tions (of lesser mass then ρN_j) which connect significantly to the ρN_j configuration in question. (Here recall the isospin factors discussed above.) For a given transition $\pi N_i \rightarrow \rho N_j$ that one with the greatest $l(\pi N_i)$ will dominate because of the centrifugal barrier. (Mathematically this is enforced by the threshold division discussed in Sec. IV.) Generally the highest spin (s_j+1) goes with the greatest l. It is this feature which accounts for the simple recurrence of resonances as s_j is increased by two units. An example was given in the Introduction.

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Dynamical Theory of Strong Interactions*

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The many-body quantum mechanics of a set of (self-consistent) composite particles is developed for use as the basis for a theory of strong interactions. The theory deals only with physical particles which may in general have an extended spatial structure. It is a bootstrap theory in which physical particles are examined in terms of superpositions of the physical multiparticle states of the theory; no auxillary quantities such as bare particles or fundamental local fields are introduced and no question of renormalization is encountered. Many-particle states are constructed which are (many-) three-momentum eigenstates and whose spatial integrity is assured via cluster-decomposition properties. The present theory is a dynamical theory in the sense that there is a Hamiltonian that respects the extended and composite structure of the particles and which, unlike S-matrix theory, allows a system to be studied during the course of its interactions. A drawback of the theory is that it is not manifestly Lorentz-covariant. The present paper deals with the theory in a simplified form in which heavy baryons interact with structureless mesons in the static limit of no baryon recoil. The self-consistent bootstrap dynamics is examined in the lowest order approximation, including some three-body effects. The conventional Born approximation to the scattering amplitude is recovered. The relation between the existence of particles and signs of forces is obtained. In particular the Cutkosky connection between attractive forces and group-theoretic structure is derived.

I. INTRODUCTION

IT is possible that the particles of the strong interactions are all composite, each formed from combinations of similar particles. The phenomenological evidence for this is the correlation which exists between the existence of a given particle and the attraction between particles whose total quantum numbers are the same as those of the single particle. This correlation was first pointed out by Chew[1] in his classic work on the structure of the pion-nucleon system. Chew showed that the existence of the nucleon and 3-3 resonance might be accounted for by a self-consistent mechanism in which the exchange of a nucleon would provide the force to cause the resonance and vice versa.

Later papers by Caruthers[2] extended this idea so that not only were the nucleon and 3-3 resonance spanned by the mechanism, but the entire system of baryon octet, baryon decuplet and many of the excited states of these objects could be understood. This and other work, especially by Cutkosky,[3] demonstrated a remarkable interplay between the group-theoretic structure of the strong interactions and the dynamical forces. The group-theoretic structure of the particle couplings seems to guarantee the attractive forces wherever they are needed to bind the particles.[4,5]

The S-matrix methods[6,7] which have been put forward to deal with this system seem to the author to not be ideally suited to formulating the necessary concepts of composite structure. Because the S matrix describes only the asymptotic states of a scattering system, ignoring the internal structure of the particles, the definition of a composite system is very indirect and far removed from physical intuition. One is forced to examine the effects of bound states on the analyticity of the S matrix in the oversimplified two-particle potential-scattering theory[8,9] and to extrapolate these

* Supported in part by U. S. Air Force Office of Scientific Research Grant No. 508-66 and AF 816-65.
[1] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[2] P. Carruthers, Phys. Rev. Letters **12**, 259 (1964).
[3] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).
[4] R. H. Capps, Phys. Rev. **132**, 2749 (1963).
[5] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[7] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).
[8] R. Omnès and M. Froissart, *Mandelstam Theory and Regge Poles* (W. A. Benjamin, Inc., New York, 1963).
[9] T. Regge, Nuovo Cimento **14**, 951 (1959).

effects to a very different kind of system.[10,11] In addition, to even formulate the dynamical equations, difficult and unsolved problems of the singularity structure of multibody amplitudes must be solved.[12,13]

The main technical advantage of the S matrix is that it deals only with finite and (we may hope) rapidly convergent quantities.

This paper is the first of a series in which a comprehensive theory of self-consistent bound states will be described. The theory uses only finite quantities but unlike the S-matrix theory, it is directly concerned with the internal structure of the particles. The self-consistent composite structure is not an artifical assumption and in many respects is the most natural assumption to make. We might add that all this is done without the aid of local fields. In Sec. II the meaning of a state containing two or more particles close together in space is discussed in the context of simple field theory and potential theory. This leads to the introduction of a basis of multiparticle-state vectors that are not orthonormal sets. The specific assumptions about the structure of the space of states are formulated in Sec. III. The analytical form of the assumption of composite structure is introduced as a statement of *linear dependence* among the noninteracting physical-particle states. This leads to a large number of equations that are the self-consistency requirements of the theory.

The present paper introduces the methods in terms of a preliminary model in which there is a multiplet of heavy particles—baryons—which interact with structureless mesons in the static limit of no recoil. The static model is introduced in Sec. IV and used in the remainder of the paper. Formal solutions to some self-consistency equations are proposed. Dynamical considerations are taken up in Sec. V. Self-consistency requirements are derived for the matrix element of the Hamiltonian and a solution is found.

In Sec. VI dynamical equations are derived, for the structures, masses and interactions of the particles. The equations are investigated in an approximation and the conventional Born-approximation pole in the scattering amplitude is isolated. The existence of particles and the sign of simple force diagrams of other methods are related. The equations have interesting algebraic properties and are similar to equations used by Cutkosky[3] to explain the relation between groups and self-consistent binding.

Section VII is summary and conclusions. Because this paper is largely introductory, no great effort is made to keep the theory logically independent of simple-field-theory and potential-theory ideas, if these ideas aid in the physical understanding. In later papers we shall be more careful in this respect.

II. SOME BASIC CONCEPTS

In order to understand what is meant by a system being composed of two or more other systems it is necessary to have a definite idea of what is meant by having two or more systems present simultaneously. If the systems (particles) are very far apart the meaning is unambiguous. Any measurement performed in the region in which particle 1 is located must agree with the results of a similar experiment performed on a state containing particle 1 but no other particle. Similar requirements on the measurements performed in the region of particle 2 apply. In any other region the result of an experiment must agree with the results in the vacuum state. This defines a state with two well-separated particles.

We assume the existence of a linear operator $p^\dagger$ which acts on the vacuum to produce a one-particle state with momentum p. In addition we assume that $p^\dagger$ is chosen so that $\int f(p)g(q)p^\dagger q^\dagger|0\rangle$ is a state with two well-separated particles in states f and g if $f(p)$ and $g(q)$ are momentum-space wave packets for two well-separated configuration-space packets.

We stress here that we are dealing with physical particles, entire extended structures, which might be quite complicated. For this reason certain ambiguities arise when the two systems are brought so close together that the extended structures overlap.

The problem is that in order to discuss deeply bound states, composites of particles which are very close together must be studied. There are two things that can be done. A description in terms of bare particles as in conventional field theory may be possible.[14] This requires introducing objects which are more elementary than the physical particles.

We are going to try something new here. Instead of trying to make the bound states out of bare particles, we shall try to generalize the states of several physical particles to include states with more than one physical particle close together in space. Then a definition of a composite bound state of several physical particles will be possible.

To deal with physical particles it is found that states describing the particles must not form an orthonormal basis. In fact it will be directly in terms of the overlaps among the physical-particle states that the self-consistent composite-structure assumptions will be formulated.

Before formulating the required ideas, some simple examples of physical-particle states in potential theory and simple field theory are in order.

We wish to describe a state in field theory containing two physical particles. A given physical particle con-

[10] G. F. Chew, S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962).

[11] M. Gell-Mann, S. C. Frautschi and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

[12] D. I. Olive, Phys. Rev. **135**, B745 (1964).

[13] We have in mind the fact that no integral representation similar to the Mandelstam Representation is available for higher particle processes.

[14] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

sists of products of bare-particle creation operators applied to the vacuum. For the first particle, the effects due to possible differences between the bare and physical vacuums are ignored.

$$|p_\alpha\rangle = p_\alpha^\dagger|0\rangle. \quad (1)$$

In Eq. (1) $|p_\alpha\rangle$ is a one-particle state with a particle of type α and momentum p, and $p_\alpha^\dagger$ is a sum of products of *creation* operators for bare particles. The operator $p_\alpha^\dagger$ is unique since it contains *only* creation operators.

We define the two-particle state $|p_\alpha,q_\beta\rangle$ to be that state which has all the "stuff" of p_α superimposed on the stuff which makes up q_β,

$$|p_\alpha,q_\beta\rangle = p_\alpha^\dagger|q_\beta\rangle = p_\alpha^\dagger q_\beta^\dagger|0\rangle. \quad (2)$$

That is, we apply all the bare particles making up $|q_\beta\rangle$ to the vacuum and follow this by superimposing all the bare particles of p_α. Since all creation operators commute (or anticommute), the order in which q_β and p_α are put into the system is irrelevant.

A second illustrative model is the nonrelativistic potential theory of multiparticle systems.[15] A system of n particles may have many combinations of bound states among the particles. Suppose the particles divide into two groups of n and n' particles such that the group n has a bound state b and the group n' has a bound state b',

$$|b\rangle = \int \psi(x_1\cdots x_n)|x_1\cdots x_n\rangle dx_1\cdots dx_n,$$
$$|b'\rangle = \int \psi'(y_1\cdots y_{n'})|y_1\cdots y_{n'}\rangle dy_1\cdots dy_{n'}. \quad (3)$$

A state with a b and a b' present is defined as

$$|b,b'\rangle = \int \psi(x_1\cdots x_n)\psi'(y_1\cdots y_{n'})|x_1\cdots x_n,y_1\cdots y_{n'}\rangle \times d^nx d^{n'}y. \quad (4)$$

If any of the particles in the group n is identical to one in n' then Eq. (4) must be properly symmetrized.

Consider a state with two physical particles in distantly separated regions of space,

$$|f,g\rangle = \int f(p)g(q)|p,q\rangle dp dq. \quad (5)$$

The functions f and g describe two spatially separated packets.

A second such state, $|f',g'\rangle$ with f' near f and g' near g has an overlap with $|f,g\rangle$ given by

$$\langle f',g'|f,g\rangle = \langle f'|f\rangle\langle g'|g\rangle. \quad (6)$$

This is so because the f's and g's are states containing the bare particles of f and g. Inner products may be taken by inserting a complete set of bare particles into Eq. (6), which corresponds to linking the bare particles of $|f,g\rangle$ with those of $|f',g'\rangle$. Since all the bare particles of f and f' are far from those of g and g' and will therefore never be linked with g, the overlap consists of only those links linking f to f' and g to g'. This is shown in Fig. 1(a).

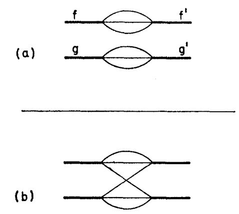


FIG. 1. Overlap of two-particle states.

Now imagine bringing g closer to f so that the bare particle distributions begin to overlap. When this occurs Eq. (6) is no longer valid. The "stuff" of the two particles lose their separate identities and in addition to the contributions from Fig. 1(a) there are now links which cross from f to g.

In terms of the momentum states $|p_1,p_2\rangle$, the above phenomenon manifests itself as a modification of the usual orthonormality conditions. Instead of $\langle q_1q_2|p_1p_2\rangle$ being just a sum of delta functions, an extra continuous connected function of the momenta arises.

$$\langle q_1,q_2|p_1,p_2\rangle = \delta(q_1-p_1)\delta(q_2-p_2) + \delta(q_1-p_2)\delta(q_2-p_1) + C(q_1,q_2;p_1p_2). \quad (7)$$

The extra connected term C is a continuous function apart from an over-all delta function of initial less final momentum. It arises because of the cross links between the particles. A typical term of this kind is shown in Fig. 1(b).

III. VECTOR SPACE OF THE STRONG INTERACTIONS

The picture adopted here will always be the Schrödinger picture. The vectors describe the possible configurations of the system at an instant and not a history of configurations as in the Heisenberg picture. All time dependence of the system is in the state vector, while the operators describing observables do not change in time.

The overlaps between vectors are overlaps between configurations at an instant and are not matrix elements of the time-development operator, Green's function, or S matrix.

The system will consist of several different kinds of particles, each of which carries a three-momentum p and a set of quantum numbers labeled by a Greek subscript. Unless necessary, the Greek subscript will be suppressed. The space of states is assumed to be spanned by the following set of vectors: $|0\rangle$, the vacuum; $|p\rangle$, a state with one physical particle of

[15] R. Rajaraman and L. Susskind, Nuovo Cimento 38, 1201 (1965).

momentum p; $|p_1,p_2\rangle$, a state with two physical particles; and so on.

According to the discussion of Sec. II, this set of vectors will not, in general, be orthonormal. The main purpose of this section is to discuss the special requirements on the overlaps in order that a self-consistent set of composites is described.

We assume that the addition of a particle is a linear operation on the space of states, performed by operating with $p^\dagger$,

$$p^\dagger|0\rangle=|p\rangle,$$
$$p^\dagger|q\rangle=|p,q\rangle,$$
$$p^\dagger|q_1,q_2\rangle=|p,q_1,q_2\rangle. \tag{8}$$

It is not trivial to assume that the addition of a particle to a system is a linear operation.

By no means do we wish to assume the linear independence of the physical-particle states at this point. Suppose a linear relation exists among the vectors $|p_1,p_2\cdots\rangle$. Then if the addition of a particle is linear, an infinite number of new linear relations may be obtained by operating on the original linear relation with operators $p^\dagger$.

In addition to being linear we assume that particle additions commute (anticommute for fermions),

$$[p_1^\dagger,p_2^\dagger]_\pm=0. \tag{9}$$

In both field theory and potential theory, the structure of the space, the inner products among physical-particle vectors, and the possibility of linear relations is determined by appeal to the bare or elementary particles. If the properties of the bare-particle states and the description of physical particles in terms of bare particles are known then it is a simple matter to compute the inner products of physical particle states.

In the present theory, we wish to assume that nothing more elementary than the physical particles exists. How then are we to learn anything about the structure of the vector space? The answer can be found in our original reasons for introducing the multi-physical-particle states in the first place; namely, to describe each physical particle as a composite. If we assume each particle is a composite of two and more physical particles, then we may hope to determine the overlaps by a self-consistent identification of each state with a state in which each particle has been broken down into a multiparticle configuration. From this point of view, the self-consistent theory is not an *ad hoc* idea. It is the only theory of a particular kind which can be worked out in terms of physical-particle states without implicit or explicit reference to bare particles.

Hence it is postulated that every one-particle state is equivalent to a particular superposition of states containing two or more particles,

$$|p\rangle=\sum_{n=2}^{\infty}\int e_p^{(n)}(q_1,q_2\cdots q_n)|q_1,q_2\cdots q_n\rangle d^nq. \tag{10}$$

This assumption distinguishes the theory from usual field theories and allows an elimination of objects more elementary than physical particles.

Equation (10) can easily be generalized to a linear relation among operators. In terms of operators Eq. (10) becomes

$$p^\dagger|0\rangle=\sum_{n=2}^{\infty}\int e_p^{(n)}(q_1\cdots q_n)q_1^\dagger\cdots q_n^\dagger|0\rangle. \tag{11}$$

Since, by assumption any state can be expanded by a suitable sum of products of the $p^\dagger$ on the vacuum we may write

$$|A\rangle=A^\dagger|0\rangle,$$

where $[A^\dagger,p^\dagger]=0$. Hence,

$$\begin{aligned}A^\dagger p^\dagger|0\rangle&=\sum_{n=2}^{\infty}\int e_p^{(n)}(q_1\cdots q_n)A^\dagger q_1^\dagger\cdots q_n^\dagger|0\rangle,\\ p^\dagger|A\rangle&=\sum_{n=2}^{\infty}\int e_p^{(n)}(q_1\cdots q_n)q_1^\dagger\cdots q_n^\dagger|A\rangle.\end{aligned} \tag{12}$$

Since $|A\rangle$ may be any state, Eq. (12) is a relation among the operators.

$$p^\dagger=\sum_{n=2}^{\infty}\int e_p^{(n)}(q_1\cdots q_n)q_1^\dagger q_2^\dagger\cdots q_n^\dagger d^nq. \tag{13}$$

This equation is useful because it gives a representation for a multibody state similar to Eq. (10).

$$\begin{aligned}|p,q\rangle&=p^\dagger q^\dagger|0\rangle\\ &=\sum_{m,n=2}^{\infty}\int e_p^{(n)}(l_1\cdots l_n)e_q^{(m)}(k_1\cdots k_m)\\ &\qquad\times l_1^\dagger\cdots l_n^\dagger k_1^\dagger\cdots k_m^\dagger|0\rangle\\ &=\sum_{m,n=2}^{\infty}\int e_p^{(n)}(l_1\cdots l_n)e_q^{m}(k_1\cdots k_m)\\ &\qquad\times|l_1\cdots l_n,k_1\cdots k_m\rangle.\end{aligned} \tag{14}$$

Equation (1) will be denoted graphically by the use of "e" bubbles as in Fig. 2(a). Equation (14) is then represented in Fig. 2(b).

Consider now, some function of an arbitrary set of momenta, $F(p_1,p_2\cdots p_m)$. Define a process called the "expansion of F by p_1" to be the replacement of $F(p_1,p_2\cdots)$ by

$$\sum_{n=2}^{\infty}\int e_{p_1}^{(n)}(q_1\cdots q_n)F(q_1\cdots q_n,p_2,p_3\cdots).$$

[See Fig. 3(a).]

A function will be called "expandable from the right" if its expansion by each of its momenta gives back the same function. Such functions have the same linear dependences as the vectors of the space of states. They are of interest because they represent the possible inner

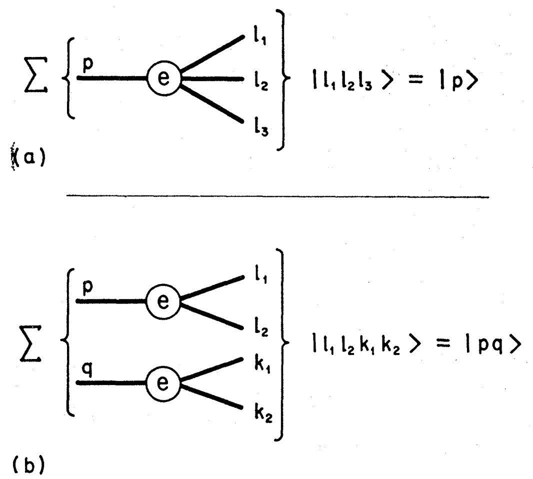


FIG. 2. Expansion of physical particles.

products of a vector $\langle\psi|$ with the particle states,

$$\langle\psi|p_1,p_2\cdots\rangle=\sum\int e_{p_1}{}^{(n)}(l_1\cdots l_n)\times\langle\psi|l_1\cdots l_n,p_2,\cdots\rangle. \quad (15)$$

The inner products are expandable from the left also.

$$\langle p_1,p_2\cdots|\psi\rangle=\sum\int e_{p_1}{}^{(n)}(l_1\cdots l_n)^*\times\langle l_1\cdots l_n,p_2\cdots|\psi\rangle. \quad (15')$$

The inner products $\langle q_1,q_2\cdots|p_1,p_2\cdots\rangle$ are expandable both left and right. In Fig. 3(b) a diagrammatic notation for inner products is given. The "P" stands for projection. In Fig. 3(b) the condition that the inner products are expandable is shown.

The linear relations of Eq. (13) require the vectors to be nonorthogonal since there can be no linear relation among an orthogonal set of vectors. The extent of this nonorthogonality can be examined as follows: First suppose (contrary to fact) that the entire extent of the overlap was the necessary overlap among vectors with different numbers of particles. The vectors with a given number of particles, we suppose, form a basis for a subspace H_n, the nonorthogonality being restricted to vectors in different H_n.

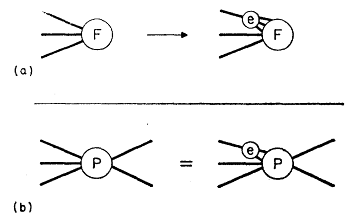
$$\langle q_1|p_1\rangle=\delta(q_1-p_1),$$
$$\langle q_1,q_2|p_1,p_2\rangle=\delta(q_1-p_1)\delta(q_2-p_2)+\delta(q_2-p_1)\delta(q_1-p_2),$$

and so on.

In order to study the consistency graphically, a straight line without a bubble in it represents a delta function. In Fig. 4 the orthonormality condition is substituted into the condition that the overlaps are expandable. The two-body overlap is expanded according to

$$\langle q_1,q_2|p_1,p_2\rangle=\int e_{q_1}(l,m)^*e_{p_1}(l',m')\langle l,m,q_2|l',m',p_2\rangle. \quad (16)$$

FIG. 3. Expandability of a matrix element.



The result is found to have two terms, a disconnected term proportional to a product of delta functions plus a connected continuous function, which contains only a delta function of over-all momentum conservation. Therefore the assumption of orthogonality within a given subspace H_n is inconsistent with Fig. 3(b). The postulated structure of the overlaps is not reproduced under expansion of the momenta.

A suitable structure which is reproduced under expansion is the "cluster-decomposable structure" similar to that of the S matrix. For a complete explanation of cluster properties and their physical meaning we refer the reader to the article by Wichmann and Crichton.[16] Here we shall just state that the cluster decomposability of amplitudes is an assumption demanded by the independence of measurements on distantly separated system. The structure consists of a sum of disconnected parts together with an over-all connected function of all the momenta containing no delta functions other than a delta of total initial less total final momentum. The structure is worked out in Fig. 5.

Expanding the overlaps as in Fig. 6 by using the expandability of Fig. 3(b) and cluster structure of Fig. 5, we find the result consists of a sum of clusters with the same cluster structure as Fig. 5.

In Fig. 7, the various terms with given cluster properties are isolated from Fig. 6 and set equal to their counterparts in Fig. 5. Only the first term of the expansion of the one-particle states have been included so that only two-body "e" bubbles appear. The reader can write out the more general equations and find the conclusions of all arguments are valid in the general case.

Figures 7(a) and 7(b), which come from isolating the disconnected parts of Fig. 6, are identical to equations obtained from the expansion of smaller "P" bubbles.

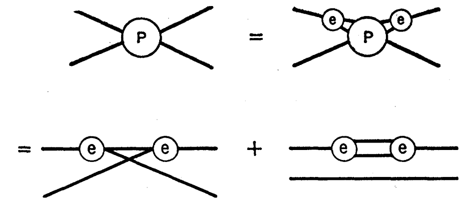


FIG. 4. Expansion of two-body overlap assuming orthonormality in a given subspace.

[16] E. H. Wichmann and J. H. Crichton, Phys. Rev. 132, 2788 (1963).

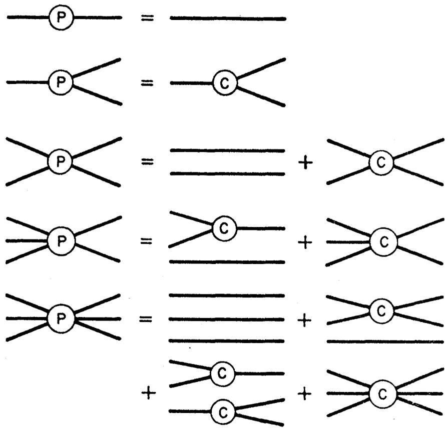


FIG. 5. Cluster of properties of inner products.

For example, Fig. 7(a) reads

$$\delta(p-q)=\int e_p(l,m)e_q(l,m)^* + \int e_p(l,m)e_q(l',m')^*C(l,m;l'm'). \quad (17)$$

The delta function can be expanded in the form

$$\delta(p-q)=\int e_p(l,m)e_q^*(l',m')\langle l,m|l',m'\rangle, \quad (18)$$

which is identical to Eq. (17).

Similarly, Fig. 7(b) reads

$$\delta(p-q)=\int e_q(l,m)^*C_p(l,m), \quad (19)$$

which by Fig. 5 is

$$\delta(p-q)=\int e_q(l,m)^*\langle l,m|p\rangle=\langle q|p\rangle. \quad (20)$$

Therefore the only new equation in Fig. 7 is 7(c), the fully connected equation.

This circumstance is completely general. Expanding an overlap and inserting the cluster properties of Fig. 5

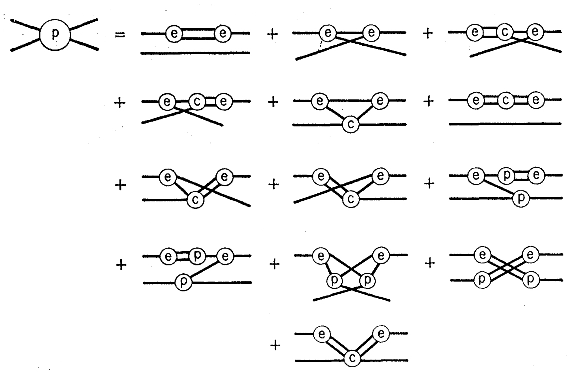


FIG. 6. Expansion of cluster decomposable overlap.

will produce a sum of terms with various different cluster properties. These are to be equated to the corresponding clusters in Fig. 5. This always gives equations identical to those obtained previously by expanding a smaller overlap except for the case of the fully connected equation. The fully connected equation in every case is new.

In this way, a huge number of relations among the connected parts may be obtained. The principal task of Sec. IV is to set up and solve these equations for the simplified static model. A solution of the equations means an algorithm for constructing all of the connected parts from knowledge of the projections of the one body states on the multibody states, together with some assurance that the result of the algorithm is cluster-decomposable and expandable.

IV. STATIC MODEL

For the remainder of this paper we shall consider only a static model. The objects of the theory are static composite nucleons and elementary mesons. The nucleon type is specified by a label α which denotes spin, isospin and whatever other quantum numbers we wish to describe. A meson is labeled by k which denotes its momentum and possible other quantum numbers. Only one nucleon will be present at a time so that a complete set of states will be generated by putting several mesons into a system containing a single nucleon. We can write

$$|\alpha,k_1,k_2\cdots k_n\rangle=k_1^\dagger k_2^\dagger\cdots k_n^\dagger|\alpha\rangle$$

where $k_j^\dagger$ is the creation operator for a meson of momentum k_j and $|\alpha\rangle$ is the state vector for a nucleon of type α.

The mesons in this model are not composite. Each nucleon is a composite of a nucleon and several mesons. As before, this means that a nucleon state is expandable in terms of states containing a nucleon and one or more mesons,

$$|\alpha\rangle=\sum_{n=1}^{\infty}\int e_{\alpha\beta}^{(n)}(k_1\cdots k_n)|\beta,k_1\cdots k_n\rangle d^nk. \quad (21)$$

Since mesons are added by a linear operator $k^\dagger$ Eq. (21) generalizes to[17]

$$|\alpha k_1'k_2'\cdots\rangle=\sum_{n=1}^{\infty} e_{\alpha\beta}^{(n)}(k_1\cdots k_n) \times|\beta,k_1\cdots k_n,k_1',k_2'\cdots\rangle. \quad (22)$$

Because the mesons are not composite, the cluster properties can be taken to be much simpler than in Sec. III. Elementarity of the mesons is expressed in terms of cluster properties by having only those connected clusters involving a nucleon on both sides. Some cluster decompositions are shown in Fig. 8.

[17] In some equations such as Eq. (22) integral and summation signs are omitted. The convention is that repeated momenta are integrated over and repeated greek indecies are summed over.

In addition to the cluster properties of

$$\langle\alpha,l_1,l_2\cdots|\beta,k_1,k_2,k_3\cdots\rangle,$$

the expandable nature of projections is available. As before, an infinite set of coupled equations among the connected parts can be derived from these two properties. These equations might be used in truncated, and iterated versions to obtain information about the connected parts. The author has examined many such methods and has found that they all lead to a particular result when suitably rearranged. We believe the result to be of more interest than the methods of iteration, truncation and matching of terms which led to it. Instead of wasting the reader's time with these methods, the particular solution will be proposed and its consistency with the two requirements studied.

A. Unit Operator and Partial Overlaps.

Two new quantities must be introduced before explaining the solution. Consider a complete basis of orthonormal vectors $|i\rangle$, which span the space of states. They resolve the identity:

$$\sum_i |i\rangle\langle i| = I = \text{unit operator}. \tag{23}$$

Each $|i\rangle$ is expandable in the $|\alpha,k_1\cdots k_n\rangle$,

$$|i\rangle = \sum_{n=0}^{\infty} \int f_i(\alpha,k_1\cdots k_n)|\alpha,k_1\cdots k_n\rangle, \tag{24}$$

$$\sum_{n=m=0}^{\infty} \int d^nkd^ml[\sum_i f_i(\alpha,k_1\cdots k_n)f_i(\beta,l_1\cdots l_m)^*] \times|\alpha,k_1\cdots k_n\rangle\langle\beta,l_1\cdots l_m| = I. \tag{25}$$

We define $\sum_i f_i(\alpha,k_1\cdots)f_i(\beta,l_1\cdots l_m)^*$ to be

$$I(\alpha,k_1\cdots k_n;\beta,l_1\cdots l_m).$$

Denoting the states $|\alpha,k_1,k_2\cdots\rangle$ simply by $|\alpha\cdots\rangle$, we may write Eq. (25) as

$$I(\alpha\cdots;\beta\cdots)|\alpha\cdots\rangle\langle\beta\cdots| = I. \tag{26}$$

Since the $|\alpha\cdots\rangle$ are not linearly independent, the I matrix is not unique. Some relations which are independent of the particular choice of I are derived below.

$$\begin{aligned}\langle\alpha\cdots|\beta\cdots\rangle &= \langle\alpha\cdots|I|\beta\cdots\rangle\\ &= \langle\alpha\cdots|\gamma\cdots\rangle I(\gamma\cdots;\delta\cdots)\langle\delta\cdots|\beta\cdots\rangle.\end{aligned} \tag{27}$$

This equation can be generalized.

$$\begin{aligned}\langle\alpha,&l_1,l_2\cdots|\beta,q_1\cdots q_n,k_1,k_2\cdots\rangle\\ &=\langle\alpha,l_1\cdots|k_1^\dagger k_2^\dagger\cdots|\beta,q_1\cdots q_n\rangle\\ &=\langle\alpha,l_1,l_2\cdots|k_1^\dagger,k_2^\dagger\cdots|\gamma\cdots\rangle\\ &\qquad\times\langle\delta\cdots|\beta,q_1\cdots q_n\rangle I(\gamma\cdots;\delta\cdots)\\ &=\langle\alpha,l_1,l_2\cdots|\gamma\cdots,k_1,k_2\cdots\rangle\\ &\qquad\times\langle\delta\cdots|\beta,q_1\cdots q_n\rangle I(\gamma\cdots;\delta\cdots)\end{aligned} \tag{28}$$

[See Figs. 9(a) and 9(b)].

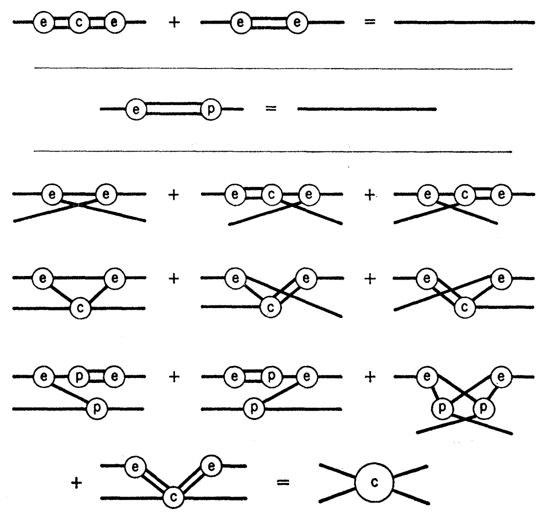


FIG. 7. Isolation of clusters from Fig. (6).

The second new quantity is the partial overlap

$$\langle(\alpha,k_1,k_2,k_3\cdots k_n)l_1,l_2\cdots|(\beta,q_1,q_2\cdots q_m)p_1,p_2\cdots\rangle$$

defined as the sum of those terms in the cluster decomposition of

$$\langle\alpha,k_1,k_2\cdots k_n,l_1,l_2\cdots|\beta,q_1,q_2\cdots q_m,p_1,p_2\cdots\rangle,$$

which do not contain disconnected delta functions connecting momenta in the group l with those in p. Figure 9(c) shows the diagrammatic notation used for a partial overlap. For example

$$\begin{aligned}\langle(\alpha,k)l|(\beta,q)p\rangle &= C(\alpha,k,l;\beta,q,p)\\ &\quad+C(\alpha,l;\beta,p)\delta(k-q)+C(\alpha,k;\beta,p)\delta(l-q)\\ &\quad+C(\alpha,l;\beta,q)\delta(k-p).\end{aligned} \tag{29}$$

The connected part and the full projection are special cases of partial overlaps.

$$\langle(\alpha,k_1,k_2\cdots)|(\beta,q_1,q_2\cdots)\rangle = \langle\alpha,k_1,k_2\cdots|\beta,q_1,q_2\cdots\rangle, \tag{30}$$

$$\langle(\alpha)l_1,l_2\cdots|(\beta)p_1,p_2\cdots\rangle = C(\alpha,l_1,l_2\cdots;\beta,p_1,p_2\cdots). \tag{31}$$

The full overlap can be expressed in terms of partial overlaps in many ways. In fact, for every partition of

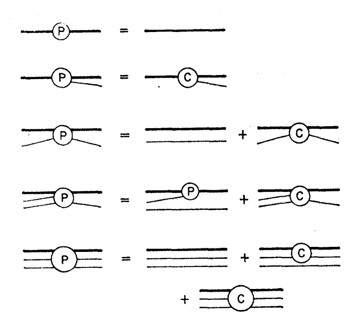


FIG. 8. Cluster properties of the static theory.

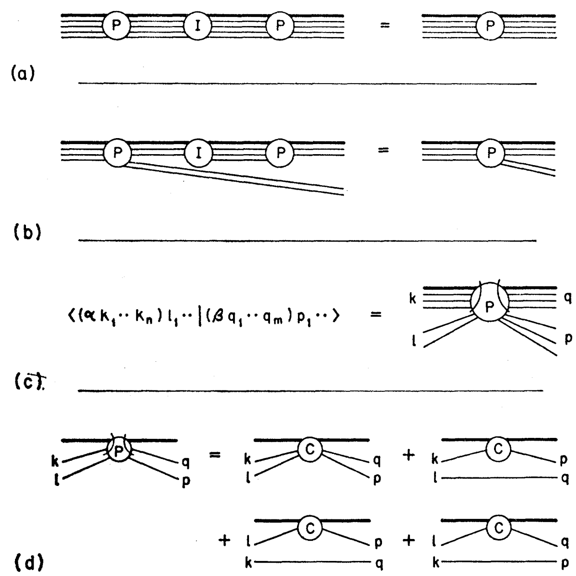


FIG. 9. The I bubble and the partial overlap.

the momenta on the left and right, a different expression exists.

$$\begin{aligned}\langle\alpha,k_1\cdots l_1\cdots|\beta,q_1\cdots p_1\cdots\rangle &= \langle(\alpha,k_1\cdots)l_1\cdots|(\beta,q_1\cdots)p_1\cdots\rangle \\ &\quad+\langle(\alpha,k_1\cdots)l_2\cdots|(\beta,q_1\cdots)p_2\cdots\rangle\delta(l_1-p_1) \quad (32)\end{aligned}$$

plus other terms with deltas of l minus p. One special partition of the momenta is to group all momenta on both sides in brackets in which case the expression for the total overlap has only one term. The other extreme is to group no mesons with the nucleon in which case the expression for the inner product is the cluster decomposition.

The proposed solution can best be expressed as a representation for partial overlaps.

$$\begin{aligned}\langle\alpha,k_1\cdots k_n)l_1\cdots|(\beta,q_1\cdots q_m)p_1\cdots\rangle &= \langle\alpha,k_1\cdots k_n|\delta',k_1',k_2'\cdots p_1,p_2\cdots\rangle \\ &\quad\times I(\delta',k_1'k_2';\gamma'q_1'q_2'\cdots) \\ &\quad\times\langle\gamma'q_1'\cdots l_1,l_2|\beta,q_1,q_2\cdots\rangle. \quad (33)\end{aligned}$$

Sums and integrals over all primed intermediate indices are implied in Eq. (33). [See Fig. 10(a).]

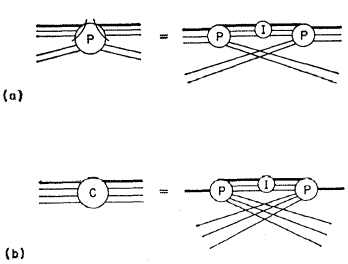


FIG. 10. A representation for partial overlaps.

As special cases, the connected bubble has the representation

$$\begin{aligned}C(\alpha,l_1,l_2\cdots;\beta,p_1\cdots) &= \langle\alpha|\delta',k_1',k_2'\cdots p_1,p_2\cdots\rangle \\ &\quad\times I(\delta',k_1'\cdots;\gamma',q_1'\cdots)\langle\gamma,q_1'\cdots l_1,l_2\cdots|\beta\rangle \quad (34)\end{aligned}$$

[see Fig. 10(b)] and the full inner product is

$$\begin{aligned}\langle\alpha,k_1,k_2\cdots|\beta,q_1,q_2\cdots\rangle &= \langle\alpha,k_1,k_2\cdots|\delta',k_1'\cdots\rangle \\ &\quad\times I(\delta',k_1'\cdots;\gamma'q_1'\cdots)\langle\gamma',q_1'\cdots|\beta,q_1,q_2\cdots\rangle. \quad (35)\end{aligned}$$

This second equation is of course Eq. (27), which is implied by the definition of I.

B. Formal Consistency.

The questions about this representation which concern us are as follows:

(1) Assuming Eq. (33) defines a unique set of overlap functions, are the resulting overlaps expandable and cluster-decomposable?

(2) Does Eq. (33) define a unique set of overlap functions?

The cluster decomposability is trivial. In terms of partial overlaps the cluster properties are the same as for the full projections except that all terms with delta functions involving two unbracketed momenta are absent. By inserting the cluster decomposition of the overlaps which occur in Eq. (33) we easily see that the correct cluster properties for the partial overlaps result.

The expandability of inner products is equally trivial. Consder whether Eq. (36) is true.

$$\begin{aligned}\langle\alpha,k_1\cdots|\beta,l_1\cdots\rangle &= \sum_{n=1}^{\infty} e_{\alpha\gamma}{}^{(n)}(p_1\cdots p_n)^* \\ &\quad\times\langle\gamma p_1\cdots p_n,k_1\cdots|\beta,l_1\cdots\rangle. \quad (36)\end{aligned}$$

The right side can be written in terms of partial overlaps by grouping all momenta in the set p with the nucleon γ. This results in an expression shown in Fig. 11(a). We encounter the quantity

$$\sum_{n=1}^{\infty} e_{\alpha\beta}{}^{n}(k_1\cdots k_n)^*\langle\beta,k_1\cdots k_n|\sigma\cdots l_1,l_2\cdots\rangle.$$

This quantity is equal to $\langle\alpha|\sigma\cdots l_1,l_2\cdots\rangle$ since the single particle α can be expanded as a multibody state.

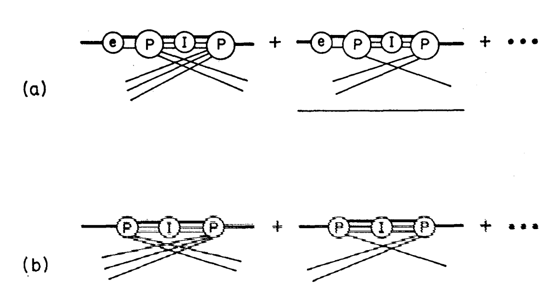


FIG. 11. Expandability of the overlaps of Fig. (10).

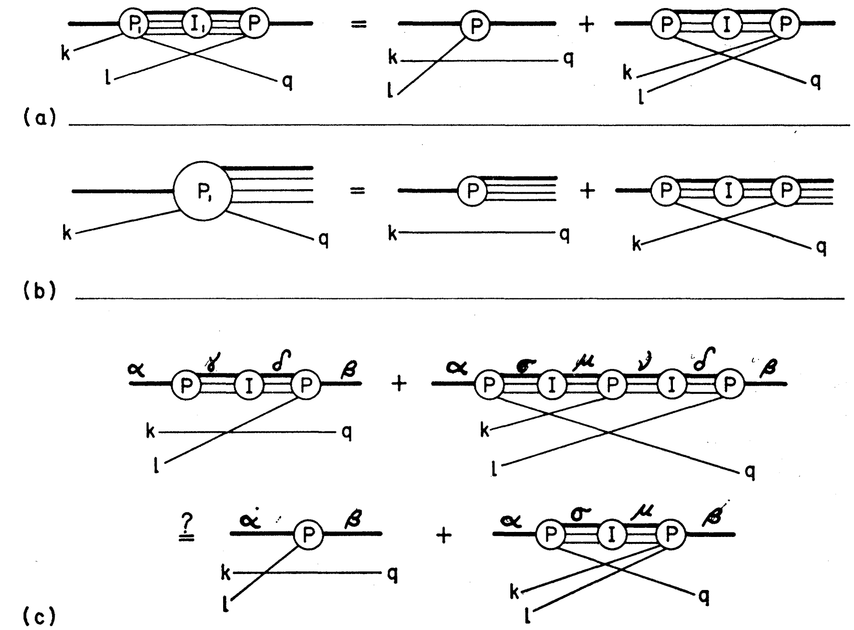


FIG. 12. Formal consistency of the overlaps of Fig. (10).

Using this in Fig. 11(a) gives Fig. 11(b). But the right-hand side is just the cluster expansion of $\langle \alpha k_1 \cdots | \beta l_2 \cdots \rangle$ thus establishing Eq. (36).

Thus the overlaps will be expandable *if* the one-particle overlaps allow an expansion of the one-particle state into multiparticle states.

The nontrivial question concerning the representation of Eq. (33) is question two. Does the representation give a unique set of overlap functions? The problem is that for every partition of the momenta there is a different expression for the full overlap in terms of partial overlaps. As an illustration, consider $\langle \alpha kl | \beta q \rangle$.

One way of expressing this inner product in terms of partial overlaps is

$$\begin{aligned}\langle \alpha,k,l | \beta,q \rangle &= \langle (\alpha,k)l |\, (\beta)q \rangle + \langle (\alpha,k) |\, (\beta) \rangle \delta(l-q) \\ &= \langle (\alpha,k)l |\, (\beta)q \rangle + \langle \alpha,k | \beta \rangle \delta(l-q). \qquad (37)\end{aligned}$$

A second expression is given by

$$\langle (\alpha)k,l |\, (\beta)q \rangle + \langle \alpha,k | \beta \rangle \delta(l-q) + \langle \alpha,l | \beta \rangle \delta(k-q). \quad (38)$$

Let us try to prove these equal. Inserting the proposed solution gives us the equality of Fig. 12(a) to be proved.

Let us decompose the P bubble labeled P_1 according to the proposed rule with all the momenta on the right of P_1 which arrive at the I bubble grouped with nucleon. This is shown in Fig. 12(b).

Inserting Fig. 12(b) into Fig. 12(a) gives Fig. 12(c) for the consistency equation.

For the first term of the left side of Fig. 12(c) to equal the first term of the right-hand side we require

$$I(\gamma \cdots ; \delta \cdots) \langle \alpha | \gamma \cdots \rangle \langle \delta \cdots l | \beta \rangle = \langle \alpha l | \beta \rangle.$$

This is satisfied as a special case of Eq. (28).

Similarly, the second terms will be equal if

$$\langle \mu \cdots k | \nu \cdots \rangle \langle \delta \cdots l | \beta \rangle I(\nu \cdots ; \delta \cdots) = \langle \nu \cdots k,l | \beta \rangle.$$

This too is a special case of Eq. (28).

A kind of internal consistency is exhibited by the representation of Eq. (33). By suitably choosing the correct partition in internal P bubbles such as P_1 consistency can always be checked. We refer to this as the *formal* consistency of the representation. It is completely general.

Arguments of the above kind show only that our proposed solution has a chance of being consistent. The question of uniqueness was transformed from the uniqueness of the over-all projection we started with, to the question of uniqueness of the bubble labeled P_1. For example, it is not clear that by making a different partition of the momenta of P_1 that a different result would not have been obtained. In order to prove

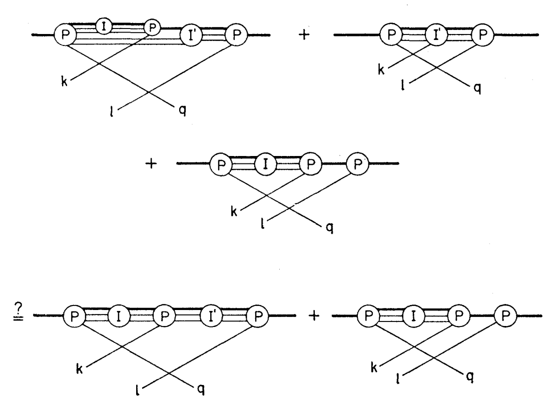


FIG. 13. Consistency by one-particle dominance in the first stage.

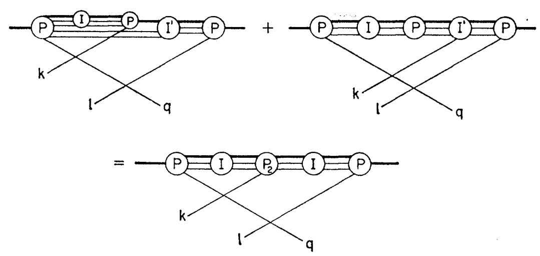


FIG. 14. Consistency requirement after one-particle contribution is removed from an I bubble.

consistency we must choose a particular *canonical* partition of momenta and then, using only that partition method in internal bubbles, prove all other partitions are equivalent. For example, let us choose the simplest partition $\langle(\alpha)k_1,k_2\cdots|(\beta)q_1,q_2\cdots\rangle$ as canonical. The P_1 bubble then must be opened up by this partition. Let us divide the sum over I into two parts, the one-particle part plus everything orthogonal to it. This can be accomplished by expressing the unit operator in the form

$$I=|a\rangle\langle\alpha|+I',$$

where I' is the projection onto the space orthogonal to single-particle states. In Fig. (13) the P_1 bubble of Fig. 12 is expanded using the canonical partition of the momenta and the I_1 bubble is decomposed into $|\alpha\rangle\langle\alpha|$ and I'. The right hand side of Fig. 12 is re-expressed with the help of Fig. 9(b).

The two one-body contributions are obviously equal. Hence the total extent of the possible difference of the left and right hand side of Fig. 13 is due to that part of the sum over a particular I bubble which is orthogonal to the one-body states. Let us suppose that the sums over I on either side are dominated by one-body states. The difference of left and right sides will then be at most small. For consistency it must be zero. Hence Fig. 14.

Another example of the formal self-consistency of Eq. (33) can now be demonstrated by expanding the P_2 bubble in a noncanonical form, with all momenta on the left of P_2 which go to I being grouped with the nucleon. This would result in a complete identification of left- and right-hand sides. However, this begs the question. P_2 must again be canonically expanded. Doing so leads to Fig. 15 as the consistency condition.

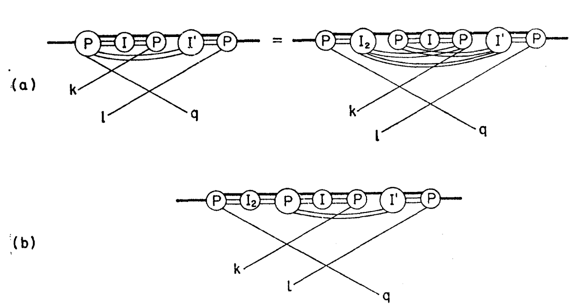


FIG. 15. Consistency by one-particle dominance in the second stage.

Using Eq. (28) the left side of Fig. 15(a) can be expressed in the form of Fig. 15(b).

Again it is apprent that for that part of the sum over I_2 due to one-body states, this is the same as the right side of Fig. 15(a). If we assume as before that the sum over I_2 is dominated by intermediate one-particle states, then only a small fraction of the previous small fraction is left to upset the consistency of the proposed solution. This process can be repeated as many times as one desires. The questionable P bubble is pushed deeper and deeper into the diagram, and if one-body states dominate the relevant sums over I then consistency is likely. We assume this to be the case.

C. The X Structure.

The particular structure of the overlaps given in Eq. (33) and Fig. 10 will be referred to as the X structure.

The X structure admits a simple interpretation connected with the elementarity of the mesons.

Consider the inner product

$$\begin{aligned}\langle\alpha,k_1\cdots l|\beta,q_1,q_2\cdots p\rangle&=\langle(\alpha,k_1\cdots)l|(\beta,q_1,q_2\cdots)p\rangle\\&\quad+\langle\alpha,k_1\cdots l|\beta,q_1,q_2\cdots\rangle\delta(l-p)\qquad(39)\\&=\langle\alpha,k_1\cdots|p^\dagger|\gamma\cdots\rangle\langle\delta\cdots|l|\beta,q_1,q_2\cdots\rangle\\&\quad\times I(\gamma\cdots,\delta\cdots)+\langle\alpha,k_1\cdots|\beta,q_1,q_2\cdots\rangle\\&\quad\times\delta(l-p),\qquad(40)\end{aligned}$$

we have

$$\langle\alpha,k_1,\cdots|l\,p^\dagger|\beta,q_1,\cdots\rangle-\langle\alpha,k_1,\cdots|p^\dagger l|\beta,q_1\cdots\rangle=\langle\alpha,k_1,\cdots|\beta,q_1,\cdots\rangle\delta(l-p),\qquad(41)$$

so that

$$[l,\,p^\dagger]=\delta(p-l).\qquad(42)$$

These are the usual commutation relations for elementary meson field operators. In fact Eq. (33) could have been derived directly from the commutation relations of Eq. (42). However, we were more interested in understanding the X rules from the independent ideas of the expandability and cluster structure of the overlaps.

Equation (33) can be used as the basis for an iteration procedure. Since it appears that the one-body states must dominate the sum over I in Eq. (33) let us start our iteration procedure with

$$C(\alpha,k_1,k_2\cdots;\beta,l_1,l_2\cdots)\sim\sum_\gamma\langle\alpha|\gamma,l_1,l_2\cdots\rangle\times\langle\gamma,k_1,k_2\cdots|\beta\rangle.\qquad(43)$$

This is obtained by writing the unit operator in the form

$$I=|\gamma\rangle\langle\gamma|+I'(\gamma,k_1\cdots;\,\delta,l_1\cdots)|\gamma\cdots\rangle\langle\delta\cdots|,$$

where I' projects onto the space orthogonal to the one-body state. In Eq. (43) I' is ignored. Now, in principle I' is known in terms of the P bubbles. It is obtained by an orthogonalization procedure among the vectors orthogonal to the single-particle vectors. Therefore by using Eq. (43) to obtain a first approximation to P we also implicitly obtain a first approximation to I'. This value of I' may now be used to complete the sum over the intermediate I bubble to obtain an improved value for the P bubbles. This process can be iterated indefinitely to obtain the P bubbles. We feel that unless the procedure is fairly rapidly convergent that one can not have much confidence in the consistency of the theory.

V. DYNAMICAL CONSIDERATIONS

A. Hamiltonian

We have not really gone very far toward a dynamics of the strong interactions in the previous sections. In nonrelativistic potential theory the analog to our present circumstance would be that we had discovered that the set of momentum vectors is an orthonormal basis. Actually this much is not even known since we only know the overlaps in terms of the one-body P bubbles about which we, as yet, know nothing.

In potential theory, the next step would be to determine how the state vector changes with time, or what is the same thing, determine the Hamiltonian. We have a great deal of freedom in potential theory because any Hermitian potential may be chosen. If the same degree of freedom existed in the present theory we would not know where to begin. Many Hamiltonians might be tried until one was found which agreed with experiment but this would not be a very satisfactory circumstance.

Fortunately the same freedom does not exist here. For example, the present theory does not admit free propagation of particles. Free propagation means that the time development of $|\alpha,k_1\cdots k_n\rangle$ is given by $\exp[it/\hbar(M_\alpha+E_{k_1}+\cdots E_{k_n})]|\alpha,k_1\cdots k_n\rangle$ where M_α is the mass of α and $E_k=(k^2+\mu^2)^{1/2}$. Hence the free-particle Hamiltonian would give

$$H_{\text{free}}|\alpha,k_1\cdots\rangle=(M_\alpha+E_{k_1}+\cdots)|\alpha,k_1\cdots\rangle.$$

Since each E_k is greater than zero every multiparticle state has energy greater than M_α, and therefore the one-particle state is orthogonal to every multibody state. This is inconsistent with the assumption that the one-body state lies in the subspace of multibody states.

The problem with free propagation is that the matrix elements of H_{free} do not reflect the linear dependences of the vectors themselves. The matrix elements of an acceptable Hamiltonian must be expandable since H is a linear operator.

This is still not enough information to determine the structure of H. The one extra piece of information that must be used is that the time development of distant systems must be independent. In terms of cluster

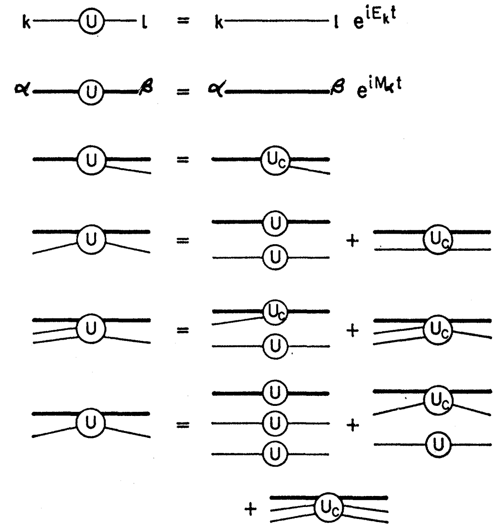


FIG. 16. Cluster properties of the time development operator.

properties of the matrix elements of $U(t)$, this can be expressed as in Fig. 16. It is more convenient to work in terms of the Hamiltonian instead of $U(t)$. Since for small times $U(t)=I+iHt$, the cluster properties of H can be obtained from those of U. The resulting cluster decomposition of H is shown in Fig. 17.

An analysis of H could be made by expanding each matrix element of H and then inserting the cluster decomposition of Fig. 17 to equations for the connected H bubbles. Instead of proceeding this way, we shall try to make use of some simple field-theoretic ideas concerning the Hamiltonian. We shall derive a repre-

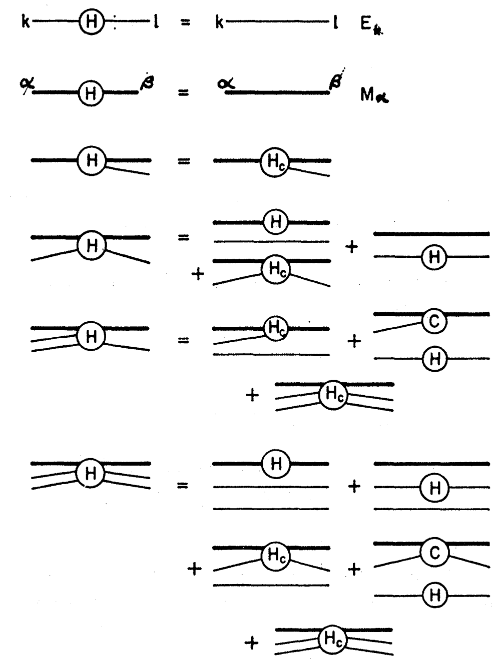


FIG. 17. Cluster properties of the Hamiltonian.

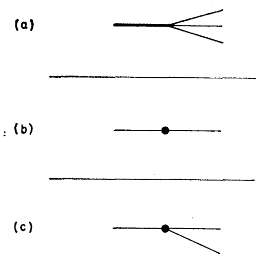


FIG. 18. Notations for matrix elements involving bare particles.

sentation for H in a simple model field theory and then investigate its consistency with the composite nature of the nucleons. We feel that such a procedure aids the physical understanding of the Hamiltonian more than an investigation of the infinite set of coupled equations for the connected parts. At any rate the important point is that the particular representation for H that we shall propose is the only one we have found which is formally self-consistent with the expandability and cluster properties of H.

B. Hamiltonian in a Simple Field Theory

Consider the static limit of a field theory in which the mesons are bare and the nucleons emit and absorb mesons. The emission and absorption processes involve

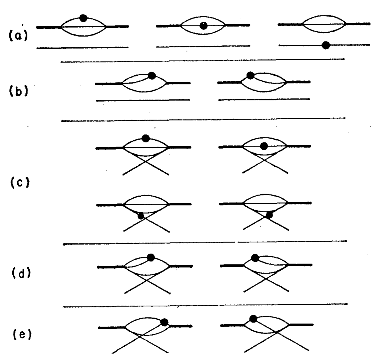


FIG. 19. Terms in the matrix element of the Hamiltonian in a simple field theory.

single mesons. The notation $|\alpha\rangle$, $|b\rangle$, etc. will be used to denote a state with several bare mesons and a bare nucleon so that as the label a varies, the entire space of bare-particle states is covered. The nucleon $|\alpha\rangle$ has a representation

$$|\alpha\rangle=\sum_a \psi_\alpha(a)\,|a\rangle. \tag{44}$$

We are interested in matrix elements of H of the form

$$\begin{aligned}\langle\alpha,k|H|\beta,l\rangle&=\psi_\alpha(a)^*\langle a|kHl^\dagger|b\rangle\psi_\beta(b)\\&=\psi_\alpha(a)^*\langle a,k|H|l,b\rangle\psi_\beta(b).\end{aligned} \tag{45}$$

We use the following diagramatic notation for matrix element involving bare particles: The expansion of a nucleon in bare states is given by a point vertex as in Fig. 18(a). The dot in Fig. 18(b) represents the kinetic energy if it is attached to a meson and a nucleon mass if attached to a nucleon. Finally, the black dot between a nucleon and a nucleon plus meson represents an emission or absorption process in the interaction term of H. In Fig. 19 various terms of the matrix element $\langle\alpha,k|H|\beta,l\rangle$ are shown.

The terms Fig. 19(a) represent kinetic-energy terms in which the external mesons are not confused with the mesons in the nucleon clouds. The terms in Fig. 19(b) are interaction terms in which the emitted or absorbed meson was a meson from the nucleon cloud and the external mesons are still not confused with cloud mesons. In Figs. 19(c) and 19(d) we find kinetic energy and interaction terms in which the external mesons on one side are projected into cloud mesons on the other side. Finally in Fig. 19(e) the terms in which the

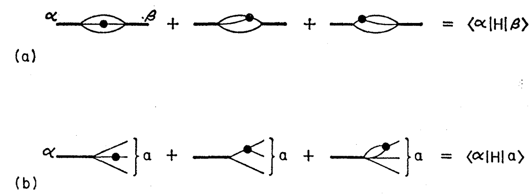


FIG. 20. Matrix elements of H involving physical particles.

external meson on one side is projected onto a meson produced by the interaction and the meson on the other side is absorbed into the nucleon cloud.

Note that in each term only one black dot is present. In particular there are no terms with the interaction acting twice. This is because we are doing nothing more than calculating a matrix element of H between two states in the Schrödinger picture.

The matrix elements $\langle\alpha|H|\beta\rangle$ which equal $M_\alpha\delta_{\alpha\beta}$ and $\langle\alpha|H|a\rangle$, where a is a bare nucleon plus bare mesons, are shown in Fig. 20. Figure 20 can be used to "telescope" the internal "dots" to the external physical-particle legs as follows: The terms in Figs. 19(a) and 19(b) are simple and just add up to the sum of the kinetic energy of the meson plus the nucleon mass. Next, consider the sum of the first three terms of Fig. 19(c) plus the two terms of Fig. 19(d) plus the second term of Fig. 19(e). Using Fig. 20 these sum up to Fig. 21(a). The H bubble is a matrix element of H between a physical nucleon and set of bare particles. The first,

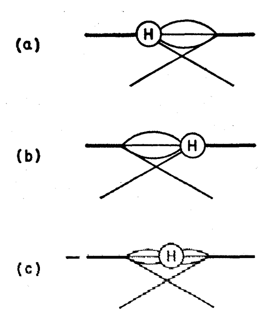


FIG. 21. "Telescoping" the terms of Fig. (20) into the physical particles.

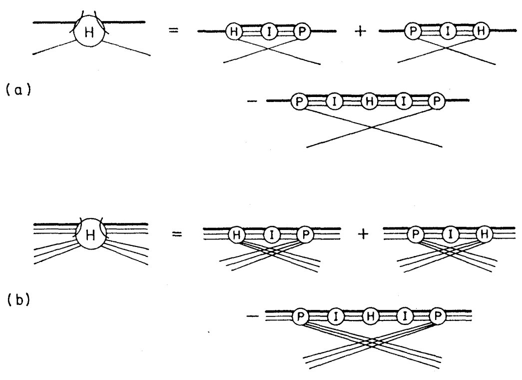


FIG. 22. A representation for the matrix elements of H.

second, and fourth terms of Fig. 19(c) together with Fig. 19(d) and the first term of Fig. 19(e) add up to give Fig. 21(b). The Figs. 21(a) and 21(b) together include term one and two of Fig. 19(c) and Fig. 19(d) twice and hence these terms must be subtracted in the form of Fig. 21(c). Figures 21(a)–21(c) give the connected two-body matrix elements of H.

Denoting bare-particle states by $|a\rangle$, $|b\rangle\cdots$, we can express the terms of Fig. 21 as

$$H_c(\alpha,k;\beta,l)=\sum_a \langle\alpha|al\rangle\langle k,a|H|\beta\rangle +\sum_a \langle\alpha|H|a,l\rangle\langle k,a|\beta\rangle -\sum_{a,b} \langle\alpha|a,l\rangle\langle a|H|b\rangle\langle b,k|\beta\rangle. \quad (46)$$

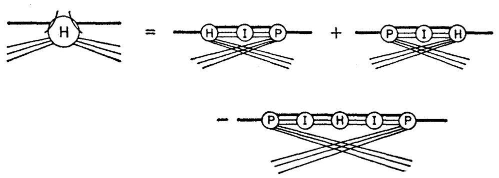


FIG. 23. The connected parts of H as a special case of Fig. (22).

This can be reduced as follows:

$$H_c(\alpha,k;\beta,l)=\sum_a \langle\alpha|l^\dagger|a\rangle\langle a|kH|\beta\rangle +\sum_a \langle\alpha|Hl^\dagger|a\rangle\langle a|k|\beta\rangle -\sum_{a,b} \langle\alpha|l^\dagger|a\rangle\langle a|H|b\rangle\langle b|k|\beta\rangle. \quad (47)$$

Using the fact that the bare-particle states form an orthonormal basis, this becomes

$$H_c(\alpha,k;\beta,l)=\langle\alpha|l^\dagger kH|\beta\rangle+\langle\alpha|Hl^\dagger k|\beta\rangle -\langle\alpha|l^\dagger Hk|\beta\rangle. \quad (48)$$

Assuming that the states $|\alpha,k_1,k_2\cdots\rangle=|\alpha\cdots\rangle$ are complete, it is possible to express this in a form involving matrix elements of H between these physical-particle states. This is accomplished by appropriate insertion of the identity, $I_{\alpha\cdots,\beta\cdots}|\alpha\cdots\rangle\langle\beta\cdots|$.

$$\begin{aligned}H_c(\alpha,k;\beta,l) &=[\langle\alpha|l^\dagger|\delta\cdots\rangle\langle\gamma\cdots|kH|\beta\rangle I(\delta\cdots;\gamma\cdots)\\ &\quad+\langle\alpha|Hl^\dagger|\delta\cdots\rangle\langle\gamma\cdots|k|\beta\rangle I(\delta\cdots;\gamma\cdots)\\ &\quad-\langle\alpha|l^\dagger|\delta\cdots\rangle\langle\gamma\cdots|H|\sigma\cdots\rangle\langle\tau\cdots|k|\beta\cdots\rangle]\\ &\qquad\times I(\delta\cdots;\gamma\cdots)I(\sigma\cdots;\tau\cdots), \quad (49)\\ &=[\langle\alpha|\delta\cdots l\rangle\langle\gamma\cdots k|H|\beta\rangle\\ &\quad+\langle\alpha|H|\delta\cdots l\rangle\langle\gamma\cdots k|\beta\rangle]I(\delta\cdots;\gamma\cdots)\\ &\quad-\langle\alpha|\delta\cdots l\rangle\langle\gamma\cdots|H|\sigma\cdots\rangle\langle\tau\cdots k|\beta\rangle]\\ &\qquad\times I(\delta\cdots;\gamma\cdots)I(\sigma\cdots;\tau\cdots). \quad (50)\end{aligned}$$

(See Fig. 22).

Equation (50) and Fig. 22(a) can be generalized to any partial matrix element of H. This is shown in Fig. 22(b). A partial matrix element of H is defined as in analogy with a partial overlap.

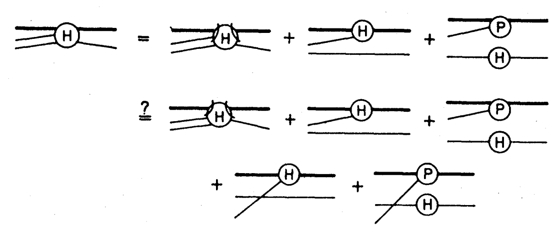


FIG. 24. Consistency of Fig. (23).

In particular, the connected parts are given in Fig. 23.

Figure 23 can be simplified by using the fact that the one-body states $|\alpha\rangle$ are eigenstates of H with eigenvalue M_α. Hence $\langle\alpha|H|\gamma\cdots\rangle=M_\alpha\langle\alpha|\gamma\cdots\rangle$ and $\langle\gamma\cdots|H|\alpha\rangle=M_\alpha\langle\gamma\cdots|\alpha\rangle$.

C. X-Structure of the Hamiltonian

We are not interested in this representation of H in the simple-field-theory model, but rather as a possible solution to the expansion and cluster requirements on the matrix elements of H. It is a simple matter to show that the representation does satisfy these requirements if only it defines a consistent set of matrix elements. In fact, the problem is identical to that of the overlaps, treated in the previous part: Does the representation of Fig. 22 define a unique and consistent set of matrix

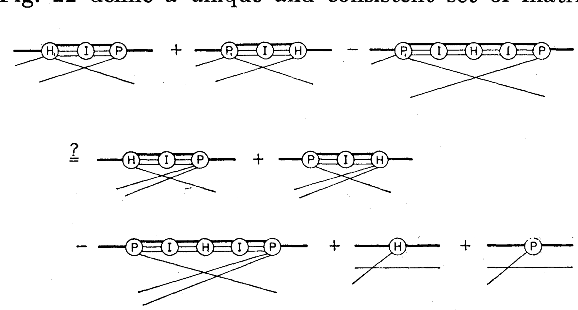


FIG. 25. Formal consistency of Fig. (23).

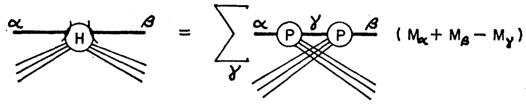


FIG. 26. The first approximation to the matrix elements of H.

elements? As with the overlaps each partition of the momenta leads to a different expression for $\langle|H|\rangle$ in terms of partial matrix elements. We must show that the use of Fig. 22 leads to a unique set of matrix elements. For example, in Fig. 24 we see two expressions for $\langle\alpha kl|H|\beta p\rangle$.

Expressing the partial overlaps through the use of Fig. 22 gives Fig. 25 as the consistency condition.

Formal consistency can now be shown by expanding the P_1 and H_1 bubbles according to Figs. 22 and 10 with the momenta partitioned so that all momenta from P_1 or H_1 which arrive at I are grouped together. The methods used are so similar to those in Sec. IV that we leave it to the reader to show formal self-consistency. Also as before, we can show that if sums over I are dominated by one-body states, then all partitions can be shown to be identical to the "canonical" partition.

The representation of Fig. 22 is called the X-structure for H. It is the only representation which we have found which is formally self-consistent with the expandability and cluster structure of H.

The X-structure can be used to obtain the matrix elements of H through an iteration method. The X-representation is first used as in Fig. 23 to express the connected parts of H. In first approximation, we replace the sums over I by one-body parts.

This gives Fig. 26 for the connected parts of H. We use

$$\langle\alpha|H|\gamma\cdots\rangle=M_\alpha\langle\alpha|\gamma\rangle,$$

and

$$\langle\gamma\cdots|H|\beta\rangle=M_\beta\langle\gamma\cdots|\beta\rangle.$$

The full matrix elements of H are now expressed in terms of the connected matrix elements and the result used to complete the sum over IHI. This procedure can be iterated to obtain improved approximations for H.

In this way it is possible to express the matrix elements of H and the structure of the space in terms of the one-body projections $\langle\alpha|\beta,l_1,l_2\rangle$. In the next section we shall show how to use these constructions to write dynamical equations for these one-body overlaps so as to complete a circle of nonlinear equations for these quantities.

VI. SELF-CONSISTENT DYNAMICS

This final section is intended as a brief survey of the dynamical and self-consistency equations. We examine the way in which the structure and interactions of a system of baryons might be generated.

A. Two-Body Dynamics

The dynamics derives from the assumption that the single-particle states $|\alpha\rangle$ are eigenstates of H,

$$H|\alpha\rangle=M_\alpha|\alpha\rangle. \tag{51}$$

Expanding the state $|\alpha\rangle$ we have

$$H\sum_{n=1}^{\infty} e_{\alpha\beta}^{(n)}(l_1,l_2\cdots l_n)|\beta,l_1,l_2\cdots l_n\rangle =M_\alpha\sum_{n=1}^{\infty} e_{\alpha\beta}^{(n)}(l_1\cdots l_n)|\beta,l_1\cdots l_n\rangle.^{18} \tag{52}$$

Taking matrix elements gives

$$\sum_{n=1}^{\infty}\langle\gamma,k_1\cdots k_m|H|\beta,l_1\cdots l_n\rangle e_{\alpha\beta}^{(n)}(l_1\cdots l_n) =M_\alpha\sum_{n=1}^{\infty}\langle\gamma,k_1\cdots k_m|\beta,l_1,l_2\cdots l_n\rangle \times e_{\alpha\beta}^{(n)}(l_1\cdots l_n). \tag{53}$$

Equation (53) provides a dynamical equation for e in terms of the matrix elements of H and the overlaps $\langle\gamma,k_1,k_2\cdots|\beta,l_1\cdots\rangle$.

We now approximate Eq. (53) by assuming that $|\alpha\rangle$ lies in the two-body subspace

$$\langle\gamma,k|H|\beta,l\rangle e_{\alpha\beta}(l)=M_\alpha\langle\gamma,k|\beta,l\rangle e_{\alpha\beta}(l). \tag{54}$$

We evaluate the matrix elements $\langle\gamma,k|H|\beta,l\rangle$ and $\langle\gamma,k|\beta,l\rangle$ through the use of the X-structure for connected parts in first approximation.

$$\langle\alpha,k|\beta,l\rangle=\delta_{\alpha\beta}\delta(k-l)+\langle\alpha|\gamma,l\rangle\langle\gamma,k|\beta\rangle, \tag{55}$$

$$\langle\alpha,k|H|\beta,l\rangle=\delta_{\alpha\beta}\delta(k-l)[M_\alpha+E_k] +\langle\alpha|\gamma,l\rangle\langle\gamma,k|\beta\rangle[M_\alpha+M_\beta-M_\gamma]. \tag{56}$$

Using Eqs. (55) and (56), Eq. (54) becomes

$$e_{\alpha\gamma}(k)[M_\gamma+E_k]+[M_\beta+M_\gamma-M_\delta] \times\int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle dl =M_\alpha[e_{\alpha\gamma}(k)+\int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle dl]$$

or

$$e_{\alpha\gamma}(k)=\int\frac{M_\alpha-M_\beta-M_\gamma+M_\delta}{M_\gamma+E_k-M_\alpha} \times e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle dl. \tag{57}$$

In the two-body approximation, Eq. (57) is the dynamical equation for e.

B. Improved Two-Body Approximation

Before analyzing the equations in greater detail, we shall improve the approximation to include some three-body effects. The structure of the diagrams we have been analyzing could be characterized as single-particle exchange. However, we have been working in the approximation in which the state $|\alpha\rangle$ is in the two-body subspace. We should like to include just those three-body effects which supplement the two-body equations and which can still be characterized as single-particle exchange. To this end, we approximate the three-body matrix elements by appropriate disconnected parts as in Fig. 27.

We introduce an expansion of the one-body state as a two- and three-body superposition

$$|\alpha\rangle = e_{\alpha\beta}(k)|\beta,k\rangle + e_{\alpha\beta}(k,l)|\beta,k,l\rangle. \quad (58)$$

The dynamics consists of a pair of coupled equations

$$e_{\alpha\beta}(k)\langle\gamma p|H|\beta k\rangle + e_{\alpha\delta}(kl)\langle\gamma p|H|\delta kl\rangle = M_\alpha[e_{\alpha\beta}(k)\langle\gamma p|\beta k\rangle + e_{\alpha\beta}(kl)\langle\gamma p|\beta kl\rangle] \quad (59)$$

and

$$e_{\alpha\beta}(k)\langle\gamma pq|H|\beta k\rangle + e_{\alpha\beta}(kl)\langle\gamma pq|H|\beta kl\rangle = M_\alpha[e_{\alpha\beta}(k)\langle\gamma pq|\beta k\rangle + e_{\alpha\beta}(kl)\langle\gamma pq|\beta kl\rangle]. \quad (60)$$

Using the matrix elements of Eq. (43) in Eq. (60) gives

$$e_{\alpha\gamma}(k,l) = e_{\alpha\beta}(k)\langle\gamma l|\beta\rangle \frac{M_\beta + E_k - M_\alpha}{M_\alpha - M_\gamma - E_k - E_l}. \quad (61)$$

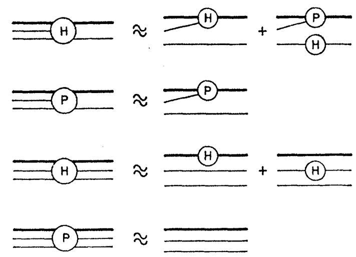


FIG. 27. Three-body matrix elements in the improved two-body approximation.

Inserting this into Eq. (59) gives us the improved two-body equation

$$e_{\alpha\gamma}(k)(M_\gamma + E_k - M_\alpha) = \int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle \times \frac{(M_\beta - M_\delta - E_k)(M_\gamma - M_\delta - E_l)}{M_\delta + E_l + E_k - M_\alpha} dl + \int e_{\alpha\beta}(k)\langle\gamma|\delta l\rangle\langle\delta l|\beta\rangle \times \frac{(M_\beta + E_k - M_\alpha)(M_\gamma + E_k - M_\alpha)}{M_\delta + E_l + E_k - M_\alpha} dl. \quad (62)$$

We define vertex functions $\mu_{\alpha\beta}$ and $v_{\alpha\beta}$ as follows:

$$v_{\alpha\gamma}(k) = e_{\alpha\gamma}(k)(M_\gamma + E_k - M_\alpha)\sqrt{E_k},$$
$$\mu_{\alpha\gamma}(k) = \langle\gamma k|\alpha\rangle(M_\gamma + E_k - M_\alpha)\sqrt{E_k}.$$

Equation (62) then takes the form,

$$v_{\alpha\gamma}(k) = \mu_{\beta\delta}(k)\int \frac{v_{\alpha\beta}(l)\mu_{\gamma\delta}(l)^* \bar{d}l}{(M_\beta + E_l - M_\alpha)(M_\delta + E_l + E_k - M_\alpha)} + v_{\alpha\beta}(k)\int \frac{\mu_{\beta\delta}(l)\mu_{\gamma\delta}(l)^*(M_\gamma + E_k - M_\alpha)}{(M_\alpha - M_\delta - E_k - E_l)(M_\beta - M_\delta - E_l)(M_\gamma - M_\delta - E_l)} \bar{d}l. \quad (63)$$

The symbol $\bar{d}l$ means dl/E_l, the invariant phase-space volume element.

C. Consistency equations

The dynamical equations, either (57) or (62), must be supplemented with equations which connect $e_{\alpha\beta}(k)$ and $\langle\alpha|\beta k\rangle$. We shall work with the improved two-body equations although many of the results apply to the strict two-body approximation. The consistency equations are

$$\sum_1^\infty \int e_{\alpha\beta}(l_1\cdots l_n)\langle\gamma,k_1\cdots|\beta,l_1\cdots l_n\rangle d^n l = \langle\gamma,k_1\cdots|\alpha\rangle. \quad (64)$$

We approximate the consistency by

$$\int dl e_{\alpha\beta}(l)\langle\gamma k|\beta l\rangle + \int e_{\alpha\beta}(l_1,l_2)\langle\gamma k|\beta,l_1 l_2\rangle dl = \langle\gamma k|\alpha\rangle, \quad (65a)$$

$$\int e_{\alpha\beta}(l)\langle\gamma k_1 k_2|\beta l\rangle + \int e_{\alpha\beta}(l_1,l_2)\langle\gamma k_1 k_2|\beta l_1 l_2\rangle dl = \langle\gamma k_1 k_2|\alpha\rangle. \quad (65b)$$

Using the matrix elements of Fig. 27 and Eq. (55) and the e bubble from Eq. (61) in Eq. (65a) gives

$$\langle\gamma k|\alpha\rangle = e_{\alpha\gamma}(k) + \int dl e_{\alpha\beta}(l)\langle\delta k|\beta\rangle\langle\gamma|\delta l\rangle + \int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle \frac{M_\beta + E_l - M_\alpha}{M_\alpha - M_\delta - E_k - E_l} + e_{\alpha\beta}(k)\int \langle\gamma|\delta l\rangle\langle\delta l|\beta\rangle \frac{M_\beta + E_k - M_\alpha}{M_\alpha - M_\delta - E_k - E_l}. \quad (66)$$

Now using Eq. (62) for the first term of Eq. (66)

gives

$$\langle\gamma k|\alpha\rangle=\int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle\langle\delta k|\beta\rangle\frac{M_\beta+M_\delta-E_k}{M_\alpha-M_\gamma-E_k}. \quad (67)$$

In terms of "vertex functions" this is

$$\mu_{\alpha\gamma}(k)=\mu_{\beta\delta}(k)\int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle dl. \quad (68)$$

Equation (68) can easily be shown to apply to the strict two-body approximation also.

In the physical case of meson-nucleon or meson-baryon systems the mesons are in $L=1$ states in the baryon bound states. This means that $\mu_{\alpha\beta}(k)$ will be of the form $\eta_{\alpha\beta}{}^i\hat{k}_i\mu_{\alpha\beta}(k)$ where $\hat{k}_i$ is the ith comment of the unit vector $\hat{k}$, and $\eta_{\alpha\beta}{}^i$ are constants. In order to simplify the equations we shall assume S-wave states so that $\mu_{\alpha\beta}(k)$ is a spherically symmetric function. The extension to p wave is not difficult.

Equation (68) is then a linear relation among the $\mu_{\alpha\beta}$ of the form

$$\mu_{\alpha\beta}(k)=\mu_{\beta\delta}(k)D_{\alpha\beta\delta\gamma},$$

where

$$D_{\alpha\beta\delta\gamma}=\int e_{\alpha\beta}(l)\langle\gamma|\delta l\rangle dl.$$

Consistency will require the linear operator D to have an eigenvalue equal to 1. We shall assume this eigenvalue to be nondegenerate. Hence $\mu_{\beta\delta}$ must have the form $\Gamma_{\beta\delta}\mu(k)$. The Γ are a set of constants which satisfy

$$\Gamma_{\alpha\gamma}=\Gamma_{\beta\delta}D_{\alpha\beta\delta\gamma},$$

and $\mu(k)$ is a spherically symmetric function of k,

$$\mu_{\alpha\beta}(k)=\Gamma_{\alpha\beta}\mu(k). \quad (69)$$

Equation (69) says that any symmetries which exist among the vertices $\mu_{\alpha\beta}(k)$ at any one energy, are present for any other value of k.

D. Singularities and the Born Approximation

From Eq. (62) we see that $e_{\alpha\gamma}(k)$ will have a pole at $E_k=M_\alpha-M_\gamma$. It is interesting to determine the poles of the projections $\langle\beta l|\alpha\rangle$ and $\langle\beta l_1l_2|\alpha\rangle$.

$$\begin{aligned}\langle\beta k|\alpha\rangle&=\int e_{\alpha\gamma}(p)\langle\beta k|\gamma p\rangle dp+\int e_{\alpha\gamma}(p,q)\langle\beta k|\gamma pq\rangle dpdq\\&=e_{\alpha\beta}(k)+\int e_{\alpha\delta}(p)\langle\beta|p\gamma\rangle\langle\gamma k|\delta\rangle dp+\int e_{\alpha\gamma}(p,k)\langle\beta|\gamma p\rangle dp. \quad (70)\end{aligned}$$

Each $\langle\beta k|\alpha\rangle$ will have the poles of the corresponding $e_{\alpha\beta}(k)$ from the first term of Eq. (70). In addition, the second term, being a linear sum of $\langle\delta|\gamma k\rangle$ will have the poles of all these other overlaps. We shall see that the third term cancels these extra poles, leaving only the poles of the first term of Eq. (70).

The third term is evaluated with the aid of Eq. (61). It is given by

$$\begin{aligned}&e_{\alpha\delta}(l)\langle\beta|\gamma l\rangle\langle\gamma k|\delta\rangle(M_\delta+E_l-M_\alpha/M_\alpha-M_\gamma-E_l-E_k)\\&\quad+e_{\alpha\delta}(k)\langle\beta|\gamma l\rangle\langle\gamma l|\delta\rangle(M_\delta+E_k-M_\alpha/M_\alpha-M_\gamma-E_l-E_k).\end{aligned}$$

This together with the second term of Eq. (70) gives

$$\int e_{\alpha\delta}(l)\langle\beta|\gamma l\rangle\langle\gamma k|\delta\rangle\frac{M_\delta-M_\gamma-E_k}{M_\alpha-M_\gamma-E_l-E_k}+\int e_{\alpha\delta}(k)\langle\beta|\gamma l\rangle\langle\gamma l|\delta\rangle\frac{M_\delta+E_k-M_\alpha}{M_\alpha-M_\gamma-E_l-E_k},$$

which does not contain the poles of $\langle\gamma k|\delta\rangle$ or $e_{\alpha\delta}(k)$. Hence we can expect $\langle\beta k|\alpha\rangle$ to have the same pole as $e_{\alpha\beta}(k)$ with the same residue. The projections $\langle\beta l_1l_2|\alpha\rangle$ can be analyzed in a similar way.

$$\begin{aligned}\langle\beta l_1l_2|\alpha\rangle&=e_{\alpha\gamma}(l_1)\langle\beta l_2|\gamma\rangle+e_{\alpha\gamma}(l_2)\langle\beta l_1|\gamma\rangle+e_{\alpha\beta}(l_1,l_2)\\&=\frac{e_{\alpha\gamma}(l_1)\langle\beta l_2|\gamma\rangle(M_\gamma-M_\beta-E_{l_2})}{(M_\alpha-M_\beta-E_{l_2}-E_{l_1})}+\text{symmetrizing term}\\&=\frac{v_{\alpha\gamma}(l_1)\mu_{\gamma\beta}(l_2)}{(M_\alpha-M_\gamma-E_{l_1})(M_\alpha-M_\beta-E_{l_1}-E_{l_2})}+\text{symmetrizing term}. \quad (71)\end{aligned}$$

This pole structure can be used to identify the conventional Born-approximation singularities in scattering amplitudes. In order to calculate a scattering amplitude, the dynamical equations must be solved for the continuum eigenstates. Suppose we are interested in an eigenstate $|\psi_E\rangle$ with energy E.

$$|\psi_E\rangle=\psi(\beta,l)|\beta,l\rangle+\psi(\beta,l_1,l_2)|\beta,l_1,l_2\rangle+\cdots. \quad (72)$$

The dynamical equations in the two-body approximation give

$$\psi_{\beta l}H|\beta,l\rangle=E\psi_{\beta l}|\beta,l\rangle.$$

Projection onto a two-body state gives

$$\langle\alpha,k|H|\beta,l\rangle\psi_{\beta l}=E\langle\alpha,k|\beta,l\rangle\psi_{\beta l}. \quad (73)$$

The matrix elements in Eq. (73) can be divided into connected and disconnected terms.

$$\begin{aligned}\langle\alpha,k|H|\beta,l\rangle&=\delta_{\alpha\beta}\delta(l-k)[M_\alpha+E_l]+H_c(\alpha,k;\beta,l)\\&=H_0(\alpha,k;\beta,l)+H_c(\alpha,k;\beta,l)\\\langle\alpha,k|\beta,l\rangle&=\delta_{\alpha\beta}\delta(k-l)+C(\alpha,k;\beta,l)\\&=\Delta(\alpha,k;\beta,l)+C(\alpha,k;\beta,l)\end{aligned}$$

where Δ is the identity matrix.

The dynamics is summed up by the equation,

$$(E\Delta-H_0)\psi=(H_c-EC)|\psi\rangle. \tag{74}$$

Since H_c and EC are fully connected, H_c-EC play the role of an energy-dependent potential V and the first approximation to the scattering from the appropriate Lippman-Schwinger equation would be V itself.[18]

$$V=H_c-EC,$$
$$V(\alpha,k;\beta,l)=\langle\alpha|\gamma,l\rangle\langle\gamma,k|\beta\rangle[M_\alpha+M_\beta-M_\gamma-E]. \tag{75}$$

Now if $E=M_\alpha+E_k=M_\beta+E_l$, then Eq. (75) is the first approximation to the S matrix element. Let us extract out of the expression for V, the part due to the poles of $\langle\alpha|\gamma,l\rangle$ and $\langle\gamma,k|\beta\rangle$ by assuming

$$\langle\alpha|\gamma,l\rangle=\frac{\mu_{\alpha\gamma}(l)^*}{(M_\alpha-M_\gamma-E_l)\sqrt{E_l}},$$
$$\langle\gamma,k|\beta\rangle=\frac{\mu_{\beta\gamma}(k)}{(M_\beta-M_\gamma-E_k)\sqrt{E_k}}.$$

Then

$$V=\frac{\mu_{\alpha\gamma}(l)^*\mu_{\beta\gamma}(k)}{(E-M_\gamma-E_k-E_l)\sqrt{E_l}\sqrt{E_k}}. \tag{76}$$

This exhibits the usual energy denominator and has the same singularity structure as the conventional Born approximation.

E. Further Considerations on the Dynamical and Self-Consistency Equations

In this subsection we assume $\mu_{\alpha\beta}(k)=\Gamma_{\alpha\beta}\mu(k)$ and we assume $\mu(k)$ is known. In fact, $\mu(k)$ and $v_{\alpha\beta}(k)$ are assumed to be smooth, slowly varying functions of E_k.

The dynamics equation is Eq. (63) which becomes

$$v_{\alpha\gamma}(k)=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\mu(k)\int d\!\!\!/l\frac{v_{\alpha\beta}(l)\mu^*(l)}{(M_\alpha-M_\beta-E_l)(M_\alpha-M_\delta-E_l-E_k)}+v_{\alpha\beta}(k)\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\int d\!\!\!/l\frac{\mu(l)\mu^*(l)(M_\gamma+E_k-M_\alpha)}{(M_\alpha-M_\delta-E_l-E_k)(M_\beta-M_\delta-E_l)(M_\gamma-M_\delta-E_l)}, \tag{77}$$

where

$$d\!\!\!/l\equiv dl/E_l.$$

Equation (77) will now be continued to values of k for which $E_k=M_\alpha-M_\gamma$. We have not made a complete study of the analytic properties of μ and v but we assume that they are analytic enough to give meaning to the required continuations. The second term of (77) vanishes at $E_k=M_\alpha-M_\gamma$.

$$v_{\alpha\gamma}(M_\alpha-M_\gamma)=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\mu(M_\alpha-M_\gamma)\times\int\frac{v_{\alpha\beta}(l)\mu^*(l)}{(M_\alpha-M_\beta-E_l)(M_\gamma-M_\delta-E_l)}d\!\!\!/l.$$

Now at $E_k=M_\alpha-M_\gamma$ we know that $e_{\alpha\gamma}$ and $\langle\gamma k|\alpha\rangle$ have poles of equal residue. This means that

$$v_{\alpha\gamma}(M_\alpha-M_\gamma)=\mu_{\alpha\gamma}(M_\alpha-M_\gamma)=\Gamma_{\alpha\gamma}\mu(M_\alpha-M_\gamma).$$

Thus

$$\Gamma_{\alpha\gamma}=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\int d\!\!\!/l\frac{v_{\alpha\beta}(l)\mu^*(l)}{(M_\alpha-M_\beta-E_l)(M_\gamma-M_\delta-E_l)}. \tag{78}$$

[18] E. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

We shall now make the simplifying approximation that the masses M_α are equal.[3] Equation (77) becomes

$$\begin{aligned}v_{\alpha\gamma}(k)&=\int d\!\!\!/l\frac{v_{\alpha\beta}(l)\mu_{\beta\delta}(k)\mu_{\gamma\delta}(l)^*}{E_l(E_l+E_k)}-v_{\alpha\beta}(k)\int\frac{\mu_{\beta\delta}(l)\mu_{\gamma\delta}(l)^*E_k}{E_l^2(E_l+E_k)}d\!\!\!/l\\&=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\mu(k)\int d\!\!\!/l\frac{v_{\alpha\beta}(l)\mu(l)^*}{E_l(E_l+E_k)}-\Gamma_{\beta\delta}\Gamma_{\gamma\delta}v_{\alpha\beta}(k)\int d\!\!\!/l\frac{\mu(l)\mu(l)^*E_k}{E_l^2(E_l+E_k)}.\end{aligned} \tag{79}$$

Since $v_{\alpha\beta}(k)=\Gamma_{\alpha\beta}\mu(0)$ at $E_k=0$ we shall take $\Gamma_{\alpha\beta}$ as a first approximation to $v_{\alpha\beta}$ and $\mu(0)$ will be normalized to one. Equation (79) becomes

$$v_{\alpha\gamma}(k)=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\Gamma_{\alpha\beta}\mu(k)\int d\!\!\!/l\frac{v(l)\mu(l)^*}{(E_l+E_k)E_l}-\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\Gamma_{\alpha\beta}v(k)\int d\!\!\!/l\frac{\mu(l)\mu(l)^*E_k}{E_l^2(E_l+E_k)} \tag{80}$$

and Eq. (78) is

$$\Gamma_{\alpha\gamma}=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\Gamma_{\alpha\beta}\int \frac{v(l)\mu(l)^*}{E_l^2}\bar{d}l. \tag{81}$$

Hence

$$\Gamma_{\alpha\gamma}=\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\Gamma_{\alpha\beta}\lambda\,, \tag{82}$$

where λ is the integral in Eq. (81).

Substituting Eq. (82) into Eq. (80) gives

$$v_{\alpha\gamma}(k)=\frac{\Gamma_{\alpha\gamma}}{\lambda}\left[\mu(k)\int \bar{d}l\frac{v(l)\mu(l)^*}{(E_l+E_k)E_l} -E_k v(k)\int \bar{d}l\frac{\mu(l)\mu(l)^*}{E_l^2(E_l+E_k)}\right]. \tag{83}$$

Hence we find that $v_{\alpha\gamma}(k)$ will continue to be proportional to $\Gamma_{\alpha\gamma}$ even when $E_k\neq 0$. Let us suppose that $v_{\alpha\gamma}(k)=\Gamma_{\alpha\gamma}v(k)$. The dynamical equations then give

$$\frac{\Gamma_{\alpha\gamma}}{\Gamma_{\beta\delta}\Gamma_{\gamma\delta}\Gamma_{\alpha\beta}}=\frac{\mu(k)}{v(k)}\int \bar{d}l\frac{v(l)\mu(l)^*}{(E_l+E_k)E_l} -E_k\int \bar{d}l\frac{\mu(l)\mu(l)^*}{E_l^2(E_l+E_k)}. \tag{84}$$

Hence the dynamics factors into two parts, a part involving Γ and a part involving the μ and v.

$$\Gamma_{\alpha\gamma}=\lambda\Gamma_{\alpha\beta}\Gamma_{\beta\delta}\Gamma_{\gamma\delta} \tag{85}$$

and

$$\frac{\mu(k)}{v(k)}\int \bar{d}l\frac{v(l)\mu(l)^*}{(E_l+E_k)E_l}-E_k\int \bar{d}l\frac{\mu(l)\mu(l)^*}{E_l^2(E_l+E_k)}=\lambda. \tag{86}$$

It can be shown that the spectrum of λ which admit solution to Eq. (86) is discrete. We are interested in the largest value of λ since this will correspond to the $|\alpha\rangle$ being the lowest-energy bound states admitted by the interactions. Hence Eq. (86) determines λ and $v(l)$ in terms of the input $\mu(l)$. Equation (85) can then be used to determine the Γ.

Equations like (86) and (85) have been suggested and studied by Cutkosky.[3] He shows that such equations are intimately connected with the group-theoretic structure of the strong interactions and that they require attractive forces for their solution. Cutkosky's starting point is a Bethe-Salpeter equation for the bound states. The static limit of Cutkosky's equations can be obtained from Eq. (85) and Eq. (86) by substituting the value $v(0)$ and $\mu(0)$ for $v(k)$ and $\mu(k)$ and continuing to $E_k=0$.

$$\lambda v(0)=\mu(0)\int \frac{v(0)\mu(0)}{E_l^2}\bar{d}l. \tag{87}$$

Since $v(0)=\mu(0)=1$ and $\bar{d}l=dl/E_l$ we get

$$\lambda=\int\frac{dl}{E_l^3}=\int\frac{dl}{[l^2+\mu^2]^{3/2}}. \tag{88}$$

Equations (88) and (85) are the static limit of Cutkosky's equations. The integral in Eq. (88) is logarithmically divergent and must be cut off although we feel that the vertex functions, $\mu(l)$ and $v(l)$ themselves will provide natural cutoffs.

In the approximation of Eq. (88) λ and therefore $\Gamma_{\alpha\beta}/\Gamma_{\alpha\beta}\Gamma_{\beta\delta}\Gamma_{\gamma\delta}$ is positive. This is the same as the condition of attractive crossing matrix elements in the S-matrix approach.

One might expect a static bootstrap to be overdetermined since a nonstatic version of the same theory would be expected to give an equation for the mass of each composite particle. In the static theory, the masses are chosen infinite and only mass differences enter the equations. Hence only if the nonstatic theory gives very large masses will the static theory be approximately consistent. Indeed, the present theory has one more equation in it which overdetermines it. The extra equation arises from the condition that the one-body states be normalized.

$$\langle\alpha|\alpha\rangle=1\,. \tag{89}$$

After all other equations (Cutkosky's approximate equations for example) have been solved giving $v(k)$, $\mu(k)$, and the Γ, consistency can be checked by demanding the equation

$$e_{\alpha\gamma}(k)\langle\gamma,k|\beta\rangle+e_{\alpha\gamma}(k,l)\langle\gamma,k,l|\beta\rangle=\delta_{\alpha\beta}. \tag{90}$$

This is as far as we shall carry the analysis of the static model. We intend to do a complete study of the coupled-nucleon, isobar-meson system in a future paper.

VII. CONCLUSIONS

The main object of the present paper was to introduce some of the ideas necessary to our formulation of the self-consistent composite quantum mechanics of the strong interactions. It seems possible to account for the connection between attractions and particles in this theory via the dynamical and self-consistency equations obtained with the aid of the Hamiltonian of Sec. V.

The final section was by no means meant to be a complete analysis of the static dynamics. Rather it was a brief survey of the kinds of equations which will be encountered in a more ambitious attempt to solve the static baryon, meson problem. The equations are of three kinds.

(1) We encounter the dynamical equations which are derived from the eigenstate character of one-body states.

$$H|\alpha\rangle=M_\alpha|\alpha\rangle.$$

(2) There are the consistency requirements which say that the one-body states are equivalent to superpositions of multibody states.

$$\sum_n e_{\alpha\beta}(k_1\cdots k_n)\langle\beta,k_1\cdots k_n|\gamma,l\cdots\rangle=\langle\alpha|\gamma,l_1,\cdots l_n\rangle.$$

(3) The third set of equations ensures that the one-body states are normalized.

$$\sum_n e_{\alpha\beta}(k_1\cdots k_n)\langle\beta,k_1\cdots k_n|\gamma\rangle=\delta_{\alpha\gamma}.$$

These equations are closely related to equations used by Cutkosky to discuss symmetries and bootstraps.

We have found that in the static limit, the function $\mu(k)$ is not determined by our equations. This is, as far as we can see, not a general situation. The nonstatic theory does not collapse to algebra to the same extent as does the static theory and it may be that no ambiguity such as an arbitrary $\mu(k)$ exists in the full theory. In order to avoid these ambiguities, it seems that one must go to the full theory in which all particles are composite.

Among the questions which must be answered, the following stand out as most important.

(1) Can the theory be generalized, to include all particles as composite and will the generalization contain all the required features of a covariant dynamics of particles?

(2) What is the exact nature of the convergence of the theory so that a consistent approximation method can be obtained?

In particular, we are faced with two seemingly independent series of approximations, the sum over the internal I bubbles in the expressions for matrix elements, and the truncation of the series expressing the single-particle state in multibody states.

We shall answer some of these questions in the next paper. However, we do not as yet have a general scheme of approximation which is convincing.

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