High-Spin Baryons. II. Dynamical Mechanisms for Regge Recurrences of Inelastic Resonances; General Theory*

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Virtual transitions of the type $\pi N_i \rightarrow \rho N_j$, where N_i represents an arbitrary nucleon isobar, are considered as the driving force for Regge recurrences of inelastic resonances. We consider only the s-wave ρN_j configuration. Since most of the low-lying isobars have even parity, we are therefore concerned mainly with odd-parity resonances. Regge trajectories of the latter are a consequence of Regge recurrences of the even-parity states. An important example is given by taking N_i to be the sequence: nucleon $\frac{1}{2}^+$ and its recurrence $\frac{5}{2}^+$ (quantum numbers P_{11} and F_{15} in conventional notation). Isospin and centrifugal-barrier considerations lead directly to the existence of the D_{13}-G_{17} sequence. In this paper the general formalism is set up. Numerical results are given separately. First, however, a simpler derivation is given of the elastic forces due to isobar exchange in the quasistatic approximation. Previous work has shown the relation of these forces to the existence of even-parity Regge trajectories. Then the unitarity relations are set up for helicity amplitudes describing the reactions $N_a\pi \rightarrow N_c\pi$, $N_a\pi \rightarrow N_c\rho$, and $N_a\rho \rightarrow N_c\rho$. Next a general explicit expression is given for the one-pion-exchange (OPE) approximation for $N_a\pi \rightarrow N_c\rho$. Expressions are given for the above amplitudes in several models, in which only the OPE coupling is considered. The pole approximation is used to solve the many-channel N/D equations. Techniques for handling the general isospin problem are developed and their relevance to the present problem discussed.

I. INTRODUCTION

EXPERIMENTS have revealed a fascinating wealth of structure in the spectrum of the excited states of baryons.[1] While no simple dynamical structure yet exists which describes the whole spectrum in a simple unified manner, much insight can be gained from a systematic study of dynamical models. There are many ways in which groups of these particles can be organized. Here we emphasize the systematic features of dynamics leading to Regge recurrences.[2] (Actually, the tool of complex angular momentum has so far proved unnecessary in our program.) In previous work we have analyzed in detail the elastic forces in meson-baryon scattering.[3] There it was shown that the elastic forces in πN scattering all collaborate in such a way as to favor the generation of even-parity Regge recurrences, one series of isospin $\frac{1}{2}$ (originating in the nucleon) having spin parity J^P of $\frac{1}{2}^+$, $\frac{5}{2}^+$, $\frac{9}{2}^+$, $\cdots$ and the second of isospin $\frac{3}{2}$, with J^P equal to $\frac{3}{2}^+$, $\frac{7}{2}^+$, $\frac{11}{2}^+$, $\cdots$. Except for the significant role played by vector-meson exchange, the dynamical framework is a generalization of Chew's reciprocal bootstrap model[4] for the nucleon and its first excited state, the 3-3 isobar. It was also found that the elastic forces in the odd-parity states are relatively small or repulsive, so that other forces must be responsible for the odd-parity resonances.[5]

The main purpose of this paper is to describe likely dynamical mechanisms for inelastic resonances[6]; Regge recurrences arise as naturally for such resonances as in the elastic force case. We shall mainly discuss the πN system, since the extension to strange meson-baryon scattering is easily done by invoking $SU(3)$ symmetry.[3,7] Fortunately, there is to some extent little overlap between channels dominated by interchannel coupling and those with strong *attractive* elastic forces. Therefore, we investigate the idealized case in which only interchannel coupling exists. Calculations including both elastic and inelastic forces are underway.[8]

The characteristic feature of the high-mass resonances is the large number of significantly coupled channels. As a first step towards isolating the portions of these many-particle configurations significant for resonance formation, we make a number of drastic simplifying assumptions. For a first try we isolate the most peripheral inelastic mechanism, namely one-pion-exchange forces. Moreover, the multiparticle configuration—nucleon plus any number of pions—is taken to be represented by either one pion plus a nucleon isobar or one rho meson plus a nucleon isobar. (Here any excited state of the nucleon, including the nucleon itself, is called an isobar.)

Thus we consider the role of the peripheral reactions

$$\pi+N_i \rightarrow \rho+N_j, \tag{1.1}$$

where the N_k are various nucleon isobars, for inelastic resonance formation. The attractive force due to the coupling of πN_i to the virtual ρN_j s-wave configuration

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[1] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **37**, 633 (1965).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1962); R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[3] P. Carruthers, Phys. Rev. Letters **10**, 540 (1963); **12**, 259 (1964); Phys. Rev. **133**, B497 (1964); *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1965), Vol. VIIb, p. 83.

[4] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[5] An exception is the case of the s waves, which requires special attention. We only discuss $\frac{1}{2}^-$ states which arise in coupled inelastic channels involving a centrifugal barrier.

[6] By inelastic resonance, we mean a resonance caused by coupling between unlike channels.

[7] E. Golowich, Phys. Rev. **139**, B1297 (1965).

[8] Nand Lal (private communication).

is used to generate resonances. In the particular case $N_i = N_j = N$, one recovers the model for the "second" πN resonance $D_{13}(1512)$ investigated by many authors.[9] Our work generalizes the general pattern developed in papers by Cook and Lee[10] and Auvil and Brehm.[11]

In Sec. II a new, simpler, derivation is given of the elastic forces due to isobar exchange. The angular momentum crossing matrix in the quasistatic approximation of Ref. 3 is rederived in a much improved fashion. In Sec. III the helicity formalism is used to formulate the requisite coupled-channel unitarity relations connecting the reactions $N_a\pi_b \to N_c\pi_d$, $N_a\pi_b \to N_c\rho_d$, and $N_a\rho_d \to N_c\rho_d$. In Sec. IV explicit expressions are given for the threshold amplitudes for reaction (1.1). For this purpose it is necessary to use expressions for the general pion-baryon-baryon vertex given in Ref. 12. Threshold factors are enforced as in Ref. 10 and expressions for the pole approximation to the left-hand discontinuities are given. Expressions are also given for the near-threshold s-wave ρ production cross sections in a channel of given J^P and isospin. Section V contains resonance conditions in the pole approximation for various two- and three-channel models. In the three-channel problem, a criterion is given to determine whether the elastic phase shift goes through 0° or 90° at resonance. Section VI is given to a discussion of general features leading to Regge behavior. Numerical results and further discussion are given in another paper.[13]

The Regge recurrences of the odd-parity resonances can be considered to be a consequence of even-parity Regge recurrences. Thus if the coupling to a closed channel $(\rho N_{J^+})_{L=0}$ gives a resonance $(J+1)^-$, the replacement of N_{J^+} by its recurrence $N_{(J+2)^+}$, $[\rho N_{(J+2)^+}]_{L=0}$ leads to the dominant configuration $(J+3)^-$. (The largest orbital momentum dominates because of the centrifugal barrier effect.) These ideas are considered in more detail in Ref. 13, but one example will show what is involved. The ρN configuration (s-wave) couples to the πN $S_{1/2}$ and $D_{3/2}$ channels. Only the D wave has a centrifugal barrier. Next consider the recurrence configuration ρF_{15}. Of the $\frac{7}{2}^-$, $\frac{5}{2}^-$, $\frac{3}{2}^-$ channels, $\frac{7}{2}^-$ involves the maximum πN orbital angular momentum. Of course, other channels have to be considered, in particular πF_{15}. Here $l=2$ for both $\frac{7}{2}^-$ and $\frac{3}{2}^-$, but the vertex factor suppresses $\frac{3}{2}^-$. The isospin factors are trivial, and one is automatically led to expect a D_{13}, G_{17} pair as a consequence of the (N,F_{15}) pair (given the existence of the π and ρ mesons.)

[9] Our notation is standard; $L_{2T,2J}$ denotes the following quantum numbers: L is the orbital angular momentum, T the isospin, and J the total angular momentum.

[10] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962); **127**, 297 (1962).

[11] P. Auvil and J. J. Brehm, Phys. Rev. **140**, B135 (1965); **145**, 1243 (1966).

[12] P. Carruthers, Phys. Rev. **152**, 1345 (1966). This paper is referred to as I in the present series.

[13] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **18**, 297 (1967).

II. ELASTIC FORCES IN PION-NUCLEON SCATTERING

In previous papers[3] we have discussed the systematic features of elastic forces due to exchange of various mesons and nucleon isobars which are favorable to the generation of even-parity Regge recurrences originating in the nucleon $(T,J^P) = (\frac{1}{2},\frac{1}{2}^+)$, $(\frac{1}{2},\frac{5}{2}^+)$, $\cdots$ and the 3-3 resonance $(\frac{3}{2},\frac{3}{2}^+)$, $(\frac{3}{2},\frac{7}{2}^+)$, $\cdots$. While the majority of the even-parity resonances seem to be accounted for by this scheme, the elastic forces are much smaller (or quite repulsive) in the odd-parity states. Hence other channels must be involved in these odd-parity states. To be sure, the inelastic channels are not negligible for the even-parity states, but the elastic forces themselves are adequate to produce resonances in these states, except perhaps for very high energies and spin values. Actually the successes of $SU(6)$ suggest that to some extent the distinction between elastic and inelastic is artificial and that a simple description obtains only when all particles in the $SU(6)$ 56- and 35-dimensional multiplets are treated on an equivalent footing. For the time being, however, we retain the more traditional detailed attack, assuming at most $SU(3)$ symmetry.

The main purpose of this paper is to supplement the previous analysis by an exploration of *inelastic* dynamical mechanisms and their Regge systematics. However, we want to report here a much simpler derivation of the structure of the angular momentum s-u channel crossing matrix than was previously given. The approximations involved in this procedure become progressively worse for higher energies and higher mass and spin of the exchanged particles. Beginning with a fixed-energy dispersion relation, the absorptive part is replaced by the imaginary part of the amplitude which in turn is approximated by a truncated partial-wave expansion. Since this expansion formally diverges, one has to appeal to the Regge-pole hypothesis to justify the procedure. But thus far no one really knows at what point the suppression of high spin states sets in. Roughly speaking, we are treating the exchanged particles as elementary. (Even in this context, there are well-known ambiguities in the field-theoretic description of high-spin-exchange forces.)

Despite all these difficulties, the pattern of forces revealed by our admittedly inaccurate crossing matrix exhibits a simple pattern in remarkable qualitative correspondence with the experimental situation. In the previous work, we studied the fixed-momentum-transfer dispersion relations. We show here that in the *quasi-static approximation* the results thereby obtained are the same in the more simple fixed-energy dispersion relations. We also are able to give a simple explicit formula. The structure is different when the orbital angular momentum l' of the exchanged baryon state is less than or greater than that of the partial wave of interest.

For $l' \leq l$, we previously gave[3] the following expression for the Born term in partial wave $h_{jl}(W)=e^{i\delta}\sin\delta/k^{2l+1}$ due to exchange of a baryon state with orbital and total angular momentum l' and j':

$$h_{jl}{}^B(W)=\tfrac{1}{2}W_r\gamma_r X_{ll'}{}^{jj'}k^{-2|l-l'|-2}Q_{|l-l'|}(y), \quad (2.1)$$

where $\gamma_r=\Gamma_r/k_r{}^{2l+1}$ is the reduced width, $X_{ll'}{}^{jj'}$ the crossing matrix in question, and Q the Legendre function of the second kind. y is defined as $1+(M_r{}^2-u_0)/2k^2$, where M_r is the mass of the exchanged particle, k the c.m. momentum, and $u_0=(M^2-\mu^2)^2/4s$. Finally M and μ are the nucleon and pion masses and $s=W^2$ is the squared total c.m. energy. Note that h^B approaches a constant as W approaches the threshold $M+\mu$. The essential approximation involved in deriving (2.1) is the neglect of the pion energy with respect to $2M$. The main advantage of the approximate form is the transparent exposure of systematic features of the exchange forces leading to Regge trajectories.

From the fixed-energy dispersion relations, we can compute the Born amplitude $h_{l\pm}$ in ($j=l\pm\frac{1}{2}$) due to exchange of a baryon isobar with $j'=l'\pm\frac{1}{2}$ (see, for example, Golowich[7])

$$\begin{aligned} h_{l\pm}{}^B=-\frac{\Gamma_r M_r(-1)^{l\pm1}}{2k_r k^{2l+2}}\{&[D_{11}(W,M_r)P'_{l'\pm1}(x_u) \\ &-D_{12}(W,M_r)P'_{l'}(x_u)]\times Q_l(y) \\ &-[D_{21}(W,M_r)P'_{l'\pm1}(x_u) \\ &-D_{22}(W,M_r)P'_{l'}(x_u)]Q_{l\pm1}(y)\}. \quad (2.2)\end{aligned}$$

In Eq. (2.2) the plus and minus signs need to be explained. Every $\pm$ symbol *except* that in $Q_{l\pm1}$ refers to whether $j'=l'\pm\frac{1}{2}$ for the exchanged particle. (Q_{l+1} goes with h_{l+} and Q_{l-1} with h_{l-}.) The "angle" x_u, the crossed cosine, is given by $x_u=1+t_r(s)/2k_r{}^2$, where $t_r(s)=2(M^2+\mu^2)-s-M_r{}^2$, and k_r is the c.m. momentum for $s=M_r{}^2$. The elements of the matrix D are given by

$$\mathbf{D}(W,W')=\frac{1}{2W}\begin{bmatrix} \left(\dfrac{E+M}{E'+M}\right)(2M+W'-W) & \left(\dfrac{E+M}{E'-M}\right)(2M-W'-W) \\ -\left(\dfrac{E-M}{E'+M}\right)(2M+W'+W) & \left(\dfrac{E-M}{E'-M}\right)(W'-W-2M) \end{bmatrix}, \quad (2.3)$$

where E is the nucleon energy, $E=(W^2+M^2-\mu^2)/2W$.

The passage to the quasistatic limit is simple but somewhat delicate. First replace the nucleon kinetic energy by its nonrelativistic value $E=M+k^2/2M$. Then write $W=M+\omega$, where ω is the pion energy. Neglecting ω with respect to $2M$, Eq. (2.3) then reduces to

$$\mathbf{D}\cong\begin{bmatrix} 1 & -2M(\omega+\omega')/(k')^2 \\ -k^2/2M & -k^2/(k')^2 \end{bmatrix}. \quad (2.4)$$

In this same limit x_u is

$$1-2M(\omega+\omega_r)/2k_r{}^2\approx -M(\omega+\omega_r)/k_r{}^2$$

and y is

$$2M(\omega+\omega_r)/2k^2-1\approx M(\omega+\omega_r)/k^2.$$

Hence x_u is equal to $-\xi y$, where $\xi\equiv k^2/k_r{}^2$. We then further write (2.4) in the form

$$\mathbf{D}\cong\begin{bmatrix} 1 & -2\xi y \\ 0 & -\xi \end{bmatrix}. \quad (2.5)$$

For these approximations to be reliable, the energy W and resonance energy M_r have to be such that $|x_u(s)|$ and y are substantially greater than unity. (Roughly speaking k_r must be rather less than $M=6.72\ \mu$.) In this domain we can use the asymptotic formulas

$$P_n(x)\sim\frac{(2n)!}{2^n(n!)^2}x^n, \quad (2.6)$$

$$Q_m(y)\sim\frac{(m!)^2 2^m}{(2m+1)!}y^{-m}Q_0(y). \quad (2.7)$$

$Q_0(y)$ is equal to $\frac{1}{2}\ln[(y-1)/(y+1)]$. Straightforward manipulation then leads to the following expressions:

$$h_{l+}{}^B=A_l\times\begin{cases} 1, & j'=l'+\frac{1}{2} \\ 2l', & j'=l'-\frac{1}{2} \end{cases}$$

$$A_l\equiv\frac{(-1)^l\Gamma_r M_r}{2k_r k^{2l+2}}\frac{(2l')!}{2^{l'}(l'!)^2}(x_u)^{l'}Q_l(y), \quad (2.8)$$

$$h_{l-}{}^B=\frac{A_l}{l}\times\begin{cases} 2ll'+l+l', & j'=l'+\frac{1}{2} \\ -l', & j'=l'-\frac{1}{2}. \end{cases} \quad (2.9)$$

For $l'\leq l$ these expressions can be conveniently reduced to the form (2.1) with the quasistatic crossing matrix given by the expression

$$\mathbf{X}_{ll'}=(-1)^{l-l'}\binom{l}{l'}^2\binom{2l+1}{2l'}^{-1} \times\begin{pmatrix} -l'/l & (2ll'+l+l')/l \\ 2l' & 1 \end{pmatrix}. \quad (2.10)$$

For $l\geq l'$, the expressions $\binom{a}{b}$ are usual combinatorial factors. More generally, if $l'>l$ we mean the following:

$$\binom{l}{l'}=\frac{l!}{(l')!(|l-l'|)!},$$

$$\binom{2l+1}{2l'}=\frac{(2l+1)!}{(2l')!(2|l-l'|+1)!}. \tag{2.11}$$

The rows and columns are in increasing order $j=l-\frac{1}{2}$, $j=l+\frac{1}{2}$ and $j'=l'-\frac{1}{2}$, $j'=l'+\frac{1}{2}$, respectively. Explicit numerical values of the crossing matrix were given in Refs. 3 and 4. For $l'>l$ the matrix structure of (2.10) is the same but the over-all factor is a little different from that given in Eq. (2.1).

The crossing matrix (2.10) is to be multiplied by the factor $[M(\omega+\omega_r)]^{2(l'-l)}$ for $l'>l$. This factor is typically large but is compensated for by three factors: (a) the mass M_r in the Q function is heavier (the larger l' the larger M_r) leading to a greater centrifugal barrier; (b) the quasistatic approximation tends to be worse for $l'>l$ than for $l>l'$ because in the former case k_r^2 is large and ω small in comparison to the latter case; (c) Regge behavior tends to damp the effect of high-spin exchanges.

Equation (2.10) has to be supplemented by the isospin crossing matrix connecting the s and u channels:

$$X_{su}=\frac{1}{3}\begin{pmatrix}-1 & 4\\ 2 & 1\end{pmatrix}. \tag{2.12}$$

This is a special case of (2.10), with $l=l'=1$. The same ordering conventions are used.

It will be noticed that the s-u channel crossing matrices are mainly off diagonal. This circumstance has a simple physical origin. The physical (s) channel and the crossed (u) channel are connected by turning the pions into antiparticles. Thus the pion isospin (orbital-angular-momentum) points in the "opposite" direction relative to the nucleon isospin (spin) when viewed from the crossed channel. The more classical the situation, the more nearly perfect the rule becomes [note that when l and l' are large and of the same order of magnitude, the diagonal elements of (2.10) are of order $1/l$ relative to the off-diagonal elements].

These results can be summarized in the following simple rule: The force due to the exchange of an isobar with $T=\frac{1}{2}(\frac{3}{2})$ and $j'=l'\pm\frac{1}{2}$ is by far the strongest in states of $T=\frac{3}{2}(\frac{1}{2})$ and $j=l\mp\frac{1}{2}$. For $l-l'=0, 2, 4, \cdots$ this strongest force is attractive and for $l-l'=1, 3, 5, \cdots$ it is repulsive. The consequences of this rule for Regge recurrences have been thoroughly discussed in Ref. 3.

The meson exchange forces (especially ρ-meson exchange) are also significant elastic forces. The qualitative effect of these forces has been discussed by the author and by Hamilton and collaborators.[3,14] The result is that ρ exchange gives a large attraction in $T=\frac{1}{2}$, $J=l-\frac{1}{2}$. A slight attraction occurs for $T=\frac{3}{2}$, $J=l+\frac{1}{2}$, and a slight repulsion in $T=\frac{1}{2}$, $J=l+\frac{1}{2}$. Repulsion occurs in $T=\frac{3}{2}$, $J=l-\frac{1}{2}$. Thus in the even-parity states, ρ exchange adds attractive forces to the baryon exchange forces, while it mainly cancels out repulsive or small attractive forces because of baryon exchange in the odd-parity states.

Thus far, no attempt has been made to convert the "forces" just discussed into actual amplitudes to be confronted with experiment. (See, however, the successful but more conservative program of Ref. 14.) The reason for this neglect is the absence of any reliable, unambiguous, method for this. The usual N/D methods involve many notorious difficulties,[15] the latter becoming more unruly as energy and spin increase. However, the author has checked that these elastic forces are quite strong enough to produce the desired resonances when conventional threshold factors and cutoffs are used.

In the rest of the paper, we actually construct unitary amplitudes for given inelastic forces. These numerical results naturally suffer from at least as many ambiguities as those for the elastic problem. However, only numerical results can give a feeling for the nature of the mechanisms and sensitivity to various parameters. No great faith should be put in the precise numbers obtained.

III. UNITARITY CONDITION FOR THE MULTICHANNEL $N_a\pi, N_b\rho$ SYSTEM

Consider the reaction, baryon $a+$ meson $b\rightarrow$ baryon $c+$ meson d. The actual particles of interest will often be unstable and possess high spin. The spins and helicities of the particles will be denoted by (s_a,s_b,s_c,s_d) and $(\lambda_a,\lambda_b,\lambda_c,\lambda_d)$. The invariant amplitude $\langle cd|M|ab\rangle$ defined in terms of the S matrix by

$$\langle cd|S|ab\rangle=\delta_{ac}\delta_{bd}-(2\pi)^4 i\delta(p_a+p_b-p_c-p_d)\times\left(\frac{M_aM_c}{4E_aE_c\omega_b\omega_d}\right)^{1/2}\langle cd|M|ab\rangle \tag{3.1}$$

obeys the unitarity condition[16,17]

$$[\langle cd|M|ab\rangle]=\sum_{ef}\pi\rho_{fe}(s)\int d\Omega'\times\sum_{\lambda_e\lambda_f}\langle cd|M|ef\rangle\langle ab|M|ef\rangle^*, \tag{3.2}$$

where the invariant phase space is given by

$$\rho_{ab}(s)=\frac{2M_a}{4(2\pi)^3}\frac{p_{ab}(s)}{s^{1/2}}\theta(s-(M_a+\mu_b)^2). \tag{3.3}$$

[14] A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964).

[15] L. M. Simmons, Jr., Phys. Rev. **144**, 1157 (1966).
[16] J. D. Bjorken, Phys. Rev. Letters **4**, 474 (1960).
[17] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

M_a is the fermion mass and μ_b the boson mass. The baryon energies are E_a, E_c and the meson energies ω_b, ω_d. The c.m. momentum $p_{ab}(s)$ depends on the square of the c.m. energy s as follows:

$$p_{ab}^2=[s-(M_a+\mu_b)^2][s-(M_a-\mu_b)^2]/4s. \quad (3.4)$$

In Eq. (3.2), the angular integration goes over the direction $\hat{p}_e$ of the baryon e in the c.m. system. The symbol F means the energy discontinuity $[F(s+i\epsilon)-F(s-i\epsilon)]/2i$. For problems involving fermions, we shall often use $W=s^{1/2}$ as the appropriate variable. We shall apply these equations in an approximate way to describe unstable particles. The degree to which this is appropriate can be surmised from a comparison with the work of Cook and Lee.[10] In this regard it should be mentioned that when energy variables other than s occur the general unitarity condition is expressed as a discontinuity in s, the other energy variables remaining fixed.

The unitarity condition takes on a simpler form for amplitudes of definite angular momentum J. The geometry is described in Fig. 1. The over-all scattering occurs in the x-z plane, with scattering angle θ. Baryons a, c, and e have momenta $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{p}''$. From the work of Jacob and Wick,[18] we have the expansions

$$M_{cd,ef}=\frac{1}{4\pi}\sum_{JM}(2J+1)d_{M\lambda'}{}^J(\theta)\times\langle\lambda_c\lambda_d|M_J|\lambda_e\lambda_f\rangle d_{M\lambda''}{}^J(\theta')e^{-i(M-\lambda'')\phi'},$$
$$M_{ab,ef}{}^*=\frac{1}{4\pi}\sum_{J}(2J+1)d_{M\lambda''}{}^J(\theta')e^{i(M-\lambda'')\phi'}\times\langle\lambda_a\lambda_b|M_J|\lambda_e\lambda_f\rangle^*. \quad (3.5)$$

Inserting these expressions into Eq. (3.2), we find

$$[\langle\lambda_c\lambda_d|M_J|\lambda_a\lambda_b\rangle]=\sum_{ef}\pi\rho_{ef}\langle\lambda_c\lambda_d|M_J|\lambda_e\lambda_f\rangle\times\langle\lambda_a\lambda_b|M_J|\lambda_e\lambda_f\rangle^*. \quad (3.6)$$

The amplitude $\langle\lambda_c\lambda_d|M_J|\lambda_a\lambda_b\rangle$ is calculated from the invariant amplitude by means of the formula

$$\langle\lambda_c\lambda_d|M_J|\lambda_a\lambda_b\rangle=2\pi\int d\cos\theta\times d_{\lambda\lambda'}{}^J(\theta)\langle cd|M|ab\rangle, \quad (3.7)$$

where $\lambda=\lambda_a-\lambda_b$, $\lambda'=\lambda_c-\lambda_d$, and the reaction $ab\to cd$ is oriented to be in the x-z plane.

For practical purposes it is useful to re-express (3.6) in terms of amplitudes of definite parity. Recall that for πN elastic scattering, M_J is a 2×2 matrix with two independent components $M_{J++}(=M_{J--})$ and M_{J+-} $(=M_{J-+})$. The unitarity condition is "diagonalized" by introducing the parity amplitudes

$$M_{(l+1)-}=M_{J++}+M_{J+-},\quad P=(-1)^{J-1/2}; \quad (3.8)$$
$$M_{l+}=M_{J++}-M_{J+-},\quad P=(-1)^{J+1/2},$$

[18] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

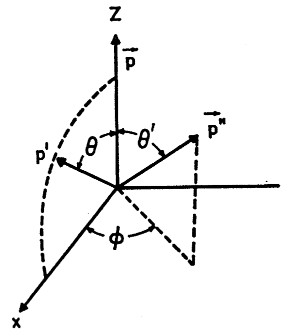

FIG. 1. The geometry of the scattering process $N_a\pi_b\to N_c\rho_d$ is shown. The z axis defines the initial direction. The scattering plane is the x-z plane.

where $l=J\pm\frac{1}{2}$ for $M_{l\mp}$ gives the "orbital" angular momentum. The J^P πN amplitudes $M_{l\pm}$ are then given by the standard relations

$$M_{l\pm}=\frac{e^{2i\delta_{l\pm}}-1}{2\pi i\rho},\quad S_{l\pm}=e^{2i\delta_l}. \quad (3.9)$$

We now wish to extend our considerations to include the reactions

$$N_a\pi_b\to N_c\pi_d, \quad (3.10)$$
$$N_a\pi_b\to N_c\rho_d, \quad (3.11)$$
$$N_a\rho_b\to N_c\rho_d, \quad (3.12)$$

where N_i is an arbitrary nucleon isobar regarded as a stable particle. According to Jacob and Wick, the parity operation has the following effect on the two-particle state:

$$P|JM\lambda_a\lambda_b\rangle=\eta_a\eta_b(-1)^{J-s_a-s_b}|JM-\lambda_a-\lambda_b\rangle. \quad (3.13)$$

Here η^i denotes the intrinsic parity of particle i. The $N_a\pi$ states with parity $(-1)^{J\pm1/2}$ are accordingly

$$[|Jm\lambda_a\rangle\pm\eta_a(-1)^{s_a-1/2}|Jm-\lambda_a\rangle]/2^{1/2}. \quad (3.14)$$

For illustration, note that η_a is 1 when N_a is the nucleon $(\frac{1}{2},\frac{1}{2}^+)$ or the 3-3 isobar $(\frac{3}{2},\frac{3}{2}^+)$ and -1 when N_a is the 1512-MeV resonance $(\frac{1}{2},\frac{3}{2}^-)$. There are $(2s_a+1)/2$ independent functions of type (3.14).

Using reflection invariance we find the J^P amplitude for $N_a\pi_b\to N_c\pi_d[P=(-1)^{J\pm1/2}]$:

$$\langle Jm\lambda_c|M|Jm\lambda_a\rangle\pm\eta_c(-1)^{s_c-1/2}\times\langle Jm-\lambda_c|M|Jm\lambda_a\rangle. \quad (3.15)$$

Next consider the $N_a\rho_b$ state with parity $(-1)^{J\mp1/2}$:

$$[|Jm\lambda_a\lambda_b\rangle\pm\eta_a(-1)^{s_a-1/2}|Jm-\lambda_a-\lambda_b\rangle]/2^{1/2}. \quad (3.16)$$

All the independent functions of type (3.16) can be obtained by keeping $\lambda_a>0$ and letting λ_b run over its three values. We obtain $3(2s_a+1)/2$ independent states. (Note that for the ρN system $\eta=-1$.)

The transition amplitude for parity $(-1)^{J\mp1/2}$ in $N_a\pi_b\to N_c\rho_d$ is now computed from Eqs. (3.14) and (3.16) to be

$$\langle Jm\lambda_c\lambda_d|M|Jm\lambda_a\rangle\pm\eta_c(-1)^{s_c-1/2}\times\langle Jm-\lambda_c-\lambda_d|M|Jm\lambda_a\rangle, \quad (3.17)$$

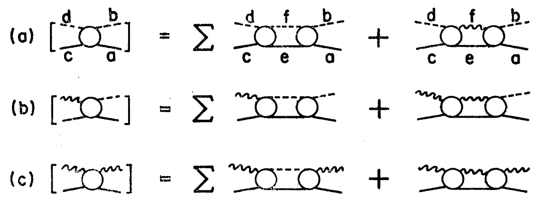

FIG. 2. Graphical representation of the unitarity conditions. Solid lines denote nucleon isobars, wiggly lines ρ mesons, and dashed lines π mesons.

while the transition amplitude for parity $(-1)^{J\mp 1/2}$ in the reaction $N_a\rho_b \to N_c\rho_d$ is

$$\langle Jm\lambda_c\lambda_d|M|Jm\lambda_a\lambda_b\rangle \pm \eta_c(-1)^{s_c-1/2} \times\langle Jm-\lambda_c-\lambda_d|M|Jm\lambda_a\lambda_b\rangle. \quad (3.18)$$

For given J and P there are respectively, $(s_a+\frac{1}{2})$-$(s_c+\frac{1}{2})$, $3(s_a+\frac{1}{2})(s_c+\frac{1}{2})$, and $9(s_a+\frac{1}{2})(s_c+\frac{1}{2})$ independent transition amplitudes for the reactions $N_a+\pi_b\to N_c+\pi_d$, $N_a+\pi_b\to N_c+\rho_d$, and $N_a+\rho_b\to N_c+\rho_d$. All these amplitudes will be coupled to each other by the unitarity condition, as indicated in Fig. 2.

From Eq. (3.6) for the unitarity condition we can derive a very similar unitarity condition connecting the independent J^P amplitudes enumerated above. The demonstration is elementary but rather lengthy. To show what is involved we sketch the procedure for $N_a\pi_b\to N_c\pi_d$ [Fig. 2(a)]. Clearly all independent amplitudes, which can be labeled by a variable $\zeta=1, 2, \cdots, (s_a+\frac{1}{2})(s_c+\frac{1}{2})$, are obtained if we restrict λ_a and λ_c in Eq. (3.15) to positive values. We restrict attention to intermediate states of the type allowed by reactions (3.10)–(3.12). First consider the $N_e\pi_f$ intermediate states. Let $\xi_a=\lambda_a$ for positive λ_a, $\xi_b=\lambda_b$ for positive λ_b. The amplitude (3.15) can be briefly denoted by $\langle\lambda_b|M|\lambda_a\rangle$. According to Eq. (3.6), the discontinuity due to $N_e\pi_f$ intermediate states is for $P=(-1)^{J+1/2}$

$$[\langle\xi_c|M|\xi_a\rangle_{J^P}]=\pi\rho_{ef}\sum_{\lambda_e}\{\langle\xi_c|M|\lambda_e\rangle\langle\xi_a|M|\lambda_e\rangle^* +\eta_c(-1)^{s_c-1/2}\langle-\xi_c|M|\lambda_e\rangle\langle\xi_a|M|\lambda_e\rangle^*\}. \quad (3.19)$$

The sum on λ_e is now split into separate parts according to positive or negative values of λ_e:

$$\begin{aligned}[\langle\xi_c|M|\xi_a\rangle_{J^P}]&=\pi\rho_{ef}\sum_{\xi_e}\{[\langle\xi_c|M|\xi_e\rangle\langle\xi_a|M|\xi_e\rangle^*+\eta_c(-1)^{s_c-1/2}\langle-\xi_c|M|\xi_e\rangle\langle\xi_a|M|\xi_e\rangle^*]\\&\qquad+[\langle\xi_c|M|-\xi_e\rangle\langle\xi_a|M|-\xi_e\rangle^*+\eta_c(-1)^{s_c-1/2}\langle-\xi_c|M|-\xi_e\rangle\langle\xi_a|M|-\xi_e\rangle^*]\}\\&=\pi\rho_{ef}\sum_{\xi_e}\{\langle\xi_c|M|\xi_e\rangle+\eta_c(-1)^{s_c-1/2}\langle-\xi_c|M|\xi_e\rangle\}\times\{\langle\xi_a|M|\xi_e\rangle+\eta_a(-1)^{s_a-1/2}\\&\qquad\times\langle-\xi_a|M|\xi_e\rangle\}^*;\end{aligned} \quad (3.20)$$

$$[\langle\xi_c|M|\xi_a\rangle_{J^P}]=\pi\rho_{ef}\sum_{\xi_e}\langle\xi_c|M|\xi_e\rangle_{J^P}\langle\xi_a|M|\xi_e\rangle_{J^P}^*. \quad (3.21)$$

To obtain the ρ_dN_c intermediate-state contribution, first note that the independent $N_c\rho_d\to N_a\pi_b$ amplitudes can be chosen to be

$$\langle\xi_a|M|\xi_c\lambda_d\rangle\mp\eta_c(-1)^{s_c-1/2}\langle\xi_b|M|-\xi_c-\lambda_d\rangle\equiv\langle\xi_a|M|\xi_c\lambda_d\rangle_{J^P} \quad (3.22)$$

for parity $(-1)^{J\pm1/2}$, where the ξ_i are all non-negative.

A calculation similar to that of Eq. (3.20) gives for the $P=(-1)^{J\pm1/2}$ contribution to the discontinuity

$$\pi\rho_{ef}\sum_{\xi_e\lambda_f}\{\langle\xi_c|M|\xi_e\lambda_f\rangle\mp\eta_e(-1)^{s_e-1/2}\langle\xi_c|M|-\xi_e-\lambda_d\rangle\}\{\langle\xi_a|M|\xi_e\lambda_f\rangle\mp\eta_e(-1)^{s_e-1/2}\langle\xi_a|M|-\xi_e-\lambda_f\rangle\}^*. \quad (3.23)$$

In this expression the phase space is of course different from that in Eq. (3.21). We therefore find the unitarity condition

$$\frac{1}{\pi}[\langle\xi_c|M|\xi_a\rangle_{J^P}]=\sum_{\xi_e}\rho_{ef}(\pi N_e)\langle\xi_c|M|\xi_e\rangle_{J^P}\langle\xi_a|M|\xi_e\rangle_{J^P}^*+\sum_{\xi_e\lambda_f}\rho_{ef}(\rho N_e)\langle\xi_c|M|\xi_e\lambda_f\rangle_{J^P}\langle\xi_a|M|\xi_e\lambda_f\rangle_{J^P}^*. \quad (3.24)$$

Here the sum on index e is supposed to run over all nucleon isobars N_e.

The unitarity relations corresponding to Figs. 2(b) and 2(c) are quite similar. It is necessary to note that all the independent J^P amplitudes of Eq. (3.18) ($\langle\xi_a\lambda_b|M|\xi_c\lambda_d\rangle_{J^P}$) are obtained by keeping the isobar helicities positive in the first amplitude while the ρ helicities run over all possible values. Then the discontinuity of the $N\pi\to N\rho$ J^P amplitude is

$$\frac{1}{\pi}[\langle\xi_c\lambda_d|M|\xi_a\rangle_{J^P}]=\sum_{\xi_e}\rho(\pi N_e)\langle\xi_c\lambda_d|M|\xi_e\rangle_{J^P}\langle\xi_a|M|\xi_e\rangle_{J^P}^*+\sum_{\xi_e\lambda_f}\rho(\rho N_e)\langle\xi_c\lambda_d|M|\xi_e\lambda_f\rangle_{J^P}\langle\xi_a|M|\xi_e\lambda_f\rangle_{J^P}^*. \quad (3.25)$$

Finally, we write down the discontinuity for the $N_a\rho_b\to N_c\rho_d$ amplitudes,

$$\frac{1}{\pi}[\langle\xi_c\lambda_d|M|\xi_a\lambda_b\rangle]=\sum_{\xi_e}\rho(\pi N_e)\langle\xi_c\lambda_d|M|\xi_e\rangle_{J^P}\langle\xi_a\lambda_b|M|\xi_e\rangle_{J^P}^*+\sum_{\xi_e\lambda_f}\rho(\rho N_e)\langle\xi_c\lambda_d|M|\xi_e\lambda_f\rangle_{J^P}\langle\xi_a\lambda_b|M|\xi_e\lambda_f\rangle_{J^P}^*. \quad (3.26)$$

The extension to completely general spins of the colliding particles should now present no problem to the reader.

In order to utilize the unitarity condition, one has to know something about the analyticity of the various amplitudes. What little is known of the analytic properties of production amplitudes, which our quasi-two-body amplitudes are supposed to approximate, is quite discouraging. However, the great success of $SU(3)$ seems to indicate that the stability of a particle is perhaps of little relevance. We shall assume that such questions can be answered by analytic continuation in the masses from a domain in which no complex singularities or anomalous thresholds occur. In such a region it is quite plausible that the helicity amplitudes obey a Mandelstam representation once the kinematical singularities have been removed.[19,20] At present all questions of construction of singularity free amplitudes, threshold subtractions, and asymptotic behavior should be examined in this domain. It should be stressed that in actual applications we shall be concerned with specific models which may have a significance independently of the questions of the existence of the full analytic structure presumed in a double-dispersion approach. In any event we cannot wait for a satisfactory solution of the many-body problem in strong interactions. A successful exploratory study may in fact aid in the construction of such a theory.

IV. ONE-PION-EXCHANGE (OPE) CONTRIBUTION TO $N_a\pi_b \to N_c\rho_d$

We now wish to consider a specific set of "forces" (left-hand singularities of the partial-wave amplitudes) due to the virtual transition $N_a\pi_b \to N_c\rho_d$ (Fig. 3). The multichannel N/D method used to satisfy the unitarity condition of Sec. II, given a definite set of forces, is discussed at the end of Sec. IV. In this section we consider the general structure of the OPE amplitude of Fig. 3. The main work is involved in the analysis of the general pion-baryon-baryon vertex (Fig. 4), which was reported in I.[12] Although there are $s+\frac{1}{2}$ independent on-mass-shell $\pi B_a B_b$ vertex functions[21] where s is the lesser spin of B_a and B_b, we keep only that vertex structure associated with the lowest multipole in the Breit frame. The additional vertices involve the transfer of at least two more units of angular momentum and will be ignored here. We describe the kinematical behavior of high-spin baryons by Rarita-Schwinger wave functions.[22,23] Effects due to the instability of the actual particles of interest are not considered. The structure of the vertex was found to depend on the relative γ

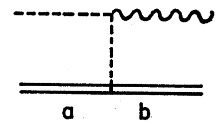

FIG. 3. Peripheral (OPE) production of rho mesons is shown. a and b label the initial and final nucleon isobars, respectively.

parity of the two baryons. When a baryon of angular momentum J decays into a pion of orbital momentum l, we find that γ is given by $(-1)^{J+l-1/2}$. If the initial baryon has spin $s_k=k+\frac{1}{2}$, γ-parity γ and the final baryon spin $s_k'=k'+\frac{1}{2}$, γ-parity γ', the effective interaction Hamiltonian density has the form

$$\mathcal{H}(x)=\mu^{k-k'}g_{k'k}{}^{\gamma'\gamma}\bar{\psi}^{\mu_1\cdots\mu,k'}(x)\Gamma\mathbf{T}\psi_{\mu_1\mu_2\cdots\mu_k}(x) \times\partial_{\mu_{k+1}}\cdots\partial_{\mu_{k'}}\pi(x)+\text{H.c.}. \quad (4.1)$$

The μ_i are four-vector indices for the Rarita-Schwinger field operators. μ is the pion mass and $g_{k'k}$ is a dimensionless coupling constant. When the relative γ parity $\gamma_R=\gamma\gamma'$ is $+1$ the Dirac matrix Γ is $i\gamma_5$; if $\gamma_R=-1$, then Γ is the unit matrix. [Equation (4.1) has been written in a form appropriate to the case $s_k'\geqslant s_k$.] The rectangular vector matrix $\mathbf{T}$ converts the bilinear form in the baryon fields to an isovector as discussed in I. In the spherical basis, the elements of the three matrices $T^{(\rho)}$, $\rho=\pm1, 0$, are defined by

$$T^{\rho}{}_{\nu\mu}=(2T'+1)^{1/2}C(T'1T;\nu\rho\mu). \quad (4.2)$$

The isospin of the "initial" baryon (spin s_k) is called T; that of the "final" baryon (spin $s_{k'}$) is called T'. When $T=T'$, the matrices T_i are proportional to the ordinary isospin matrices I_i.

As explained in Refs. 10 and 11 it is appropriate and convenient to study the threshold production amplitude. This kinematical position dominates the virtual production process (as shown in Paper IV) and also makes the calculation of helicity and J^P amplitudes elementary, provided one uses the results of I.[12] There it was found that when $\gamma_R=+1$, the invariant pion vertex between baryons of spin s and $s+n$ (neglecting the isospin factor T_i) is given by

$$\Gamma_+(s+n,s;\quad p_0'\lambda'p\lambda)=(-ip)^n i\sinh\tfrac{1}{2}\omega(-1)^{\lambda'+1/2}\epsilon_{s\lambda}{}^{+} \times d_{\lambda'\lambda}{}^{s+n}(\theta)\prod_{k=1}^{n}C(s+k-1,\,1,\,s+k;\,\lambda 0) \quad (4.3)$$

when the final baryon (spin $s+n$) is at rest. p is the three-momentum of the initial baryon (spin s) and ω is given by $p=M_s\sinh\omega$ where M_s is the mass of the baryon having spin s. λ and λ' are the initial and final helicities. The C factors are Clebsch-Gordan coefficients,

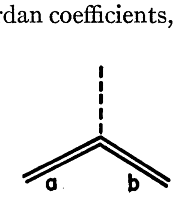

FIG. 4. The general pion-baryon-baryon vertex is shown.

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[19] Y. Hara, Phys. Rev. **136**, B507 (1964).
[20] L-L. C. Wang, Phys. Rev. **142**, 1187 (1966).
[21] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).
[22] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[23] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1955).

in the convention of Rose.[24] The d functions are standard rotation matrices.[24]

For odd relative γ parity, the vertex is

$$\Gamma_-(s+n,\, s;\, p_0'\lambda' p\lambda) = (-ip)^n \cosh\tfrac{1}{2}\omega\, (-1)^{\lambda'-\lambda}\epsilon_{s\lambda}^- \times d_{\lambda'\lambda}^{s+n}(\theta)\prod_{k=1}^{n} C(s+k-1,\, 1,\, s+k;\, \lambda 0)\,. \quad (4.4)$$

The angle θ is that between the z axis (initial baryon direction $\mathbf{p}$) and that of the final baryon (in the x-z plane, the latter then taken to zero momentum (see Fig. 1).

The $\epsilon_{s\lambda}^{\pm}$ are polynomials in the variable $c=E/M_s$. Quoting from I, we find

$$\begin{aligned}
&\epsilon_{1/2,1/2}^{\pm}=1\,,\\
&\epsilon_{3/2,3/2}^{\pm}=\pm 1\,,\\
&\epsilon_{3/2,1/2}^{\pm}=\pm\tfrac{1}{3}(1\mp 2c)\,,\\
&\epsilon_{5/2,5/2}^{\pm}=1\,,\\
&\epsilon_{5/2,3/2}^{\pm}=\tfrac{1}{5}(1\mp 4c)\,,\\
&\epsilon_{5/2,1/2}^{\pm}=\tfrac{1}{5}(1\mp 2c+2c^2)\,,\\
&\epsilon_{7/2,7/2}^{\pm}=\pm 1\,,\\
&\epsilon_{7/2,5/2}^{\pm}=\pm\tfrac{1}{7}(1\mp 6c)\,,\\
&\epsilon_{7/2,3/2}^{\pm}=\pm\tfrac{1}{7}(1\mp 2c+4c^2)\,,\\
&\epsilon_{7/2,3/2}^{\pm}=\pm(1/35)(3\mp 12c+12c^2\mp 8c^3)\,.
\end{aligned} \quad (4.5)$$

To compute the amplitude of Fig. 3 we now need to define the $\rho\pi\pi$ coupling

$$\mathcal{H}_{\rho\pi\pi}=f\epsilon_{ijk}\rho_i^{\mu}\pi_j\partial_\mu\pi_k\,. \quad (4.6)$$

If the final ρ has isospin index m and the initial pion isospin index n, the invariant OPE amplitude is

$$\langle p_c\lambda_c p_d\lambda_d m|M|p_a\lambda_a p_b n\rangle=\Gamma_{\gamma'\gamma}(s_{k'}s_k;\, p_c\lambda_c p_a\lambda_a) \times\frac{2i\epsilon_{mnl}T_l f g_{k'k}{}^{\lambda'\lambda} p_b\cdot e(p_d,\lambda_d)^*}{t-\mu^2}\,, \quad (4.7)$$

where e is the polarization four-vector of the outgoing ρ meson.

In order to investigate the role of this mechanism in resonance formation, we examine the J^P content of (4.7) at the threshold for ρN_c formation. The computation of the partial-wave projections is especially simple at this point because (1) the propagator is independent of $\cos\theta$ at threshold, and (2) the remaining factors are simply proportional to the Jacobi polynomials.

For brevity we write Eqs. (4.3)–(4.4) in the form

$$\begin{aligned}
\Gamma_+(s's;\, p_0'\lambda' p\lambda)&=(-1)^{\lambda'+1/2}E_{s's\lambda}^{+}d_{\lambda'\lambda}^{s'}(\theta)\,, \quad (4.8)\\
\Gamma_-(s's;\, p_0'\lambda' p\lambda)&=(-1)^{\lambda'-\lambda}E_{s's\lambda}^{-}d_{\lambda'\lambda}^{s'}(\theta)\,.
\end{aligned}$$

[24] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

The angular momentum amplitudes defined by Eq. (3.7) involve the integrals

$$\begin{aligned}
\int d\cos\theta\quad & d_{\lambda_a\lambda_c-\lambda_d}{}^{J}(\theta)d_{\lambda_c\lambda_a}{}^{s'}(\theta)d_{0\lambda_d}{}^{1}(\theta)\\
&=\tfrac{2}{3}(-1)^{\lambda_d}C(Js'1;\, \lambda_a-\lambda_a)C(Js'1;\, \lambda_c-\lambda_d,\, -\lambda_c)\\
&\equiv(-1)^{\lambda_d}I(Js';\, \lambda_c\lambda_d\lambda_a)\,. \quad (4.9)
\end{aligned}$$

The J^P amplitudes for parity $(-1)^{J\mp 1/2}$ are found from Eq. (3.17) to be

$$\begin{aligned}
M_+{}^J&=M_0E_{s_cs_a\lambda_a}{}^{+}(-1)^{\lambda'+1/2}I(Js_c;\, \lambda_c\lambda_d\lambda_a)\times(1\pm\eta_c(-1)^{J+1/2})\,, \quad (4.10)\\
M_-{}^J&=M_0E_{s_cs_a\lambda_a}{}^{-}(-1)^{\lambda'-\lambda}I(Js_c;\, \lambda_c\lambda_d\lambda_a)\times(1\pm\eta_c(-1)^{J+1/2})\,,
\end{aligned}$$

with $\lambda'=\lambda_c'+\lambda_d'$, $\lambda=\lambda_a$. The subscripts denote the relative γ parity, and M_0 is

$$M_0=-4\pi ip\epsilon_{mnl}(T_l f g_{k'k}{}^{\gamma'\gamma}/(t_0-u^2)) \quad (4.11)$$

when t_0 is the value of t at threshold.

The function of the common factor $1\pm\eta_c'(-1)^{J+1/2}$ is to enforce the obvious consequences of conservation of parity and angular momentum. At threshold the ρN_c orbital angular momentum is zero, and so $J=s_c+1$, s_c, s_c-1 (except when $s_c=\frac{1}{2}$). If $\eta_c'=+1$ these states have negative parity. If $\eta_c'=-1$, the nonvanishing amplitudes have positive parity. The factor in question equals 2 in the allowed configurations and zero otherwise.

It will be noticed that the dependence of the amplitudes on the helicity quantum numbers factors into parts *separately* dependent on the initial and the final sets of helicities necessary to specify these states. This factorization of the amplitude is crucial for solution of the N/D equations. The factorization is exhibited by

$$\begin{aligned}
M_{\gamma R}{}^J&=\tfrac{2}{3}M_0[1\pm\eta_c(-1)^{J+1/2}]\\
&\times[(-1)^{\lambda'+1/2)}C(Js_c1;\, \lambda_c-\lambda_d,\, -\lambda_c)]\\
&\times\begin{cases}E_{s_cs_a\lambda}{}^{+}C(Js_c1;\, \lambda_a-\lambda_a)\,, & \gamma_R=+1\\ (-1)^{\lambda+1/2}E_{s_cs_a\lambda}{}^{-}C(Js_c1;\, \lambda_a-\lambda_a)\,, & \gamma_R=-1\,.\end{cases} \quad (4.12)
\end{aligned}$$

The isospin content can be made explicit by use of the projection operators P_T for $N_a\pi_b\to N_c\rho_d$:

$$i\epsilon_{mnl}T_l=\sum_T c_T(P_T)_{mn}\,. \quad (4.13)$$

Using well-known symmetry properties of the Clebsch-Gordan coefficients, one sees that the dependence of the threshold amplitudes on the helicities can be specified by giving the values of the independent J^P state labels ζ, ζ'.

In order to enforce the threshold behavior in the solutions to the N/D equations, we multiply by the threshold factors g_l for a πN state ζ containing minimum orbital angular momentum l, and G_L for the ρN

state of minimum orbital momentum L. Thus the amplitude $F_{lL}{}^{JT}$, hopefully devoid of kinematical singularities, is given by[10]

$$(F_{lL}{}^{JT})_{\zeta'\zeta}=g_l{}^{1/2}M_{\zeta'\zeta}{}^{JT}G_L{}^{1/2}. \tag{4.14}$$

The nonuniqueness of the threshold factors and the attendant ambiguities have been mentioned earlier. We add nothing to the problem here. For purposes of comparison to previous work we have mainly used the factors of Cook and Lee:

$$\begin{aligned} g_l&=\frac{2M_a}{E_a+M_a}\left(\frac{E_a+M_a}{p_{ab}}\right)^{2l},\\ G_L&=\frac{2M_c}{E_c+M_c}\left(\frac{E_c+M_c}{p_{cd}}\right)^{2L}. \end{aligned} \tag{4.15}$$

As discussed in the Introduction, we restrict our attention in this paper to virtual excitation of the $N_c\rho_d$ configuration. Thus only $L=0$ will enter our considerations.

We now must subject our amplitudes to the brutal approximations required to obtain manageable equations. The discontinuity $F_{\zeta'\zeta}$ is approximated by a single pole at W_1:

$$[F_{\zeta'\zeta}]=\pi\alpha_{\zeta'}{}^*\beta_\zeta\delta(W-W_1), \tag{4.16}$$

where W_1 is chosen at or near the threshold $M_a+\mu_b$. Since the actual singularity is not a simple pole, the residue in

$$F_{\zeta'\zeta}=\frac{\alpha_{\zeta'}{}^*\beta_\zeta}{W_1-W}, \tag{4.17}$$

which presumably represents $F_{\zeta'\zeta}$ below threshold, cannot be computed from $-F_{\zeta'\zeta}'(W_1)$. A very convenient procedure is to match (4.17) with the OPE amplitude, assumed to be the dominant force, at the production threshold $W_2=M_c+\rho_d$. The decisive virtue of this choice is the simplicity of the evaluation of the OPE diagram at threshold. In addition, it is even sensible because the threshold kinematical conditions are in the range of momenta expected to be most important. We therefore obtain

$$\alpha_{\zeta'}{}^*\beta_\zeta=-(W_2-W_1)F_{\zeta'\zeta}(W_2). \tag{4.18}$$

Comparison with Eqs. (4.12) and (4.14) indicates that α_ζ, and β_ζ can be chosen as follows:

$$\alpha_{\zeta'}{}^*=-(W_2-W_1)\frac{4M_0{}^T}{3}(g_lG_0)^{1/2}(-1)^{\lambda'+1/2} \times C(Js_c1;\lambda_c-\lambda_d,-\lambda_c), \tag{4.19}$$

$$\beta_\zeta=\begin{cases} E_{s_cs_a\lambda}{}^+C(Js_c1;\lambda-\lambda), & \gamma_R=1\\ (-1)^{\lambda+1/2}E_{s_cs_a\lambda}{}^-C(Js_c1;\lambda-\lambda), & \gamma_R=-1, \end{cases} \tag{4.20}$$

$$M_0{}^T=-4\pi c_Tpfg_{k'k}{}^{\gamma'\gamma}/(t_0-\mu^2). \tag{4.21}$$

The resonance conditions and cross-section expressions involve the quantities

$$\alpha^2\equiv\sum_{\zeta'}|\alpha_{\zeta'}|^2,\quad \beta^2\equiv\sum_{\zeta}|\beta_\zeta|^2. \tag{4.22}$$

The sum on ζ may be explicitly performed for α, noting that

$$\textstyle\sum_{\zeta'}\rightarrow\frac{1}{2}\sum_{\lambda_c\lambda_d},$$

$$\alpha^2=\frac{128\pi^2}{3}\left(\frac{W_2-W_1}{t_0-\mu^2}\right)^2c_T{}^2f^2(g_{k'k}{}^{\gamma'\gamma})^2p^2g_l(W_2). \tag{4.23}$$

Note that $G_0(W_2)$ is unity. β^2 has to be dealt with case by case:

$$\beta^2=\sum_\zeta[E_{s_cs_a\lambda}{}^\pm C(Js_c1;\lambda-\lambda)]^2. \tag{4.24}$$

In the especially interesting case where $s=\frac{1}{2}$, there is only one value of ζ and

$$\beta^2=\tfrac{1}{2}|E_{s_c1/2}{}^\pm|^2=\tfrac{1}{2}p_0{}^{2s_c-1}[(s+\tfrac{1}{2})!/(2s)!] \times[\sinh^2(\tfrac{1}{2}\omega_0)\text{ or }\cosh^2(\tfrac{1}{2}\omega_0)],$$

where ω_0 is the value of E/M for the initial baryon at the final state threshold.

It is also simple to obtain a general formula for the OPE cross section near threshold, by using the real phase space and the threshold value of the transition amplitude. One obtains thereby the usual s-wave rise, with σ proportional to the ρ_dN_c momentum.

Denoting the maximum inelastic cross section $2\pi(J+\frac{1}{2})/p^2$ by $\sigma_{J^P\max}$, we have for the transition $\zeta'\zeta$

$$\bar{\sigma}_{\zeta'\zeta}\equiv\frac{\sigma_{J^P\zeta'\zeta}}{\sigma_{J^P\max}}\cong\frac{p_{ab}p_{cd}M_aM_c}{(8\pi^2W)^2}|M_{\zeta'\zeta}{}^{J^P}|^2. \tag{4.25}$$

For each J^P we have to sum over the independent final ζ' amplitudes and average over the initial ζ variables. The calculation is similar to those leading Eqs. (4.23) and (4.24), and yields

$$\bar{\sigma}_{J^P}\cong\frac{4}{3}\frac{p_{ab}p_{cd}}{2s_a+1}\frac{M_aM_c(M_0{}^T)^2}{(8\pi^2W)^2} \sum_{\lambda_a}|E_{s_cs_a\lambda}{}^\pm|^2\times C^2(Js_c1;\lambda_a-\lambda_a). \tag{4.26}$$

Note that when $s=\frac{1}{2}$, the final sum can be performed with the result that $\bar{\sigma}_{J^P}$ is *independent* of J. Thus for $\pi N\rightarrow\rho N$ near threshold, $\bar{\sigma}$ is the same for the $D_{3/2}$ and $S_{1/2}$ amplitudes. Of course, the total cross sections are weighted by the factor $(2J+1)$.

V. RESONANCE CONDITIONS FOR "PURELY INELASTIC" MODELS

At present, it is clearly out of the question to take explicit account of the enormous number of coupled channels. However, one may hope to isolate the most significant factors, even at the present primitive stage

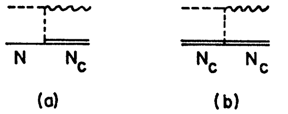

FIG. 5. The direct coupling of the πN channel to the $N_c\rho$ configuration is shown in (a). In (b) the important inelastic amplitude $N_c\pi \to N_c\rho$ is shown.

of dynamical theory. For the present, we are concerned with effects noticeable in experiments in which pions hit nucleons. This criterion excludes many resonances that might occur in many-particle systems (represented here as $N_a\pi_b$, $N_c\rho_d$, etc.) which are for various reasons weakly coupled to the πN system. Those channels coupled to the πN system may be further classified according to the character of the elastic, or potential, forces in the relevant channel. According to the analysis of Sec. II, substantial cancellation of elastic forces occurs in the odd-parity states (D, G, $\cdots$ waves) so that coupling to other channels is expected to be of decisive importance here. For the even-parity recurrences of the nucleon and its first excited state, the situation is more complicated owing to the presence of strong attractive "elastic" forces. However, in this case the "peripheral inelastic" mechanisms are mostly suppressed by parity-angular momentum considerations, since the s-wave (ρN_c) configuration has odd parity when N_c has even parity.

For clarity, we mention briefly a few examples. In D_{13} the elastic forces are rather small, so that coupling to ρN (virtual) and πN^* (real) are most important. The (virtual) ρN channel is assumed to be most important for resonance formation, although the state decays largely through the πN^* channel. In D_{33} the situation is similar except that the isospin factors are such as to make the OPE ρ production smaller, and comparable to the nucleon pole terms in the $\pi N \to \rho N$ amplitude. In D_{15} there is a residual weak long-range attraction due to nucleon exchange. One expects this attraction to lower somewhat the energy of the resonance found by Auvil and Brehm to occur in the channel $\pi N^* \to \rho N^*$. In D_{35} there is similarly a long-range weak repulsion. In the F states, the situation is rather different. In F_{15} and F_{37} a resonance is already in the making because of the strong elastic forces. (The partial waves F_{17} and F_{35} have insignificant elastic forces, leaving them open to domination by inelasticity.) It is of some interest, however, to investigate in detail the influence of the channels ρN and ρN^*, whose thresholds lie at the energies of the F_{15} and F_{37} resonances, respectively.

In this paper we consider those channels in which the elastic forces are not large. That is, we avoid the recurrence states of the nucleon and $N^*(1238)$.

In accordance with the preceding discussion we consider three different "purely inelastic" models: In model A there are two channels, $N\pi$ and $N_c\rho$, coupled by OPE [Fig. 5(a)]; in model B there are two channels $N_a\pi$ and $N_a\rho$, where the dominant process is OPE in an "inelastic" channel $N_a\pi \to N_c\rho$ [Fig. 5(b)]; and model C in which all three channels, $N\pi$, $N_c\rho$, and $N_a\pi$, are coupled by both of the processes shown in Fig. 5. Of course model C includes A and B as special cases, but it is often revealing and always simpler to solve a two-channel problem.

Auvil and Brehm[25] have given explicit solutions to a three-channel problem slightly more general than model C in that off-diagonal transitions of the type $N\pi \to N_c\pi$ were allowed. However, it is useful to give the solutions separately. For model A, denoting the $N\pi$ channel by subscript 1 and the $N_c\rho$ channel by 2, the N/D solution in the pole approximation of Sec. V is

$$F_{11}=\frac{\alpha^2\mathcal{G}_2(W)}{(W-W_1)[1-\alpha^2\mathcal{G}_1(W)\mathcal{G}_2(W)]}, \tag{5.1}$$

$$F_{21}{}^{\zeta}=\frac{-\alpha_\zeta{}^*}{(W-W_1)[1-\alpha^2\mathcal{G}_1(W)\mathcal{G}_2(W)]}, \tag{5.2}$$

$$F_{22}{}^{\zeta'\zeta}=\frac{\alpha_{\zeta'}{}^*\alpha_\zeta\mathcal{G}_1(W)}{(W-W_1)[1-\alpha^2\mathcal{G}_1(W)\mathcal{G}_2(W)]}, \tag{5.3}$$

$$\mathcal{G}_1(W)=(W-W_1)\int_{W_1}^{\infty}\frac{\rho_1'(W')dW'}{(W'-W)(W'-W_1)^2},$$

$$\rho_1'=\rho_{ab}/g_l, \tag{5.4}$$

$$\mathcal{G}_2(W)=(W-W_1)\int_{W_2}^{\infty}\frac{\rho_2'(W')dW'}{(W'-W)(W'-W_1)^2},$$

$$\rho_2'=\rho_{cd}/G_0. \tag{5.5}$$

The resonance condition is

$$1=\alpha^2\,\mathrm{Re}[\mathcal{G}_1(W_r)\mathcal{G}_2(W_r)]. \tag{5.6}$$

This result satisfies the various physical and mathematical assumptions only when W_r lies between W_1 and W_2. In this case, only $\mathcal{G}_1$ is complex. The results (5.1)–(5.6) can be regarded as special cases of the solutions given by Cook and Lee, who allowed the ρ to have a finite width. The effect of the zero-width approximation can be seen clearly in the work of Goldberg and Lomon.[26] They show that when the width is decreased to zero, the D_{13} resonance width narrows slightly and the resonance position increases about 30 MeV. Such small refinements are not pertinent to the present panoramic survey, and hence will be neglected.

For model B, the solution for the discontinuity of Eq. (3.14) is given by

$$F_{11}{}^{\zeta'\zeta}=\frac{\alpha_{\zeta'}{}^*\alpha_\zeta\beta^2\mathcal{G}_2(W)}{(W-W_1)[1-\alpha^2\beta^2\mathcal{G}_1(W)\mathcal{G}_2(W)]}, \tag{5.7}$$

$$F_{21}{}^{\zeta'\zeta}=\frac{\alpha_{\zeta'}{}^*\beta_\zeta}{(W-W_1)[1-\alpha^2\beta^2\mathcal{G}_1(W)\mathcal{G}_2(W)]}, \tag{5.8}$$

[25] P. Auvil and J. J. Brehm, Ann. Phys. (N. Y.) 34, 505 (1965).
[26] H. Goldberg and E. L. Lomon, Phys. Rev. 134, B659 (1964).

$$F_{22}{}^{\zeta'\zeta}=\frac{\beta_{\zeta'}{}^*\beta_\zeta\alpha^2\mathcal{J}_1(W)}{(W-W_1)[1-\alpha^2\beta^2\mathcal{J}_1(W)\mathcal{J}_2(W)]}. \quad (5.9)$$

The integrals $\mathcal{J}_1$ and $\mathcal{J}_2$ are formally the same as previously except that $W_1=M_a+\mu_b$ and $W_2=M_c+\mu_d$. The subscripts 1 refer to the $N_a\pi_b$ channel and 2 to the $N_c\rho_d$ channel.

The resonance condition for model B is

$$1=\alpha^2\beta^2\,\mathrm{Re}[\mathcal{J}_1(W_r)\mathcal{J}_2(W_r)]. \quad (5.10)$$

Next, consider model C. The discontinuities are (the $N_c\pi$ channel is called channel 3)

$$[F_{21}{}^{\zeta'\zeta}]=\pi\alpha_{\zeta'}{}^*\beta_\zeta\delta(W-W_1), \quad (5.11)$$
$$[F_{23}{}^{\zeta'\zeta}]=\pi\alpha_{\zeta'}{}^*\gamma_\zeta\delta(W-W_1).$$

The singularities of both amplitudes have been located at the same energy for computational convenience. The solution to the N/D equations for model C involving channel 1 are

$$F_{11}{}^{\zeta'\zeta}=\frac{-\beta_{\zeta'}{}^*\beta_\zeta\alpha^2\mathcal{J}_2(W)}{(W_1-W)[1-\alpha^2\beta^2\mathcal{J}_1\mathcal{J}_2-\alpha^2\gamma^2\mathcal{J}_2\mathcal{J}_3]},$$
$$F_{21}{}^{\zeta'\zeta}=\frac{\alpha_{\zeta'}{}^*\beta_\zeta}{W_1-W}\,\frac{1}{1-\alpha^2\beta^2\mathcal{J}_1\mathcal{J}_2-\alpha^2\gamma^2\mathcal{J}_2\mathcal{J}_3}, \quad (5.12)$$
$$F_{31}{}^{\zeta'\zeta}=\frac{\gamma_{\zeta'}{}^*\beta_\zeta}{W_1-W}\,\frac{1}{1-\alpha^2\beta^2\mathcal{J}_1\mathcal{J}_2-\alpha^2\gamma^2\mathcal{J}_2\mathcal{J}_3}.$$

$\mathcal{J}_3$ is defined in analogy to $\mathcal{J}_1$ and $\mathcal{J}_2$, with the lower limit of integration being the πN_c threshold. The resonance condition is a simple generalization of that found for models A and B:

$$1=\alpha^2\beta^2\,\mathrm{Re}\mathcal{J}_1(W_r)\mathcal{J}_2(W_r)+\alpha^2\gamma^2\,\mathrm{Re}\mathcal{J}_2(W_r)\mathcal{J}_3(W_r). \quad (5.13)$$

Defining a function F by the right-hand side of Eq. (5.13), one finds the width Γ to be given by

$$\tfrac{1}{2}\Gamma=\mathrm{Im}F(W_r)/[d\,\mathrm{Re}F(W_r)/dW_r]; \quad (5.14)$$

$$\Gamma=\frac{2\pi[\rho_1'(W_r)\theta(W_r-W_1)+\rho_3'(W_r)\theta(W_r-W_3)]\mathcal{J}_2(W_r)}{(W_r-W_1)\{d\,\mathrm{Re}[\alpha^2\beta^2\mathcal{J}_1\mathcal{J}_2+\alpha^2\gamma^2\mathcal{J}_2\mathcal{J}_3]/dW_r\}}, \quad (5.15)$$

where the step functions involve the thresholds $W_1=M+\mu$ and $W_3=M_c+\mu$. (We have assumed $\mathcal{J}_2$ real, i.e., $W_r<W_2=M_c+\rho$.)

It should be realized that these expressions cannot be expected to give an accurate value for the width. Experience shows that the widths are typically too narrow. One reason for this is the neglect of other coupled channels; while a given mechanism such as that under study may be mainly responsible for the *existence* of a resonance the actual decay may occur via other modes which broaden the "natural" width. Another reason, sometimes more significant than the former, is that the real situation to which we must compare our results involves unstable particles. The validity of describing resonances by unstable particles has not been established. It would seem that the actual width would be at least as great as that of the constituent particles (in open channels) making up the resonance. For instance, in model B of Fig. 5(b), we expect the total width to exceed that of unstable baryon N_c. In model C, where the resonance is shared among the $N\pi$ and $N_c\pi$ channels we expect a smaller width depending on the fraction of the time the resonance belongs to the $N_c\pi$ channel.

We now discuss model C, which is more realistic than its specializations A and B except when both baryons are nucleons. The relative importance of the $N\pi$ and $N_c\pi$ channels determines whether the elastic phase shift passes through 0° or 90° at resonance. Employing the usual phase shift and absorption parameter η, we write the J^P πN amplitude in question as

$$F_{11}=(\eta e^{2i\delta}-1)/2\pi i\rho_l'. \quad (5.16)$$

If $\delta=0$ at resonance,

$$2\pi\rho_l'\,\mathrm{Im}F_{11}(W_r)=1-\eta\leq 1 \quad (5.17)$$

while if $\delta=\pi/2$ at resonance,

$$2\pi\rho_l'\,\mathrm{Im}F_{11}(W_r)=1+\eta\geqslant 1. \quad (5.18)$$

Thus we can tell whether $\delta(W_r)$ is 0 or $\pi/2$ by seeing whether $2\pi\rho_l'\,\mathrm{Im}F_{11}(W_r)$ is less than or greater than unity.

We can calculate $2\pi\rho_l'\,\mathrm{Im}F_{11}$ from Eq. (5.12) with β independent of ζ for nucleons (thus $\beta^2=\beta_\zeta{}^2$). We find

$$2\pi\rho_l'\,\mathrm{Im}F_{11}(W_r)=\frac{2\beta^2\rho_1'(W_r)}{\beta^2\rho_1'(W_r)+\gamma^2\rho_3'(W_r)}. \quad (5.19)$$

Thus we find the criteria

$$\beta^2\rho_1'(W_r)<\gamma^2\rho_3'(W_r),\quad \delta(W_r)=0 \quad (5.20)$$
$$\beta^2\rho_1{}^1(W_r)>\gamma^2\rho_3'(W_r),\quad \delta(W_r)=\pi/2.$$

In each case the absorption parameter is

$$\eta=\left|\frac{\beta^2\rho_1'(W_r)-\gamma^2\rho_3'(W_r)}{\beta^2\rho_1'(W_r)+\gamma^2\rho_3'(W_r)}\right|. \quad (5.21)$$

The physics behind the criterion of Eq. (5.20) is fairly clear; the measure of the relative importance of the two channels is the product of the strength parameter squared (e.g., β^2) and the reduced phase space available.

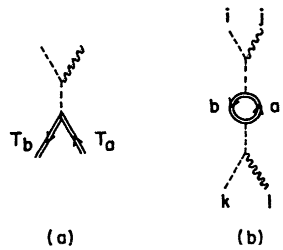

FIG. 6. The annihilation amplitude $N_bN_a \to \pi_i\rho$ is shown in (a). The squared amplitude (b) gives the isospin-one $\pi\rho \to \pi\rho$ amplitude; this condition is used to determine a normalization constant.

VI. GENERAL FEATURES OF THE MODEL

Consider first the isospin content of the graph of Fig. 3. The matrix amplitude is proportional to $i\epsilon_{ijk}T_k$, where T_k is the transition isospin matrix leading from T_a to T_b [cf. Eq. (4.1)] according to the coupling $\bar{\psi}_{T_b}\mathbf{T}\psi_{T_a}\cdot\boldsymbol{\pi}+$H.c. This matrix can be resolved into isospin projection operators by using crossing symmetry. Note that this matrix represents pure isospin 1 in the crossed channel $\bar{N}_bN_a \to \pi_j\rho_i$ [Fig. 6(a)]. We choose the constant C such that $Ci\epsilon_{ijk}T_k \equiv \mathcal{P}^1$ is the properly normalized projection operator. The magnitude of C is found by satisfying the identity

$$\mathcal{P}^1(\mathcal{P}^1)^\dagger = \mathcal{P}_{\pi\rho}{}^1, \tag{6.1}$$

where

$$\mathcal{P}^1 = \sum_{T_z(T=1)} |\pi\rho T_z\rangle\langle N_{T_a}\bar{N}_{T_b}T_z| \tag{6.2}$$

is the usual projection operator, and $\mathcal{P}_{\pi\rho}{}^1$ is the isospin-1 projection operator for $\pi_i\rho_j \to \pi_k\rho_l$:

$$(\mathcal{P}_{\pi\rho}{}^1)_{kl,ij} = \tfrac{1}{2}\epsilon_{mkl}\epsilon_{mij} = \tfrac{1}{2}(\delta_{ki}\delta_{lj} - \delta_{kl}\delta_{li}). \tag{6.3}$$

We thus require $(\mathcal{P}^1)_{ij} = Ci\epsilon_{ijk}T_k$ and find

$$\begin{aligned}\mathrm{Tr}(\mathcal{P}^1)_{ij}(\mathcal{P}^{1+})_{kl} &= |C|^2\epsilon_{mlk}\epsilon_{mji}\,\mathrm{Tr}T_mT_n{}^+\\ &= \frac{2(2T_a+1)(2T_b+1)}{3}|C|^2(\mathcal{P}_{\pi\rho}{}^1)_{kl,ij}\end{aligned} \tag{6.4}$$

on using the trace properties explained in Ref. 12.

Hence we can write the projection operator for

$$\bar{N}_bN_a \to \pi_j\rho_i$$

as

$$\mathcal{P}_{ij}{}^1 = \left(\frac{3}{2(2T_a+1)(2T_b+1)}\right)^{1/2} - i\epsilon_{ijk}T_k. \tag{6.5}$$

Now we can use the general isospin crossing matrices to express (6.5) in terms of s-channel projection operators. Using the notations of Ref. 27, we find

$$\mathcal{P}_t = \sum_s (X^{-1})_{ts}\mathcal{P}_s, \tag{6.6}$$

s and t being the isospin in the s and t channels, respectively. Explicit crossing matrices $(X^{-1})_{ts}$ have been tabulated in Ref. 27 for many isospin values of interest. We have finally

$$-i\epsilon_{ijk}T_k = \sum_s \left(\frac{2(2T_a+1)(2T_b+1)}{3}\right)^{1/2}(X^{-1})_{1s}(\mathcal{P}_s)_{ij}. \tag{6.7}$$

For the trivial case $T_a = T_b = \frac{1}{2}$, $\mathbf{T} = (\frac{2}{3})^{1/2}\boldsymbol{\tau}$ and (6.7) give the well-known result,

$$\mathcal{P}_{ij}{}^1 = \tfrac{1}{2}i\epsilon_{ijk}\tau_k = (\mathcal{P}_{1/2})_{ij} - \tfrac{1}{2}(\mathcal{P}_{3/2})_{ij}. \tag{6.8}$$

This expression shows the dominance of isospin $\frac{1}{2}$ over $\frac{3}{2}$ by 4:1 in squared amplitude.

For $T_a = T_b = \frac{3}{2}$ we have

$$i\epsilon_{ijk}T_k = 4(5/3)^{1/2}(-\tfrac{1}{2}\mathcal{P}_{1/2} - \tfrac{1}{5}\mathcal{P}_{3/2} + \tfrac{3}{10}\mathcal{P}_{5/2})_{ij}. \tag{6.9}$$

In terms of the usual isospin-$\frac{3}{2}$ matrix $I = (15)^{1/2}\mathbf{T}/4$, we have

$$i\epsilon_{ijk}I_k = (-\tfrac{5}{2}\mathcal{P}_{1/2} - \mathcal{P}_{3/2} + \tfrac{3}{2}\mathcal{P}_{5/2})_{ij}, \tag{6.10}$$

which shows clearly the dominance of isospin $\frac{1}{2}$.

For $T_a = \frac{1}{2}$, $T_b = \frac{3}{2}$ we find

$$i\epsilon_{ijk}T_k = \frac{2}{3^{1/2}}(\mathcal{P}_{1/2})_{ij} + \left(\frac{10}{3}\right)^{1/2}(\mathcal{P}_{3/2})_{ij}. \tag{6.11}$$

Thus when the isospin flips, $T = \frac{3}{2}$ is favored by $\frac{5}{2}$ in squared amplitude.

The ratios of various isospin amplitudes in general are easily found by reading down the isospin-1 column of the crossing matrix $(X^{-1})_{ts}$. (When the baryon isospin does not change, the lowest isospin dominates.)

When the standard matrices T_k are used, the largest coefficients $|C_T|^2$ of Sec. IV are, for $(T_a,T_b) = (\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, and $(\frac{1}{2},\frac{3}{2})$, respectively, 8/3, 20/3, and 10/9.

In paper IV of this series[13] we study the actual numerical value of various parameters which determine the size of the various transition amplitudes. The important point to be made here is that isospin $\frac{1}{2}$ is favored when the isobar isospin does not change, and that when it does change, isospin $\frac{3}{2}$ is favored when the isospin changes from $\frac{1}{2}$ to $\frac{3}{2}$.

Although a full discussion is best given with the detailed numerical computations,[13] a few remarks are in order about spin and parity. In analyzing a given problem, first focus on the final ρN_j s-wave configuration. The relevant spin-parity values are $(s_j \pm 1)^-$, s_j^- for $s_j > \frac{1}{2}$. Next consider the various πN_i configura-

[27] P. Carruthers and J. P. Krisch, Ann. Phys. (N. Y.) 33, 1 (1965). In Table I of this reference the matrices X_{ts} and $X^{-1}{}_{st}$ for the processes $N\pi \to N\pi$ and $N\bar{N} \to \pi\pi$ should be multiplied by -1.

tions (of lesser mass then ρN_j) which connect significantly to the ρN_j configuration in question. (Here recall the isospin factors discussed above.) For a given transition $\pi N_i \to \rho N_j$ that one with the greatest $l(\pi N_i)$ will dominate because of the centrifugal barrier. (Mathematically this is enforced by the threshold division discussed in Sec. IV.) Generally the highest spin (s_j+1) goes with the greatest l. It is this feature which accounts for the simple recurrence of resonances as s_j is increased by two units. An example was given in the Introduction.

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Dynamical Theory of Strong Interactions*

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The many-body quantum mechanics of a set of (self-consistent) composite particles is developed for use as the basis for a theory of strong interactions. The theory deals only with physical particles which may in general have an extended spatial structure. It is a bootstrap theory in which physical particles are examined in terms of superpositions of the physical multiparticle states of the theory; no auxillary quantities such as bare particles or fundamental local fields are introduced and no question of renormalization is encountered. Many-particle states are constructed which are (many-) three-momentum eigenstates and whose spatial integrity is assured via cluster-decomposition properties. The present theory is a dynamical theory in the sense that there is a Hamiltonian that respects the extended and composite structure of the particles and which, unlike S-matrix theory, allows a system to be studied during the course of its interactions. A drawback of the theory is that it is not manifestly Lorentz-covariant. The present paper deals with the theory in a simplified form in which heavy baryons interact with structureless mesons in the static limit of no baryon recoil. The self-consistent bootstrap dynamics is examined in the lowest order approximation, including some three-body effects. The conventional Born approximation to the scattering amplitude is recovered. The relation between the existence of particles and signs of forces is obtained. In particular the Cutkosky connection between attractive forces and group-theoretic structure is derived.

I. INTRODUCTION

It is possible that the particles of the strong interactions are all composite, each formed from combinations of similar particles. The phenomenological evidence for this is the correlation which exists between the existence of a given particle and the attraction between particles whose total quantum numbers are the same as those of the single particle. This correlation was first pointed out by Chew[1] in his classic work on the structure of the pion-nucleon system. Chew showed that the existence of the nucleon and 3-3 resonance might be accounted for by a self-consistent mechanism in which the exchange of a nucleon would provide the force to cause the resonance and vice versa.

Later papers by Caruthers[2] extended this idea so that not only were the nucleon and 3-3 resonance spanned by the mechanism, but the entire system of baryon octet, baryon decuplet and many of the excited states of these objects could be understood. This and other work, especially by Cutkosky,[3] demonstrated a remarkable interplay between the group-theoretic structure of the strong interactions and the dynamical forces. The group-theoretic structure of the particle couplings seems to guarantee the attractive forces wherever they are needed to bind the particles.[4,5]

The S-matrix methods[6,7] which have been put forward to deal with this system seem to the author to not be ideally suited to formulating the necessary concepts of composite structure. Because the S matrix describes only the asymptotic states of a scattering system, ignoring the internal structure of the particles, the definition of a composite system is very indirect and far removed from physical intuition. One is forced to examine the effects of bound states on the analyticity of the S matrix in the oversimplified two-particle potential-scattering theory[8,9] and to extrapolate these

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