Solution of the Bethe-Salpeter Equation in the Inelastic Region*

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The Bethe-Salpeter equation is solved numerically for a ϕ^3 theory in the ladder approximation up to the second inelastic threshold. Below the first inelastic threshold, the results agree with previous work. Between the first and second inelastic thresholds, it is found that there are coupling strengths which cause unitarity ($\sigma_{\text{total}} \geq \sigma_{\text{elastic}}$) to be violated.

I. INTRODUCTION

In the past few years the N/D equations have been applied extensively to the study of the dynamics of strongly interacting particles. Among other advantages, one practical advantage of the N/D equations is that they can readily be solved numerically. On the other hand, the N/D equations suffer from several difficulties. One difficulty is that even for a completely renormalizable theory one needs cut-off parameters in order to get finite results. Moreover, some recent study[1] indicates that these equations yield unphysical results. As the calculations have become more refined, it has also become apparent that it is hard, in general, to include accurately certain forces which may be important. In particular, forces arising from three-particle intermediate states are usually left out. Recently Mandelstam[2] has given a set of N/D equations for three particles; but they are very complicated, and, in practice, the left-hand singularities are not well known.

An alternative way of including inelastic effects is the use of off-shell equations from field theory. As an example, the Bethe–Salpeter (BS) equation[3] in the ladder approximation includes some multiparticle processes. It is of some interest to see whether the solutions are physically reasonable.

Recently Schwartz and Zemach succeeded with the use of a variational technique in obtaining the positions of the bound states[4] and the values of the phase shifts in the elastic region[5] of the BS equation in the ladder approximation for a ϕ^3 theory. Their method, however, breaks down beyond the inelastic threshold. Recently we described a method[6] to reduce the BS equation to a more manageable form which is suitable for extending the BS calculation into the inelastic region. In this paper we present the numerical results for the phase shifts, obtained with the use of this method, for s and p waves up to the second inelastic threshold for a range of coupling constants and for two values of the mass of the exchanged particle.

Of course, it is not known how important other sets of graphs are; the ladder results may not be a very good approximation to the complete theory. Further investigation along this line should be undertaken. In fact, we find that in the inelastic region the unitarity inequality

$$\sigma_{\text{total}} \geq \sigma_{\text{elastic}}$$

can be violated if only simple ladder graphs are included. Preliminary results indicate that this can be easily cured by including other graphs.

In Sec. II we review for completeness the equations given previously and discuss their extension to the inelastic region in more detail. In Sec. III we present the results of the calculations, and we furthermore compare them with the approximation to the BS equation suggested by Blankenbecler and Sugar.[7]

II. NONSINGULAR FORM FOR THE BS EQUATION

The BS equation given formally by

$$T = V + VGT$$

describes relativistic two-particle scattering in terms of the T matrix, an interaction V, and the free two-particle Green's function G. We limit ourselves here to the scattering of two distinguishable spinless bosons of equal mass m. While V may include any number of irreducible interactions, we further restrict ourselves to the exchange of a single spinless boson of mass μ. While these restrictions exclude any physically interesting cases, they do present a reasonable starting point for solving more complicated problems. Since the total angular momentum is conserved, we may decompose T

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[1] R. F. Sawyer, Phys. Rev. **142**, 991 (1966).
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[4] C. Schwartz, Phys. Rev. **137**, B717 (1965).
[5] C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966).
[6] M. Levine, J. Tjon, and J. Wright, Phys. Rev. Letters **16**, 962 (1966).
[7] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966); to be called BBS.

and V into partial wave components T_l and V_l to obtain[8]

$$T_l(p,\omega;p',\omega';s) = V_l(p,\omega;p',\omega') - i\int_{-\infty}^{\infty} d\omega'' \int_0^{\infty} dp'' V_l(p,\omega;p'',\omega'') \times G(p'',\omega'';s) T_l(p'',\omega'';p',\omega';s), \quad (2.1)$$

where

$$V_l(p,\omega;p',\omega') = (2\lambda/\pi) Q_l\{[p^2+p'^2+\mu^2-(\omega-\omega')^2-i\epsilon]/2pp'\}.$$

The Green's function G, being the product of two free-particle propagators, is given by

$$G(p,\omega;s) = \{[p^2+m^2-i\epsilon-(\omega+\tfrac{1}{2}\sqrt{s})^2] \times [p^2+m^2-i\epsilon-(\omega-\tfrac{1}{2}\sqrt{s})^2]\}^{-1}.$$

The total center-of-mass energy is $(s)^{1/2}$, and $\hat{p}=[\frac{1}{4}s-m^2]^{1/2}$. In the c.m. system, p and ω are half of the relative momentum and energy, respectively. The phase shift is obtained from the fully on-shell T matrix by

$$t_l(s) = (2/\pi)^2 \hat{p}(\hat{p}^2+m^2)^{1/2} e^{i\delta} \sin\delta = T_l(\hat{p},0;\hat{p},0;s).$$

The solution of the BS equation is considerably complicated by the singularities of the kernel: the 4 poles of the Green's function at

$$\omega'' = \pm\{\tfrac{1}{2}\sqrt{s} \pm [p''^2+m^2]^{1/2} - i\epsilon\}$$

and the four branch points of the potential at

$$\omega'' = \omega \pm \{[(p \pm p'')^2+\mu^2]^{1/2} - i\epsilon\}.$$

The most serious singularity arises from the pinching of the integration contour by the poles $\pm\bar{\omega}$, with

$$\bar{\omega} = \tfrac{1}{2}(s)^{1/2} - (p''^2+m^2)^{1/2} + i\epsilon.$$

This happens when $p''=\hat{p}$. In analogy with the techniques of Kowalski[9] and Noyes,[10] because the potential also appears in the inhomogeneous term, we can introduce a zero into the potential term of the kernel at $p''=\hat{p}$ and $\omega''=0$. We do not attempt to remove all of the propagator singularities as do Graves-Morris[11] and Taylor[12] since the equations are soluble by ordinary numerical techniques without this further refinement and the accompanying complexity. We define $f_l(p,\omega;s)$ by factoring the fully on-shell behavior out of the half off-shell T-matrix

$$T_l(p,\omega;\hat{p},0;s) = f_l(p,\omega;s) t_l(s).$$

As a result we have

$$f_l(\hat{p},0;s) = 1.$$

Letting $V_l(s) = V_l(\hat{p},0;\hat{p},0)$, suppressing some s and all l dependence, we may use Eq. (2.1) to find t in terms of f:

$$t(s) = V(s)\left[1 + i\int_{-\infty}^{\infty} d\omega' \int_0^{\infty} dp' V(\hat{p},0;p',\omega') \times G(p',\omega') f(p',\omega')\right]^{-1}. \quad (2.2)$$

The resulting equation for f is

$$f(p,\omega) = \frac{V(p,\omega;\hat{p},0)}{V(s)} + i\int_{-\infty}^{\infty} d\omega' \int_0^{\infty} dp' \times \left\{\frac{V(p,\omega;\hat{p},0)V(p',\omega';\hat{p},0)}{V(s)} - V(p,\omega;p',\omega')\right\} \times G(p',\omega') f(p',\omega'). \quad (2.3)$$

It is desirable to rotate the contour of the ω' integration[13] so that it lies along the imaginary axis, well away from the singularities except at the origin. Even there, the potential term introduces a zero making the integration manageable. If we restrict ourselves to the elastic scattering region $(2m)^2 < s < (2m+\mu)^2$, we find that the only singularities in the first and third quadrants are the propagator poles, $\omega''=\pm\bar{\omega}$ for $p''=\hat{p}$. Because of the strong convergence for $|\omega'| \to \infty$, we may therefore take the ω' integration contour along the imaginary axis and add the residues at these poles. Letting $g(p)=f(p,\bar{\omega})$, and noticing that only $f(p,\omega)+f(p,-\omega)$ enters into $t(s)$, we find a set of coupled equations

$$f(p,i\omega) = \frac{V(p,i\omega;\hat{p},0)}{V(s)} - \int_0^{\infty} d\omega' \int_0^{\infty} dp' G(p',i\omega') f(p',i\omega') \left\{\frac{2V(p,i\omega;\hat{p},0)V(p',i\omega';\hat{p},0)}{V(s)} - V(p,i\omega;p',i\omega') - V(p,i\omega;p',-i\omega')\right\} + \frac{\pi}{2\sqrt{s}} \int_0^{\hat{p}} dp' \left\{\frac{2V(p,i\omega;\hat{p},0)V(\hat{p},0;p',\bar{\omega}(p'))}{V(s)} - V(p,i\omega;p',\bar{\omega}(p')) - V(p,i\omega;p',-\bar{\omega}(p'))\right\} \frac{g(p')}{\bar{\omega}(p')[(p')^2+m^2]^{1/2}} \quad (2.4)$$

[8] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962). The coupling constant λ is the same as in Ref. 5 and is related to Lee and Sawyer's by $g^2/(2\pi)^3 = 2\lambda/\pi$.
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[10] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).
[11] P. R. Graves-Morris, Phys. Rev. Letters **16**, 201 (1966).
[12] J. G. Taylor, Nuovo Cimento Suppl. **1**, 1002 (1963); A. Pagnamenta and J. G. Taylor, Phys. Rev. Letters **17**, 218 (1966).
[13] G. C. Wick, Phys. Rev. **96**, 1124 (1954); W. Kemmer and A. Salam, Proc. Roy. Soc. (London) **A230**, 266 (1955).

and

$$g(p)=\frac{V(p,\bar{\omega}(p);\hat{p},0)}{V(s)}-\int_0^\infty d\omega'\int_0^\infty dp'G(p',i\omega')f(p',i\omega')\left\{\frac{2V(p,\bar{\omega}(p);\hat{p},0)V(p',i\omega';\hat{p},0)}{V(s)}\right.$$
$$\left.-V(p,\bar{\omega}(p);p',i\omega')-V(p,\omega(p);p',-i\omega')\right\}+\frac{\pi}{2\sqrt{s}}\int_0^{\hat{p}}\left\{\frac{2V(p,\bar{\omega}(p);\hat{p},0)V(\hat{p},0;p',\bar{\omega}(p'))}{V(s)}\right.$$
$$\left.-V(p,\bar{\omega}(p);p',\bar{\omega}(p'))-V(p,\bar{\omega}(p);p',-\bar{\omega}(p'))\right\}\frac{g(p')}{\bar{\omega}(p')[(p')^2+m^2]^{1/2}}. \quad (2.5)$$

At $s=(2m+\mu)^2$, the first inelastic threshold, it becomes possible for the pole at $\omega'=\bar{\omega}$ to pinch a branch point of the potential. This pinching introduces a branch point in the scattering amplitude $t(s)$. In the region $(2m+\mu)^2<s<(2m+2\mu)^2$, this is the only additional singularity problem; since this potential branch point is only logarithmic, it does not destroy the Fredholm properties of the kernel and reasonable numerical care suffices. For $s\geq(2m+2\mu)^2$, an additional logarithmic branch point moves into the first quadrant. We have not carried our calculations above this second inelastic threshold.

Finally, the integral in Eq. (2.2) can also be done along the imaginary ω' axis, if the pole contribution is added. It is convenient to add and subtract the contribution of the box diagram to this integral so as to introduce a zero whenever two of the poles in the Green's function coincide:

$$t(s)=V(s)[1+t_1(s)+t_2(s)+b(s)]^{-1} \quad (2.6)$$

with

$$t_1(s)=-\int_{-\infty}^\infty d\omega'\int_0^\infty dp'V(\hat{p},0;p',i\omega')G(p',i\omega')\times\left[f(p',i\omega')-\frac{V(p',i\omega';\hat{p},0)}{V(s)}\right], \quad (2.7)$$

$$t_2(s)=\frac{\pi}{2\sqrt{s}}\int_0^{\hat{p}}dp'V(\hat{p},0;p',\bar{\omega}(p'))\bar{\omega}(p')^{-1}\times[(p')^2+m^2]^{-1/2}\left[g(p')-\frac{V(p',\bar{\omega}(p');\hat{p},0)}{V(s)}\right], \quad (2.8)$$

$$b(s)=+i\int_{-\infty}^\infty d\omega'\int_0^\infty dp'[V(\hat{p},0;p',\omega')]^2\times G(p',\omega')/V(s). \quad (2.9)$$

III. NUMERICAL RESULTS

The coupled equations (2.4) and (2.5) for f and g were solved numerically by transforming the integrations to a finite range, approximating the kernels and inhomogeneous terms by finite matrices and then solving the matrix equations on a computer. There is an integrable singularity at $p'=\hat{p}$, $\omega'=0$, which was eliminated by a change in variables. The elements of the matrices were obtained by averaging the value of the function at several points near each mesh point. When the logarithmic singularity of the potential lies within the range of integration, the kernel is replaced by a smoothed function obtained by averaging the logarithm analytically in the neighborhood of each mesh point assuming the other factors are constant. The number of mesh points in any single variable varied between 8 and 12. Typically, the kernel in Eq. (2.4) was approximated by a $10\times10\times10\times10$ array. This yields an accuracy of about 2% in the real part of the phase shifts and 5–10% in the imaginary part as estimated by varying the mesh size and point distribution. In general, the percentage error in the imaginary part increases as the magnitude of the imaginary part decreases.

The real part of the phase shift is shown in Figs. 1, 2, and 3 for $m=1$ and for various values of μ, l, and λ as a function of s. These results agree with those of Zemach and Schwartz in the elastic region for $\mu=1$.

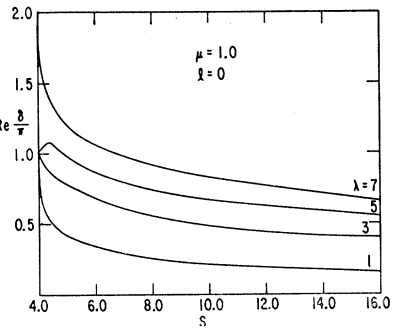

FIG. 1. Real part of the s-wave phase shift for exchange mass = 1 in the Bethe-Salpeter equation.

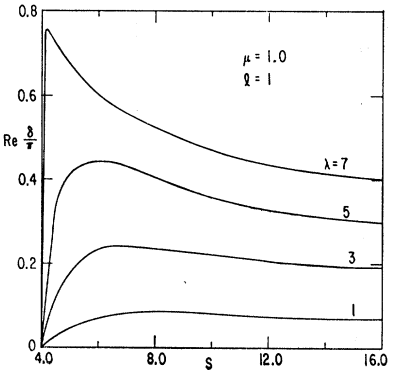

FIG. 2. Real part of the p-wave phase shift for exchange mass = 1 in the Bethe-Salpeter equation.

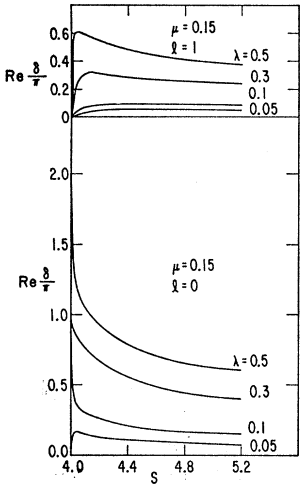

FIG. 3. Real part of the s- and p-wave phase shifts for exchange mass $=0.15$ in the Bethe–Salpeter equation.

The imaginary part of the phase shift is similarly shown in Figs. 4, 5, 6, and 7. It is to be noticed that these phase shifts can have a negative imaginary part for larger coupling constants. As a result the corresponding S-matrix elements become greater than one in absolute magnitude. Hence, the solutions of the BS equation in the ladder approximation exhibit some unphysical features.

A similar situation can be found in the well-known Lee model.[14] The ladder approximation to the V-θ scattering in this model amounts to omitting the V-particle self-energy corrections. In doing this we have found a similar behavior for the phase shifts of the V-θ scattering. The coupling constant for which the V-θ S-matrix element starts to become greater than 1 in magnitude lies below the critical value for the existence of the V-particle ghost state. Hence, the occurrence of this anomalous behavior in the V-θ sector in the ladder approximation is not at all related to the presence of ghost states in the N-θ sector. The effect of including certain self-energy diagrams in the BS equation in order to satisfy unitarity exactly up to the second

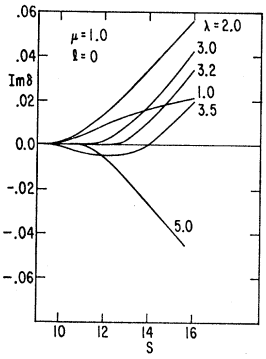

FIG. 4. Imaginary part of the s-wave phase shift for exchange mass $= 1$ in the Bethe–Salpeter equation.

[14] G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

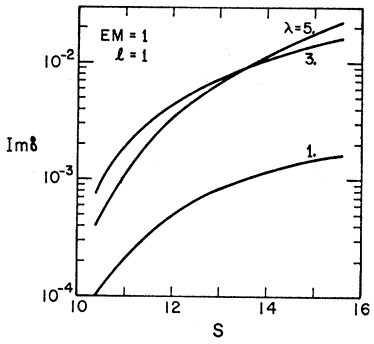

FIG. 5. Imaginary part of the p-wave shift for exchange mass $= 1$ in the Bethe-Salpeter equaton.

threshold is now being investigated. In this way we should at least be able to restore the unitarity inequality

$$\sigma_{\text{total}} \geq \sigma_{\text{elastic}}.$$

For comparison, we consider an interesting approx-

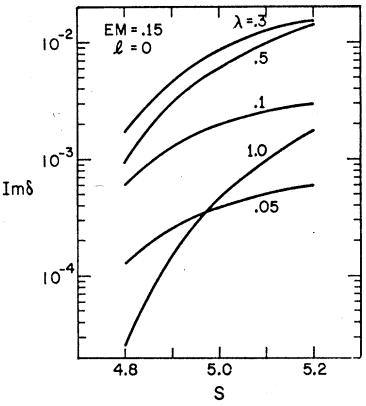

FIG. 6. Imaginary part of the s-wave phase shift for exchange mass $=0.15$ in the Bethe–Salpeter equation.

imation to the BS equation recently suggested by Blankenbecler and Sugar.[7] It has the important practical feature that the resulting equation is more amenable to numerical calculation than the BS equation. The approximation amounts to replacing the

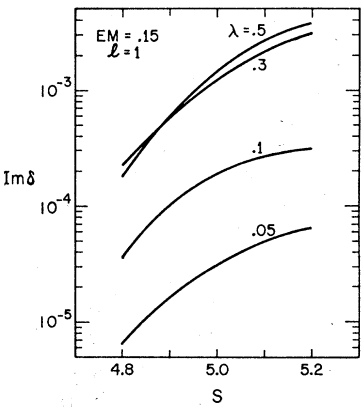

FIG. 7. Imaginary part of the p-wave phase shift for exchange mass $=0.15$ in the Bethe–Salpeter equation.

free Green's function in Eq. (2.1) by the expression

$$E_2=2\pi i\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\delta[p''^2+m^2-(\tfrac{1}{2}\sqrt{s}+\omega'')^2]\times\delta[p''^2+m^2-(\tfrac{1}{2}\sqrt{s}-\omega'')^2].$$

On substituting this into Eq. (2.1), the BS equation reduces to

$$T_l(p,0;p',0;s)=V_l(p,0;p',0)+\frac{\pi}{2}\int_0^{\infty}dp''\times[(p''^2+m^2)^{1/2}(p''^2+m^2-\tfrac{1}{4}s)]^{-1}\times V_l(p,0;p'',0)T_l(p'',0;p'0;s). \quad (3.1)$$

We have solved Eq. (3.1) and determined the corresponding phase shifts. These results are shown in Figs. 8, 9, and 10. These results approximate the Bethe-Salpeter results to within 10–30%. The agreement is worse for higher l and for smaller exchange mass. Of the two, the BBS phase shifts are consistently higher for equal λ. Finally, the positions of the first bound states in the two cases[15] are given as a function of λ in Fig. 11. An alternative is to compare the phase shifts for λ's giving the same position for the first bound state rather than for equal λ. It is then possible to bring the phase shifts into a somewhat better agreement with each other for low coupling strengths. However, the situation becomes markedly worse for higher coupling constants.

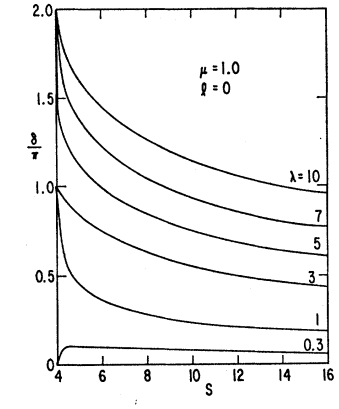

FIG. 8. s-wave phase shift for exchange mass = 1 in the Blankenbecler-Sugar approximation.

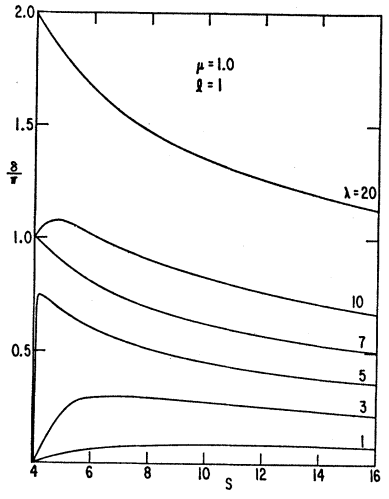

FIG. 9. p-wave shift for exchange mass = 1 in the Blankenbecler-Sugar approximation.

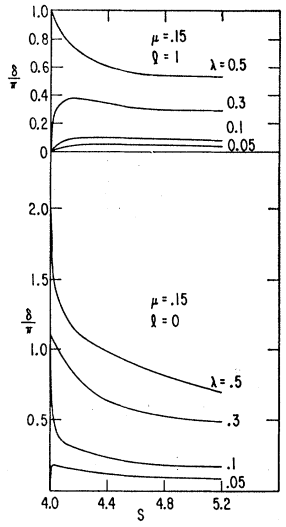

FIG. 10. s- and p-wave shifts for exchange mass = 0.15 in the Blankenbecler–Sugar approximation.

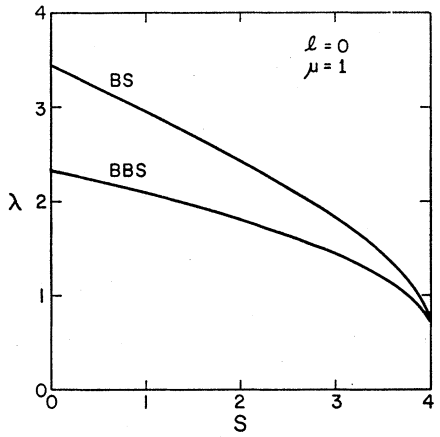

FIG. 11. Position of the first bound state for the Bethe–Salpeter equation and the Blankenbecler–Sugar approximation.

ACKNOWLEDGMENT

One of us (J.A.W.) wishes to thank Dr. M. Scadron for many helpful conversations during the early stages of this work.

[15] The bound-state positions for the BS equation were taken from Refs. 4 and 5.