdots, s_\nu$ on the left-hand cut, $\mathrm{Im} g(s+i0)$ also changes its sign at the same set of points. Without loss of generality we may assume that $s_\nu<s_{\nu-1}<\cdots<s_1<s_0$.

Next we construct from $g(s)$ a new function whose imaginary part is non-negative on the real axis. It is easy to see that the function

$$G(s)=(s-s_0)^{\alpha_0}\prod_{k=1}^{\nu}(s-s_k)g(s) \quad (2)$$

has such a property if we choose $\alpha_0=0$ or 1 according to whether $\mathrm{Im} g(s+i0)$ is greater or less than zero in the interval (s_1,s_0). In general this function is not a Herglotz function because positiveness of the imaginary part on the real axis does not necessarily imply positiveness of the imaginary part in the entire upper half of the s plane. As is shown in Appendix A, however, the former implies the latter in the particular case where the phase of the function is confined to the interval $(0,\pi)$ everywhere on the real axis. This leads us to the following construction of Herglotz functions.

To avoid unnecessary complication we shall assume that the change of phase of $g(s)$ does not exceed π when we go around the small semicircle $s-s_k=\epsilon e^{i\theta}$ $(\theta=0\to\pi$, $\epsilon>0)$ at each s_k $(k=0, 1, \cdots, \nu)$.[9] Then the function $h(s)$, defined by

$$h(s)=(s-s_0)^\alpha\prod_{k=1}^{\nu}(s-s_k)^{\eta_k}g(s)\,, \quad (3)$$

where $\alpha=1, 0$, or -1 according as $\arg g(s+i0)$ is in $(-\pi,0)$, $(0,\pi)$, or $(\pi,2\pi)$ for $s_1<s<s_0$, and $\eta_k=1$ or -1 [1 if the slope of the function $G(s)$ at s_k is positive, -1 if negative], has the property that the phase of $h(s+i0)$ lies between 0 and π on the entire real axis. According to Lemma 2 of Appendix A, $\mathrm{Im} h(s)$ is therefore positive everywhere in the upper half of the s plane. Thus $h(s)$ defined by (3) is a Herglotz function.

Once one has succeeded in constructing a Herglotz function, one can take advantage of its well-known mathematical properties. The most important is the property that $h(s)$ can be represented in the form[10]

$$h(s)=A+Bs+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+xs)\,\mathrm{Im}h(x)dx}{(1+x^2)(x-s)}\,, \quad (4)$$

where A and B are real constants, $B\geq 0$, $\mathrm{Im}h(x)\geq 0$, and the integral

$$\int_{-\infty}^{\infty}\frac{\mathrm{Im}h(x)dx}{1+x^2} \quad (5)$$

is convergent. From this representation and the fact that $-1/h(s)$ is also Herglotz it follows that there exist positive constants C_1 and C_2 such that

$$C_1|s|^{-1}\leq|h(s)|\leq C_2|s| \quad (6)$$

holds in any complex direction as $|s|\to\infty$. Of course such precise information cannot be expected about the behavior of $h(s)$ along the real axis. However, it has

[8] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[9] It is easy to remove this assumption insofar as the phase change is finite at any s_k. Then the value of η_k is not restricted to 1 and -1.

[10] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments, Mathematical Surveys, No. 1* (American Mathematical Society, New York, 1943), p. 23. For more recent literature, see N. Aronszain and W. F. Donoghue, Jr., J. Anal. Math. (Israel) **5**, 321 (1956–57); **7**, 113 (1964).

been shown by Martin[11] that for any positive number C larger than C_2 one can find an infinite sequence $\{s_i\}$, $s_i \to +\infty$, such that

$$|h(s_i)| \leq C|s_i|. \tag{7}$$

In the same manner we can find a sequence $\{s_i\}$, $s_i \to -\infty$, along the negative real axis for which (7) holds. Since $-1/h(s)$ is Herglotz, we can also find a sequence $\{s_i\}$, $s_i \to +\infty$ or $-\infty$, such that

$$C'|s_i|^{-1} \leq |h(s_i)|. \tag{8}$$

Applying (8) to the Herglotz function (3) constructed from $f_l(s)$, we obtain

$$C|s_i|^n \leq |f_l(s_i)| \tag{9}$$

for sufficiently large s_i in the sequence $\{s_i\}$, where

$$n = l + p + 2q - \alpha - 1 - \Delta,$$

and $\Delta = \sum_{k=1}^{\nu} \eta_k$ is the number of points s_k with $\eta_k = +1$ *minus* the number of points s_k with $\eta_k = -1$. α is defined in (3).

Now, because of unitarity, $f_l(s)$ is bounded by a constant as $s \to +\infty$ on the right-hand cut. Combining this with (9) we obtain the restriction

$$n \leq 0, \quad \text{i.e.,} \quad \Delta \geq l + p + 2q - \alpha - 1. \tag{10}$$

This is a generalization of the result of Jin and Martin[8] which was obtained under the more restrictive assumption that $F_l(s)$ has no zero and all η_k are equal to 1. It shows that the total number of zeros of $f_l(s)$ cannot exceed an upper limit set by Δ and hence by ν.[12] Or, conversely, given the total number of zeros of $f_l(s)$, the left-hand-cut discontinuity must change its sign at least as many times as is given by (10).

The crucial point in our consideration is that the inequality (10), which is derived from the behavior of $f_l(s)$ in the limit $s \to +\infty$, can also be used to determine the behavior of $f_l(s)$ in the limit $s \to -\infty$. In fact it is easily seen from (7) and (10) that

$$|f_l(s_i)| < C|s_i|^2 \tag{11}$$

holds for sufficiently large $|s_i|$ in the sequence $\{s_i\}$, $s_i \to -\infty$. As far as the behavior of $\mathrm{Im} f_l(s+i0)$ on the left-hand cut is concerned, we obtain a more precise information from (5) and (10):

$$\int_{-\infty}^{-C} |s^{-3}\, \mathrm{Im} f_l(s)|\, ds < \infty. \tag{12}$$

It is also seen from (6) that $f_l(s)$ satisfies

$$|f_l(s)| \leq C|s|^2 \tag{13}$$

for large $|s|$ in any complex direction. Altogether we can therefore conclude that we need at most three subtractions in writing down a dispersion relation for $f_l(s)$. Actually this can be easily improved by a closer examination of the constants A and B in the representation (4).

For this purpose we note that $f_l(s)$ satisfies the inequality

$$|f_l(s)| \leq C\left|\exp\left\{\frac{(\ln s)^2}{i(\pi-\delta)}\right\}\right| \tag{14}$$

in the domain defined by $0 \leq \arg s \leq \pi - \delta$ (except for a small neighborhood of the origin), where $\delta (>0)$ can be chosen arbitrarily small. This can be proved easily by applying the Phragmén-Lindelöf theorem[13] to this domain. It follows from (14) that, as $|s| \to \infty$ along the ray $s = |s|e^{i\theta}$, for $0 < \theta < \pi$, $f_l(s)$ is bounded by

$$C|s|^{2\theta/(\pi-\delta)}. \tag{15}$$

Thus the power of $|s|$ in (15) can be made as small as we wish by choosing a small enough θ.

The asymptotic behavior of $f_l(s)$ may now be discussed case by case for values of n equal to 0, -1, or ≤ -2.

Case $n=0$. In this case $h(s) \sim f_l(s)/s$ for large $|s|$. From (15) we see that we can choose a nonvanishing θ such that $h(s)$ is bounded by $s^{-1+\epsilon}$, $0 < \epsilon < 1$, along this ray. Taking the limit $|s| \to \infty$ in (4) while keeping θ fixed, we find that $B = 0$ and

$$\int_{-\infty}^{\infty} x\, \mathrm{Im} h(x)(1+x^2)^{-1} dx/\pi$$

is convergent and is equal to A. This means that the dispersion relation for $h(s)$ requires no subtraction. Thus $f_l(s)$ satisfies a dispersion relation with at most one subtraction. Furthermore, since $f_l(s)$ does not vanish in the limit $|s| \to \infty$, as is seen from (6), we have to make exactly one subtraction in this case.

[11] This is an improvement of the result given in Ref. 8. It can be proved as follows: Let $f(s)$ be $h(s)$ minus the integral of (4) on the *left-hand cut only*. Then $f(s)$ is regular in the s plane with the right-hand cut only, and is bounded by $C|s|$ in any complex direction as well as along the negative real axis, as is seen from (4), (5), and (6). Suppose now that along the right-hand cut $|f(s)|$ is never smaller than $C'|s|$, $C' > C$, for sufficiently large s. Then, applying the Phragmén-Lindelöf theorem (Ref. 13) to the function $1/f(s)$, which is regular in the s plane minus the positive real axis and a finite circle centered at the origin, we find that $|f(s)|$ must be larger than $C'|s|$ in the complex direction too. This is a contradiction. Thus, for any $C' > C$, there must be a sequence $\{s_i\}$, $s_i \to +\infty$, along the positive real axis such that $|f(s_i)| < C's_i$ holds for this sequence. Since the integral on the left-hand cut divided by $|s|$ tends to zero as $s_i \to +\infty$, we obtain $|h(s_i)| < C's_i$ for the same sequence. Clearly, a similar result is obtained by interchanging the role of the right-hand and left-hand cuts. I should like to thank A. Martin for informing me of this unpublished result and for allowing me to describe it here.

[12] Δ is roughly equal to the number of times $\mathrm{Re} f_l(s)$ changes its sign when s takes the *discrete* values $s_1, s_2, \cdots, s_\nu$ in this order. For instance, if $\mathrm{Re} f_l(s)$ is of one sign everywhere on the left-hand cut, η_k takes the values $+1$ and -1 alternately as k takes the values $1, 2, \cdots, \nu$ successively. Thus the values that $\Delta = \sum_{k=1}^{\nu} \eta_k$ can take is limited to $+1$, 0, and -1. Clearly, Δ is also related to the number of times that $f_l(s+i0)$ goes around the origin of the complex f_l plane when s goes to $-\infty$ along the negative real s axis.

[13] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954), p. 3.

Case n=−1. In this case $h(s)\sim f_l(s)$ for large $|s|$. Thus $h(s)$ must be bounded by s^ϵ, $0<\epsilon<1$, for some nonvanishing θ. By the same argument as above we find that $B=0$. The dispersion relation for $f_l(s)$ requires at most one subtraction.

Case n≤−2. As is seen from (5), the integral

$$\int_{-\infty}^{\infty}\frac{|x\,\mathrm{Im}f_l(x)|dx}{1+x^2}$$

is convergent in this case. We note further that $|f_l(s)|\leq C|s|^{2+n}$ for $|s|\to\infty$. Thus the dispersion relation for $f_l(s)$ does not require any subtraction for $n\leq-3$. When $n=-2$, there is likewise no subtraction needed, except possibly when $l=0$.

In summary we can therefore say that when ν is finite, the dispersion relation for $f_l(s)$ requires no more than one subtraction under any circumstance. Of course, when $l\geq1$, we can avoid introduction of an arbitrary constant of subtraction by choosing the subtraction point at the threshold $s=4\mu^2$. Thus the only place where an arbitrary constant may be introduced is in the s-wave amplitude.

III. CASE $\nu=\infty$

We shall now discuss the case where $\mathrm{Im}f_l(s)$ changes its sign infinitely many times on the left-hand cut. Since the relations like (9) and (10) are no longer well defined, the consideration of Sec. II must be modified or generalized in several ways. First of all, we have to make sure that the procedure by which the Herglotz function is constructed from $f_l(s)$ can be generalized to infinite ν. Thus we must examine the convergence property of the infinite products such as

$$\prod_{k=1}^{\infty}[1-(s/s_k)] \tag{16}$$

and

$$\prod_{j=1}^{\infty}[1-(s/c_j)][1-(s/c_j^*)]. \tag{17}$$

From the assumptions (a), (c), and (d) alone it seems to be impossible to find any reason why these products should converge. If they are divergent, we have to include convergence factors in defining these products. However, since the problem of subtraction itself is likely to become meaningless in such a general case, it would be reasonable for us to assume that the products (16) and (17) are in fact convergent. This assumption means that the entire functions defined by (16) and (17) should not grow faster than $\exp(|s|^{1-\epsilon})$, $\epsilon>0$, as $|s|\to\infty$.

It turns out, however, that even this assumption is still too general and we have to make much stronger assumptions if we want to derive a result similar to that of the finite ν case. This arises from the fact that, unlike polynomials, infinite products grow at different rates in the directions $s\to+\infty$ and $s\to-\infty$. Thus, if we want to make use of relationships similar to (11), (12), and (13), we have to impose some restrictions on the growth rates in different directions.

These considerations have led us to the investigation of the assumption of the form

$$\left|\prod_{k=1}^{\infty}[1-(s/s_k)]\right|\leq\exp[C'(\ln|s|)^{2-\epsilon}] \quad \text{for } |s|\to\infty, \tag{18}$$

which characterizes the asymptotic behavior of the left-hand-cut discontinuity in a certain way and is apparently the weakest assumption satisfying the above criteria. As is seen from (B4), this is equivalent to the assumption (e). For simplicity we shall put

$$P(s)=(s-s_0)^\alpha\prod_{k=1}^{\infty}[1-(s/s_k)]^{\beta_{k,+}} \tag{19}$$

and

$$Q(s)=(s-4\mu^2)^l\prod_{k=1}^{\infty}[1-(s/s_k)]^{\beta_{k,-}} \times\prod_{i=1}^{p}(s-r_i)\prod_{j=1}^{\infty}[1-(s/c_j)][1-(s/c_j^*)], \tag{20}$$

where $\beta_{k,\pm}=\frac{1}{2}(1\pm\eta_k)$ and α, η_k are defined in the same way as in (3). To avoid unnecessary complications we have assumed that all s_k are negative and that $\alpha=0$ or 1. If $\alpha=-1$, we should divide both $P(s)$ and $Q(s)$ by $(s-s_0)^\alpha$.

Under the assumed convergence of the infinite products $P(s)$ and $Q(s)$, the construction of the Herglotz function $h(s)$ described in Sec. II can be carried out without difficulty in the case $\nu=\infty$. Thus we find that the product $f_l(s)P(s)Q^{-1}(s)$ is a Herglotz function and satisfies the inequality

$$C_1|s|^{-1}\leq|f_l(s)P(s)Q^{-1}(s)|\leq C_2|s| \tag{21}$$

for sufficiently large $|s|$ in any complex direction. From (18), (21), and assumption (d) we see that the function $Q(s)$ has the property

$$|Q(s)|\leq\exp[C''(\ln|s|)^{2-\epsilon}] \tag{22}$$

as $|s|\to\infty$ in any complex direction. This bound can be easily extended to the real axis making use of the Phragmén-Lindelöf theorem.[13]

Along the real axis we can find, as in (8), a sequence $\{s_i\}$, $s_i\to+\infty$, such that

$$Cs_i^{-1}\leq|f_l(s_i)P(s_i)Q^{-1}(s_i)|. \tag{23}$$

Combining this with the requirement of unitarity (c) we obtain

$$|Q(s_i)|\leq C'|s_iP(s_i)| \tag{24}$$

for the same sequence $\{s_i\}$. However, for our purpose we need a relation stronger than (24) which holds for all large positive s. This can be derived in the following manner.

We first note that, according to the assumptions (a), (c), (d), and the Phragmén-Lindelöf theorem, $f_l(s)$ satisfies the inequality

$$|f_l(s)|<C\left|\exp\left\{\frac{(\ln s)^3}{3\pi i}\right\}\right| \qquad (25)$$

for all $|s|\geq R$ in the upper half of the s plane, where C is a constant larger than 1 and R is a point of the sequence $\{s_i\}$ such that the assumption (d) holds for all $|s|\geq R$. It follows from (25) and a similar inequality in the lower half of the s plane that $f_l(s)$ is bounded by a constant in the small neighborhood of the semi-infinite interval $s\geq R$ defined by

$$|\arg s|\leq\gamma(\ln|s|)^{-2},\quad |s|\geq R, \qquad (26)$$

where γ is a positive constant. On the other hand, it is easily seen from the representation (4) that the inequality

$$C'|s^{-1}(\ln s)^{-2}|\leq|f_l(s)P(s)Q^{-1}(s)|\leq C''|s(\ln s)^2|$$

holds on the boundary curve of the domain (26).[14] From these results we find that the function $Q(s)/[s(\ln s)^2P(s)]$ is bounded by a constant on the boundary of the domain (26). In fact, since this function is regular in (26), the inequality

$$|Q(s)/[s(\ln s)^2P(s)]|<C \qquad (27)$$

must hold everywhere in (26). To prove this we have only to consider this function in the z plane, where z is related to s by

$$z=\exp[\pi(\ln s)^3/6\gamma],$$

and apply the Phragmén-Lindelöf theorem to the right half of the z plane.

What we would like to find out is the nature of restrictions which the "unitarity bound" (27) imposes on the asymptotic behavior of $f_l(s)$ in complex directions and along the negative real axis. For this purpose let us define

$$M(r)=\operatorname*{Max}_{|s|=r}|P(s)|. \qquad (28)$$

Then, as is proved in Appendix B, the assumption (18) leads us to the inequality

$$|P(s)|\geq M(r)\exp[-\Delta(r)(\ln r)^{1-\epsilon}],\quad \epsilon>0, \qquad (29)$$

on the circle $|s|=r$, outside a set of circles the sum of whose radii can be made arbitrarily small. $\Delta(r)$ is a function of r which grows to infinity as slowly as we wish. Note that, in the case of $P(s)$, all these excluded points are located in the small neighborhood of points $\{s_k\}$ on the negative real axis. Thus, if we define the domain D by

$$-\pi+\sigma<\arg s<\pi-\sigma,\quad |s|>R, \qquad (30)$$

where σ is a small positive number, then we can choose, according to (29), a small fixed number $\delta_P(>0)$ such that

$$|P(s)|\geq r^{-\delta_P}P(r),\quad r=|s|, \qquad (31)$$

holds in D. Here we have taken account of the fact that the maximum of $|P(s)|$ for fixed $|s|$ occurs on the positive real axis. In the case of $Q(s)$ we do not know where on the circle $|s|=r$ the maximum value is taken. Thus we can only deduce from the property (22) that there exists a small $\delta_Q(>0)$ such that

$$|Q(s)|\leq r^{\delta_Q}Q(r),\quad r=|s|,$$

holds except when $s=r$ happens to lie within one of the excluded circles along the positive real axis. However, since the formula (29) may be applied equally well to a circle whose center is not at the origin, we can always find for any given complex s a finite positive number a such that $|s+a|-a$ does not fall in any of the excluded circles. For such an a we have

$$|Q(s)|\leq|s+a|^{\delta_Q}Q(|s+a|-a). \qquad (32)$$

Now, if we define the function $\varphi(s)$ by

$$\varphi(s)=Q(s)P^{-1}(s)s^{-1-\delta_P-\delta_Q}(\ln s)^{-2}, \qquad (33)$$

which is regular in the domain D defined by (30), it is bounded in D by the ratio $|P(|s+a|-a)/P(|s|)|$ as is seen from (27), (31), and (32). Since this ratio itself is less than 1, $P(s)$ being monotonically increasing along the positive real axis, $\varphi(s)$ is bounded by a constant everywhere in D. It follows that

$$|P(s)/Q(s)|>\text{const}|s^{-1-\delta_P-\delta_Q}(\ln s)^{-2}| \qquad (34)$$

holds in D. Combining this with (21) we find that $f_l(s)$ satisfies

$$|f_l(s)|<C|s^{2+\delta_P+\delta_Q}(\ln s)^2| \qquad (35)$$

in the domain D.

Of course the inequality (35) cannot be used on the negative real axis which is outside of the domain D. However, as is seen from (5), we have on the negative real axis the relation

$$\int_{-\infty}^{-R}\frac{|P(x)\operatorname{Im}f_l(x)|dx}{|Q(x)|(1+x^2)}<\infty. \qquad (36)$$

In order to estimate the magnitude of $|P(x)/Q(x)|$ on the real axis, we note that since $P(s)$ and $Q(s)$ satisfy (31) and (32), the lower bound (34) can also be used on the negative real axis if we avoid the small neighborhood of the points s_k where $P(s)$ vanishes. If we disregard this exception and use the inequality (34) everywhere in $(-\infty,-R)$, then we overestimate the contribution from the neighborhood of s_k. However, since $\operatorname{Im}f_l(x)$ vanishes at $x=s_k$ and since the set $\{s_k\}$ is very

[14] We may be able to improve the factor $(\ln s)^2$ in this formula by more careful estimates. However, such an improvement is not necessary for our purpose.

sparsely distributed according to assumption (e), the overestimation in the neighborhood of the set $\{s_k\}$ can be bounded by a finite number. Thus we obtain from (34) and (36) the inequality

$$\int_{-\infty}^{-R} \frac{|\mathrm{Im} f_l(x)|dx}{|x^{3+\delta_P+\delta_Q}(\ln x)^2|} < \infty . \tag{37}$$

Since $\delta_P+\delta_Q$ can be chosen as small as we wish, we see from (35) and (37) that we have to make at most three subtractions in writing down a dispersion relation for $f_l(s)$. Using the technique similar to the one employed following Eq. (14), it is now easy to show that the dispersion relation for $f_l(s)$ does not actually need more than one subtraction.

We note that, if $f_l(s)$ grows more rapidly than is allowed by the assumptions (d) and (e), it is not possible to choose a finite $\delta_P+\delta_Q$ in the formula (35). Thus we cannot replace $2-\epsilon$ in assumptions (d) and (e) by any larger number insofar as our method is based on the inequality (29).

Finally, we should like to discuss the possibility of replacing the assumption (18) by a slightly different assumption:

$$|Q(s)| = O(\exp[C(\ln|s|)^{2-\epsilon}]) \quad \text{as} \quad |s| \to \infty . \tag{38}$$

We have seen already that (38) or (22) follows from (18) under assumption (d). But the reverse is not always true. Thus (38) offers us a somewhat different approach to our problem.

As is seen from (27), the function $P(s)$ must grow at least as rapidly as $Q(s)/[s(\ln s)^2]$ along the positive real s axis. This means that as an entire function $P(s)$ must grow like $\exp[C(\ln|s|)^{2-\epsilon}]$ or more rapidly as $|s| \to \infty$. In the first case we obviously obtain the same result as before. If it grows more rapidly, for instance, if

$$P(s) = O(\exp[C'|s|^{\rho}]) , \quad 0<\rho<\tfrac{1}{2} , \tag{39}$$

and if the zeros of $P(s)$ are distributed in some regular fashion along the negative real axis, then we have[15,16]

$$\ln P(s) \sim Cs^{\rho}/(\sin\pi\rho) \quad \text{as} \quad s \to +\infty , \tag{40}$$

and

$$\ln|P(s)| \sim C(\cot\pi\rho)|s|^{\rho} \quad \text{as} \quad s \to -\infty , \tag{41}$$

except in the neighborhood of $s_1, s_2, \cdots$, where $P(s)$ vanishes. It is now seen from (21) and (36) that $f_l(s)$ goes to zero very rapidly as $|s| \to \infty$ in all directions. Thus the dispersion relation for $f_l(s)$ requires no subtraction in this case. However, if $P(s)$ grows even more rapidly as $s \to +\infty$, for instance if $\frac{1}{2} \le \rho < 1$ in (39), it has to go to zero very rapidly in the direction $s \to -\infty$, as is seen from (41). We cannot say anything about the number of subtractions in such a case.

[15] This assumption on the distribution of zeros is not essential for our consideration. See Ref. 13, pp. 40 and 43.

[16] See Ref. 13, p. 56.

Thus the assumption (38) enables us to treat the problem of subtraction in a more general manner than the assumption (18). Unfortunately, the significance of this improvement is not very clear because the physical implication of the assumption (38) [which is mainly concerned with the distribution of zeros of $\mathrm{Im} f_l(s+i0)$ with $\eta_k=-1$ on the left-hand cut] is as obscure as that of the assumption (18), which is concerned with the distribution of zeros of $\mathrm{Im} f_l(s+i0)$ with $\eta_k=+1$. Nevertheless, it may be useful to point out that (38) also implies that the number of complex zeros of $f_l(s)$ within the circle of radius $|s|$ should not increase more rapidly than $(\ln|s|)^{1-\epsilon}$ as $|s| \to \infty$. If we could relate this restriction to assumption (d) on the asymptotic behavior, and if we could properly take account of the fact that the parts of $P(s)$ and $Q(s)$ associated with the oscillation of $\mathrm{Im} f_l(s+i0)$ on the left-hand cut tend to cancel each other, then we might be able to weaken assumption (e) considerably.

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APPENDIX A. LEMMAS ON HERGLOTZ FUNCTIONS

We discuss here some lemmas on Herglotz functions adapted to the specific need of our problem. There is nothing new about them.[17]

Lemma 1. Let $f(z)$ be a real analytic function defined in the cut z plane with two nonoverlapping cuts $(-\infty, L)$ and $(R, +\infty)$ on the real axis. Suppose $f(z)$ has these properties: (i) It has neither zero nor pole within the cut plane; (ii) it is continuous and does not vanish or diverge on the cuts; (iii) $\mathrm{Im} f(z+i0)$ is positive on the cuts except at the end points L and R, where it vanishes; (iv) It is bounded by $\exp(C|z|^{1-\epsilon})$, $\epsilon>0$, as $|z| \to \infty$. Then $\mathrm{Im} f(z)>0$ everywhere in the upper half of the z plane.

Proof. Let $g(z)=\ln f(z)$. For definiteness we shall choose the branch of logarithm such that $\mathrm{Im} g(z)=0$ or π in the gap (L,R). Then $g(z)$ is regular in $\mathrm{Im} z\neq 0$ by (i), and satisfies $0 \le \mathrm{Im} g(z+i0) \le \pi$ on the real axis by (ii) and (iii). Since $\mathrm{Im} g(z'+i0)$ is bounded for any real z', the function

$$G(z) \equiv g(z) - \frac{z}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im} g(z'+i0)dz'}{z'(z'-z)} , \tag{A1}$$

where $z=0$ is assumed to be a regular point, is well-defined and regular in the whole z plane, and has the

[17] See, for instance, K. Symanzik, J. Math. Phys. 1, 249 (1960), Appendix B.

property[18]

$$\mathrm{Re}G(z)<C|z|^{1-\epsilon}+C'\ln|z|-C'\ln|\mathrm{Im}z| \quad (\mathrm{A2})$$

according to (iv). Thus $\mathrm{Re}G(z)$ is bounded by $|z|^{1-\epsilon}$ in the z plane except possibly on the real axis. However, since $G(z)$ is actually regular at any real point x, we can obtain a better bound:

$$\mathrm{Re}G(x)=\frac{1}{2\pi}\oint d\theta\,\mathrm{Re}G(x+e^{i\theta}) \le C(|x|+1)^{1-\epsilon}+C'\ln(|x|+1)+C'', \quad (\mathrm{A3})$$

which is obtained with the help of Poisson's formula, and where C'' is a finite constant. Thus, $\mathrm{Re}G(z)$ is bounded by $|z|^{1-\epsilon}$ on the real axis, too. But an entire function with such a property must be a constant.[19] This means that $g(z)$ satisfies the dispersion relation with one subtraction:

$$g(z)=g(0)+\frac{z}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}g(z'+i0)dz'}{z'(z'-z)}. \quad (\mathrm{A4})$$

For $\mathrm{Im}z\neq 0$ we obtain from this the relation

$$\mathrm{Im}g(z)=\frac{\mathrm{Im}z}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}g(z'+i0)dz'}{(z'-\mathrm{Re}z)^2+(\mathrm{Im}z)^2}. \quad (\mathrm{A5})$$

From (A5) we can easily conclude the following: Since $\mathrm{Im}g(z+i0)>0$ on the cuts, $\mathrm{Im}g(z)>0$ everywhere in the upper half of the z plane; since $\mathrm{Im}g(z+i0)<\pi$ on the cuts, we obtain $\mathrm{Im}g(z)<\pi$ for any $\mathrm{Im}z>0$. These properties together show that $\mathrm{Im}f(z)>0$ everywhere in the upper half of the z plane. Thus $f(z)$ is a Herglotz function. Q. E. D.

It is obvious from this proof that assumptions (ii) and (iii) can be replaced by somewhat weaker ones. For instance, Lemma 1 is valid even if $\mathrm{Im}f(z+i0)$ vanishes at isolated points on the cuts insofar as $\mathrm{Re}f(z+i0)$ does not vanish at the same time, which is guaranteed by (ii). We may even relax (ii) and allow $f(z)$ to vanish or diverge at isolated points on the cuts if it does not violate the crucial inequality $0\le\mathrm{Im}g(z+i0)\le\pi$ anywhere on the real axis. Thus we are led to

Lemma 2. Let $f(z)$ be a real analytic function defined in the domain given in Lemma 1, having the following properties: (i) It has neither zero nor pole in this domain; (ii) it has no essential singularity at any finite point on the cuts; (iii) the phase of $f(z)$ is continuous except at some isolated points and

$$\psi_\pm(z)\equiv\lim_{\delta\to\pm 0}\arg f(z+\delta+i0)$$

satisfies $0\le\psi_\pm(z)\le\pi$ everywhere on the real axis; (iv) it is bounded by $\exp(C|z|^{1-\epsilon})$, $\epsilon>0$, as $|z|\to\infty$ except at the isolated points on the real axis [where $f(z)$ diverges] which are separated from each other by finite distances. Then $f(z)$ is a Herglotz function.

Proof. Under assumptions (i), (ii), and (iii), we can still construct an entire function $G(z)$ as in (A1). Although the inequality (A3) no longer holds for all large $|z|$, it holds at least for an increasing sequence of values of $|z|$ according to (iv). This is sufficient to show that the entire function $G(z)$ is a constant. The rest of the proof is the same as that of Lemma 1. Q. E. D.

APPENDIX B. PROOF OF INEQUALITY (29)

This inequality can be obtained by a slight modification of the similar inequality given on page 50 of Ref. 13. From Jensen's theorem expressed in the form[20]

$$\int_0^r t^{-1}n(t)dt\le\ln M(r), \quad (\mathrm{B1})$$

where $M(r)$ is defined by (28) and $n(r)$ is the number of zeros of $P(s)$ within the circle $|s|=r$, we obtain

$$n(r)\ln r=n(r)\int_r^{r^2}t^{-1}dt\le\int_r^{r^2}t^{-1}n(t)dt\le\ln M(r^2). \quad (\mathrm{B2})$$

Since $\ln M(r)=O((\ln r)^{2-\epsilon})$ according to (18) and (19), we have

$$n(r)=O((\ln r)^{1-\epsilon}). \quad (\mathrm{B3})$$

From this it follows that

$$r\int_r^\infty t^{-2}n(t)dt=O\left(r\int_r^\infty t^{-2}(\ln t)^{1-\epsilon}dt\right) =O\left(r^{1/2}(\ln r)^{1-\epsilon}\int_r^\infty t^{-3/2}dt\right) =O((\ln r)^{1-\epsilon}). \quad (\mathrm{B4})$$

Thus, according to the formula (3.5.11) of Ref. 13, we obtain

$$\ln|P(s)|\ge\ln M(r)-\Delta(r)(\ln r)^{1-\epsilon} \quad (\mathrm{B5})$$

outside a set of circles the sum of whose radii is at most δr, where δ can be chosen arbitrarily small. $\Delta(r)$ is a function of r which tends to ∞ arbitrarily slowly. This proves the inequality (29).

[18] I should like to thank A. Martin for supplying this proof.
[19] See Ref. 13, p. 3, Theorem 1.3.4.
[20] See Ref. 13, p. 2.