Uniqueness of Mass Formulas in Static Strong-Coupling Theory

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The question of the general mass operator satisfying the unitarity equation in the static strong-coupling theory is examined. For the symmetric scalar-meson theory, the only mass operator allowed by the unitarity equation is proportional to I^2. For the case of symmetric pseudoscalar-meson theory, the question is restricted to the two physically interesting representations only—the nucleon and the hyperon isobar series. The result in both these cases is that only the mass operator proportional to aI^2+bJ^2 is consistent with the unitarity equation.

I. INTRODUCTION

RECENTLY Cook, Goebel, and Sakita[1] derived the Lie group structure of the static strong-coupling theory for the meson-baryon scattering. The basic equations of the strong-coupling theory are given by

$$[A_i,A_j]=0; \tag{1.1}$$

$$\Lambda_{ij}=\sum_k \Lambda_{ik}\Lambda_{kj}; \tag{1.2a}$$

$$\Lambda_{ij}=[A_i,[\mathfrak{M},A_j]]. \tag{1.2b}$$

These are abstract operator equations and operate on the baryon isobar states. Here A_i is the Hermitian current operator of meson i. $\mathfrak{M}$ is a diagonal operator which is proportional to the isobar mass differences: $\lambda^{-2}\mathfrak{M}\propto M-M_0$, where M is the isobar mass operator, M_0 is the degenerate baryon mass, and λ is the coupling parameter. Λ_{ij} is proportional to the scattering amplitude. Equation (1.2a) is the unitarity equation. Implications of Eqs. (1.2) are twofold. For an allowed mass operator they restrict the representations of the group, whereas for a given representation of the group they limit the form of the mass operator. CGS have investigated the first aspect of the problem. In the present paper we investigate the second implication; we begin with a representation[2] and investigate the most general mass operator, considered as an arbitrary function of the Casimir operators of the invariance group K, that satisfies the unitarity equation.

In Sec. II, we first obtain the "reduced unitarity equation" for the group $G=SU(2)_I\times T_3$ of charge-symmetric scalar-meson theory. From the property of the unitarity equation (1.2a), we then obtain a necessary condition on the mass operator. This is a difference equation which can be solved for $\mathfrak{M}$ for a given representation. We generalize the reduced unitarity equation and the necessary condition on the mass operator for the group $\mathcal{G}=[SU(2)_I\otimes SU(2)_J]\times T_9$ appropriate for the pseudoscalar symmetric meson theory.

In Sec. III, we show that for the charge-symmetric scalar-meson theory the only mass operator allowed by the unitarity condition is the one proportional to I^2 alone. This result is obtained for all the representations of the group G. In Sec. IV we study the problem for the pseudoscalar-meson theory. Here we restrict ourselves to the two physically interesting representations only: the nucleon isobar series which is characterized in terms of its isospin-spin content by $I=J=\frac{1}{2},\frac{3}{2},\cdots,\infty$ (part A) and the hyperon isobar series[3] characterized by $J=I\pm\frac{1}{2}=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots,\infty$ (part B). In both these cases we find that the most general mass operator is $\mathfrak{M}=aI^2+bJ^2$.

II. REDUCED UNITARITY EQUATION AND A NECESSARY CONDITION ON THE MASS OPERATOR

We first obtain the reduced unitarity equation and a necessary condition on the mass operator for the group G and then generalize it to the group $\mathcal{G}$.[4]

Let the baryon state be denoted by $|I,\tau\rangle$, where I is the isospin and τ the third component of the isospin. From the commutation relation for the group G [1] we see that the A_α's ($\alpha=+1,0,-1$) transform as vector operators with respect to the isospin group. Using the Wigner–Eckart theorem, we may then write

$$\langle I',\tau'|A_\alpha|I,\tau\rangle=C\begin{pmatrix} I & 1 & I' \\ \tau & \alpha & \tau' \end{pmatrix}A_{I'}{}^{I}, \tag{2.1}$$

where

$$C\begin{pmatrix} I & 1 & I' \\ \tau & \alpha & \tau' \end{pmatrix}$$

are the $SU(2)$ Clebsch–Gordan (CG) coefficients and the $A_{I'}{}^{I}$'s are the reduced matrix elements. The unitarity of the representation demands that

$$A_{I'}{}^{I}=(-)^{I-I'}\left(\frac{2I+1}{2I'+1}\right)^{1/2}A_I{}^{I'*}. \tag{2.2}$$

We call this property the vertex-symmetry relation.[3] We next turn to the unitarity equation itself.

[1] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965); hereafter referred to as CGS.

[2] Subsequently whenever the word representation occurs it should be taken to mean unitary irreducible representation only.

[3] V. Singh and B. M. Udgaonkar, Phys. Rev. **149**, 1164 (1966).

[4] The author is much indebted to Professor V. Singh for pointing out the possibility of obtaining the "reduced unitarity equation" and also for the help in deriving it.

Unitarity equation (1.2a) in the spherical basis reads[5]

$$\Gamma_{\alpha\beta}=\sum_{\mu=-1}^{+1}(-)^{\mu}\Gamma_{\alpha\mu}\Gamma_{-\mu,\beta}\,, \tag{2.3}$$

where $\Gamma_{\alpha\beta}=[A_{\alpha},[\mathfrak{M},A_{\beta}]]$. Operating with $\Gamma_{\alpha\beta}$ on $|I,\tau\rangle$ and using Eq. (2.1), we obtain

$$\Gamma_{\alpha\beta}|I,\tau\rangle=\sum_{I_1=I-1}^{I+1}\sum_{I_2=I_1-1}^{I_1+1}A_{I_1}{}^{I}A_{I_2}{}^{I_1}\left\{C\begin{pmatrix}I&1&I_1\\ \tau&\beta&\tau+\beta\end{pmatrix}C\begin{pmatrix}I_1&1&I_2\\ \tau+\beta&\alpha&\tau+\beta+\alpha\end{pmatrix}[-\mathfrak{M}(I_2)-\mathfrak{M}(I)+\mathfrak{M}(I_1)]\right.$$
$$\left.+C\begin{pmatrix}I&1&I_1\\ \tau&\alpha&\tau+\alpha\end{pmatrix}C\begin{pmatrix}I_1&1&I_2\\ \tau+\alpha&\beta&\tau+\beta+\alpha\end{pmatrix}\mathfrak{M}(I_1)\right\}|I_2,\tau+\beta+\alpha\rangle. \tag{2.4}$$

Note that the order in which the indices α and β occur in the second term in Eq. (2.4) is opposite to that in the first term. We re-express the product of the CG coefficients in the second term so as to bring it into the form of the first term. For this purpose we use the following relation[6]:

$$C\begin{pmatrix}j_1&j_2&j_{12}\\ m_1&m_2&m_1+m_2\end{pmatrix}C\begin{pmatrix}j_{12}&j_3&j\\ m_1+m_2&m-m_1-m_2&m\end{pmatrix}$$
$$=\sum_{j_{23}}(-)^{j_1+j_2+j_3+j}[(2j_{12}+1)(2j_{23}+1)]^{1/2}\begin{Bmatrix}j_1&j_2&j_{12}\\ j_3&j&j_{23}\end{Bmatrix}C\begin{pmatrix}j_2&j_3&j_{23}\\ m_2&m-m_1-m_2&m-m_1\end{pmatrix}C\begin{pmatrix}j_1&j_{23}&j\\ m_1&m-m_1&m\end{pmatrix},$$

where $\{\ \}$ is the standard 6-j symbol. From the above equation one obtains, after using some standard symmetry properties of the CG coefficients, the equation

$$C\begin{pmatrix}I&1&I_1\\ \tau&\alpha&\tau+\alpha\end{pmatrix}C\begin{pmatrix}I_1&1&I_2\\ \tau+\alpha&\beta&\tau+\beta+\alpha\end{pmatrix}$$
$$=\sum_{I'=I-1}^{I+1}(-)^{2I-I_1+I'}[(2I_1+1)(2I'+1)]^{1/2}\begin{Bmatrix}1&I&I_1\\ 1&I_2&I'\end{Bmatrix}C\begin{pmatrix}I&1&I'\\ \tau&\beta&\tau+\beta\end{pmatrix}C\begin{pmatrix}I'&1&I_2\\ \tau+\beta&\alpha&\tau+\beta+\alpha\end{pmatrix}. \tag{2.5}$$

Using Eqs. (2.4) and (2.5), we get

$$\langle I'',\tau+\beta+\alpha|\Gamma_{\alpha\beta}|I,\tau\rangle=\sum_{I_1=I-1}^{I+1}A_{I_1}{}^{I}A_{I''}{}^{I_1}$$
$$\times\left\{C\begin{pmatrix}I&1&I_1\\ \tau&\beta&\tau+\beta\end{pmatrix}C\begin{pmatrix}I_1&1&I''\\ \tau+\beta&\alpha&\tau+\beta+\alpha\end{pmatrix}[-\mathfrak{M}(I'')-\mathfrak{M}(I)+\mathfrak{M}(I_1)]\right\}+\sum_{I_1=I-1}^{I+1}A_{I_1}{}^{I}A_{I''}{}^{I_1}$$
$$\times\left\{\sum_{I'=I-1}^{I+1}(-)^{-I_1-I'}[(2I_1+1)(2I'+1)]^{1/2}\begin{Bmatrix}1&I&I_1\\ 1&I''&I'\end{Bmatrix}C\begin{pmatrix}I&1&I'\\ \tau&\beta&\tau+\beta\end{pmatrix}C\begin{pmatrix}I'&1&I''\\ \tau+\beta&\alpha&\tau+\beta+\alpha\end{pmatrix}\mathfrak{M}(I_1)\right\},$$

where in the second term we have rewritten the phase factor by noting that $2I-I_1+I'+I_1+I'=$ even number. Next, changing the summation variable in the first term from I_1 to I', using the vertex symmetry relation (2.2), and carrying out the changes in the notation, namely, $I_1\to I_2$, $I'\to I_1$, $I''\to I'$ (in this order), we finally obtain

$$\langle I',\tau+\beta+\alpha|\Gamma_{\alpha\beta}|I,\tau\rangle=\sum_{I_1=I-1}^{I+1}C\begin{pmatrix}I&1&I_1\\ \tau&\beta&\tau+\beta\end{pmatrix}C\begin{pmatrix}I_1&1&I'\\ \tau+\beta&\alpha&\tau+\beta+\alpha\end{pmatrix}R(I,I';I_1)\,, \tag{2.6}$$

[5] Note that in the spherical basis

$$\Lambda_{\alpha\beta}=[A_{\alpha}{}^{\dagger},[\mathfrak{M},A_{\beta}]]=(-)^{\alpha}[A_{-\alpha},[\mathfrak{M},A_{\beta}]]=(-)^{\alpha}\Gamma_{-\alpha,\beta}.$$

[6] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 95, Eq. (6.2.6).

where

$$R(I,I';I_1)\equiv\Big[A_{I_1}{}^{I}A_{I_1}{}^{I'*}[-\mathfrak{M}(I')-\mathfrak{M}(I)+\mathfrak{M}(I_1)]+\sum_{I_2=I-1}^{I+1}(-)^{-2I_1}A_{I_2}{}^{I}A_{I_2}{}^{I'*}\times(2I_2+1)\begin{Bmatrix}1 & I & I_2\\ 1 & I' & I_1\end{Bmatrix}\mathfrak{M}(I_2)\Big](-)^{I_1-I'}\left(\frac{2I_1+1}{2I'+1}\right)^{1/2}$$

$$=\Big[A_{I_1}{}^{I}A_{I_1}{}^{I'*}[-\mathfrak{M}(I')-\mathfrak{M}(I)+\mathfrak{M}(I_1)]+\sum_{I_2=I-1}^{I+1}A_{I_2}{}^{I}A_{I_2}{}^{I'*}C_{I_1I_2}{}^{II'}\mathfrak{M}(I_2)\Big](-)^{I_1-I'}\left(\frac{2I_1+1}{2I'+1}\right)^{1/2}. \tag{2.7}$$

Here $C^{II'}\equiv C^{I\,I'}_{I_1I_2}$ is the u-channel–to–s-channel crossing matrix and $C_{I_1I_2}{}^{II'}$ gives the contribution of an exchanged baryon I_2 in the u channel to the baryon I_1 in the s channel in the scattering process (baryon I)+(isovector meson) $\rightarrow$ (baryon I')+(isovector meson). In writing the second equation in (2.7), we have made use of the fact that a crossing matrix is essentially a 6-j symbol (or equivalently a Racah coefficient),[7] and the relationship between the two in the present case is[8]

$$C_{I_1I_2}{}^{II'}=\epsilon(2I_2+1)\begin{Bmatrix}I & 1 & I_2\\ I' & 1 & I_1\end{Bmatrix}, \tag{2.8}$$

where $\epsilon=(-)^{2I+2}=(-)^{2I'+2}$. Using the property that a 6-j symbol is left invariant under any permutation of its columns and the fact that $2(I_1-I)$ is an even number, the second line in Eq. (2.7) follows. Note that $C^{II'}=C^{I'I}$ for the crossing matrices under consideration.

Up to this point there is no simplification introduced by the above procedure. However, simplification enters when we write down the unitarity condition (2.3) with the matrix elements of $\Gamma_{\alpha\beta}$ given by Eqs. (2.6) and (2.7). Taking the matrix elements of Eq. (2.3) and transferring the CG coefficients occurring on the left side on to the right side, we obtain

$$R(I,I'';I_1)\delta_{I_1'I_1}=\sum_{\mu=-1}^{+1}(-)^{\mu}\sum_{I'=I_1-1}^{I_1+1}C\begin{pmatrix}I_1 & 1 & I'\\ \tau+\beta & -\mu & \tau+\beta-\mu\end{pmatrix}C\begin{pmatrix}I' & 1 & I_1'\\ \tau+\beta-\mu & \mu & \tau+\beta\end{pmatrix}R(I',I'';I_1')R(I,I';I_1).$$

Using the appropriate symmetry properties of the CG coefficients in the above equation, we may carry out the summation over μ to obtain

$$R(I,I'';I_1)=\sum_{I'=I_1-1}^{I_1+1}(-)^{I'-I_1}\left(\frac{2I'+1}{2I_1+1}\right)^{1/2}R(I',I'';I_1)R(I,I';I_1).$$

If, now, we define

$$S(I,I';I_1)\equiv(-)^{I'-I_1}\left(\frac{2I'+1}{2I_1+1}\right)^{1/2}R(I,I';I_1),$$

then the unitarity equation reads

$$S(I,I'';I_1)=\sum_{I'=I_1-1}^{I_1+1}S(I,I';I_1)S(I',I'';I_1), \tag{2.9}$$

where

$$S(I,I';I_1)=A_{I_1}{}^{I}A_{I_1}{}^{I'*}[-\mathfrak{M}(I')-\mathfrak{M}(I)+\mathfrak{M}(I_1)]+\sum_{I_2=I-1}^{I+1}A_{I_2}{}^{I}A_{I_2}{}^{I'*}C_{I_1I_2}{}^{II'}\mathfrak{M}(I_2). \tag{2.10}$$

We shall call Eq. (2 9) the "reduced unitarity equation." Note that $S(I,I';I_1)$ is meaningful only when $\Delta(I1I_1)$ and $\Delta(I_11I')$ pertain. Further, note that in the present case $S^*(I,I';I_1)=S(I',I;I_1)$; this follows from the fact, noticed earlier, that $C^{II'}=C^{I'I}$ for the crossing matrices occurring in the problem. Equation (2.9) is a much more manageable form of the unitarity equation.

We next obtain a necessary condition on the mass operator. The unitarity equation (2.3) can be rewritten as

$$\Lambda_{\alpha\beta}=\sum_{\mu}\Lambda_{\alpha\mu}\Lambda_{\mu\beta}, \tag{2.11}$$

where $\Lambda_{\alpha\beta}=[A_\alpha{}^\dagger,[\mathfrak{M},A_\beta]]$. The unitarity condition is a nonlinear equation in the unknown function $\mathfrak{M}$ and is consequently difficult to handle. To obtain a linear equation in $\mathfrak{M}$ we notice that Eq. (2.11) is an idempotency equation for $\Lambda_{\alpha\beta}$. Its eigenvalues are, therefore, 0 or 1 only. Since each index α and β takes only three values (in the scalar theory), the sum of the "eigenvalues" can only be $k=0$, 1, 2, or 3. Here k gives the

[7] See, for instance, J. de Swart, Nuovo Cimento **31**, 420 (1964).

[8] V. Singh and B. M. Udgaonkar (Ref. 3). Also see C. J. Goebel, *Proceedings of the 12th Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965) Vol. 1, p. 255.

number of times the eigenvalue 1 is repeated. Hence, if we construct the "trace" $\sum_\alpha \Lambda_{\alpha\alpha}$, then it can only be k times a unit operator:

$$\sum_\alpha \Lambda_{\alpha\alpha}=k\mathbf{1}\,,\quad k=0,1,2,3. \tag{2.12}$$

This "trace" in the meson indices is still an operator in the baryon representation space. Here $\mathbf{1}$ is the (infinite-dimensional) unit matrix in the isobar space. Let us assume that the mass operator has the form

$$\mathfrak{M}=f(I)/2A^2,$$

where $f(I)$ is an *arbitrary* function of I^2 and $A^2=\sum_\alpha(-)^\alpha A_\alpha A_{-\alpha}$ is the Casimir operator of the group G which can be normalized to unity. Then Eq. (2.12) becomes

$$\sum_\alpha (-)^\alpha \frac{A_\alpha}{(A^2)^{1/2}} f(I) \frac{A_{-\alpha}}{(A^2)^{1/2}} - f(I)=k\mathbf{1}\,,\qquad k=0,1,2,3. \tag{2.13}$$

This is a linear difference equation in $f(I)$ (i.e., in $\mathfrak{M}$). It is a necessary condition on $f(I)$ in order to satisfy the unitarity equation. Since Eq. (2.13) incorporates only a part of the information contained in Eq. (2.11), it yields spurious solutions which may be eliminated by using the unitarity condition. The solution corresponding to $k=0$ gives a vanishing scattering amplitude and we shall not be interested in this case in the following.

Taking the matrix elements of Eq. (2.13) and using Eq. (2.1), the symmetry property

$$C\begin{pmatrix} I & 1 & I_1 \\ \tau & -\alpha & \tau-\alpha \end{pmatrix} = (-)^{\alpha+I_1-I}\left(\frac{2I_1+1}{2I+1}\right)^{1/2} C\begin{pmatrix} I_1 & 1 & I \\ \tau-\alpha & \alpha & \tau \end{pmatrix}$$

and the orthogonality relations of CG coefficients, Eq. (2.13) can be reduced to

$$\sum_{I_1=I-1}^{I+1} (-)^{I_1-I} f(I_1) A_I{}^{I_1} A_{I_1}{}^{I} \left(\frac{2I_1+1}{2I+1}\right)^{1/2} - f(I)=k\,,\qquad k=1,2,3.$$

Using the vertex symmetry relation (2.2), the above equation can be written as

$$\sum_{I_1=I-1}^{I+1} f(I_1)\,|A_I{}^{I_1}|^2 - f(I)=k\,,\quad k=1,2,3. \tag{2.14}$$

Equations (2.9), (2.10), and (2.14) can be immediately generalized to the group $\mathcal{G}\equiv[SU(2)_I\otimes SU(2)_J]\times T_9$. The commutation relations for this group are given by CGS.[9] Let the isobar state be denoted by $|I,J;\tau,m\rangle$, where I and J are the isospin and spin of the isobar, and τ and m are the corresponding third components. This labeling is complete for the representations of $\mathcal{G}$ under consideration. The meson current operators $A_{r\alpha}$ now carry two indices: the Latin index r and the Greek index α representing the spin index and the isospin index, respectively. Since the $A_{r\alpha}$'s transform as vector operators with respect to each of the two $SU(2)$ groups, we have by the Wigner–Eckart theorem

$$\langle I',J';\tau',m'|A_{r,\alpha}|I,J;\tau,m\rangle = C\begin{pmatrix} I & 1 & I' \\ \tau & \alpha & \tau' \end{pmatrix} C\begin{pmatrix} J & 1 & J' \\ m & r & m' \end{pmatrix} A_{I'J'}{}^{IJ}\,, \tag{2.15}$$

where C's are the $SU(2)$ CG coefficients and $A_{I'J'}{}^{IJ}$ are the reduced matrix elements. As before, the unitarity of the representations gives the following vertex-symmetry relation[3]

$$A_{I'J'}{}^{IJ} = (-)^{I+J-I'-J'}\left[\frac{(2I+1)(2J+1)}{(2I'+1)(2J'+1)}\right]^{1/2} A_{IJ}{}^{I'J'}. \tag{2.16}$$

The unitarity equation for the present case reads

$$\Gamma_{r,\alpha;s,\beta}=\sum_{t,\mu}(-)^{t+\mu}\Gamma_{r,\alpha;t,\mu}\Gamma_{-t,-\mu;s,\beta}\,, \tag{2.17}$$

and the reduced unitarity equation is

$$S(I,J;I'',J'':I_1,J_1)=\sum_{I'=I_1-1}^{I_1+1}\sum_{J'=J_1-1}^{J_1+1} S(I,J;I',J':I_1,J_1)S(I',J';I'',J'':I_1,J_1)\,, \tag{2.18}$$

where

$$S(I,J;I',J':I_1,J_1)=A_{I_1J_1}{}^{IJ}A_{I_1J_1}{}^{I'J'}[-\mathfrak{M}(I',J')-\mathfrak{M}(I,J)+\mathfrak{M}(I_1,J_1)] + \sum_{I_2=I-1}^{I+1}\sum_{J_2=J-1}^{J+1} A_{I_2J_2}{}^{IJ}A_{I_2J_2}{}^{I'J'}C_{I_1I_2}{}^{II'}C_{J_1J_2}{}^{JJ'}\mathfrak{M}(I_2,J_2)\,, \tag{2.19}$$

and $C^{II'}$, $C^{JJ'}$ are the same crossing matrices that occurred in the scalar theory.

[9] For the commutation relations in the spherical basis see V. Singh, Phys. Rev. 144, 1275 (1966).

The necessary condition (2.14) generalized to the group $\mathcal{G}$ reads

$$\sum_{I'=I-1}^{I+1} \sum_{J'=J-1}^{J+1} f(I',J')(A_{IJ}{}^{I'J'})^2 - f(I,J) = k, \quad k=1,2,\cdots,9, \tag{2.20}$$

where, again, k denotes the number of times the eigenvalue 1 is repeated. In writing down Eq. (2.20) we have assumed

$$\mathfrak{M} = f(I,J)/2\mathcal{P}_2,$$

where $f(I,J)$ is an *arbitrary* function of the Casimir operators I^2 and J^2 and $\mathcal{P}_2 = \sum_{r,\alpha} (-)^{r+\alpha} A_{r,\alpha} A_{-r,-\alpha}$ is a Casimir operator of the group $\mathcal{G}$ which we have normalized to unity.

III. CHARGE-SYMMETRIC SCALAR-MESON THEORY

In this section we discuss the s-wave scattering of an isovector scalar meson by a baryon isobar. The relevant group is $G \equiv SU(2)_I \times T_3$. The representations of G are specified by two parameters, I_0 and c. However, only the dependence on the parameter I_0 is essential. All the representations of G are infinite dimensional, and each representation contains an infinite number of irreducible representations of the subgroup $SU(2)_I$ of G [10]:

$$I = I_0, I_0+1, \cdots \infty.$$

The reduced matrix elements for the representations of G, specified by I_0 and c, are given by[11]

$$A_{I-1}{}^I = c\left[\frac{2(I^2-I_0^2)}{I(2I-1)}\right]^{1/2}, \quad A_I{}^I = iI_0 c\left(\frac{2}{I(I+1)}\right)^{1/2}, \quad A_{I+1}{}^I = c\left(\frac{2[(I+1)^2-I_0^2]}{(I+1)(2I+3)}\right)^{1/2}. \tag{3.1}$$

Vertex-symmetry relation demands c to be pure imaginary and normalization of A^2 to unity requires $|c|^2 = \frac{1}{2}$.

Substituting for $A_I{}^{I_1}$ in Eq. (2.14) from Eq. (3.1), we obtain the following difference equation for the function $f(I)$:

$$\frac{(I^2-I_0^2)}{I(2I+1)} F(I-1) - \frac{[I(I+1)-I_0^2]}{I(I+1)} F(I) + \frac{[(I+1)^2-I_0^2]}{(I+1)(2I+1)} F(I+1) = 1, \tag{3.2}$$

where $F(I) = f(I)/k$. The solution of this difference equation is (see Appendix A, part I)

$$f(I) = k\{C_2 + \tfrac{1}{2}I(I+1) + \tfrac{1}{2}(I_0^2 - C_1)[\Psi(I+1+I_0) + \Psi(I+1-I_0)]\}, \quad k=1,2,3, \tag{3.3}$$

where C_1 and C_2 are constants of integration and $\Psi(x)$ is given by

$$\Psi(x) = -\gamma - \sum_{S=0}^{\infty} \left(\frac{1}{x+S} - \frac{1}{1+S}\right), \tag{3.4}$$

where γ = Euler's constant = $0.5722\cdots$. The eigenvalues of the mass operator are then

$$\mathfrak{M}(I) = \tfrac{1}{2}f(I) = \tfrac{1}{2}k\{C_2 + \tfrac{1}{2}I(I+1) + \tfrac{1}{2}(I_0^2 - C_1)[\Psi(I+1+I_0) + \Psi(I+1-I_0)]\}, \quad k=1,2,3. \tag{3.5}$$

In passing, one may notice that CGS expression for the mass operation $\mathfrak{M} = I^2/2A^2$ corresponds to $C_1 = I_0^2$ and $k=2$ in Eq. (3.5).

In Eq. (3.5) we have arrived at an explicit form of the mass operator, using the necessary condition. As remarked earlier, this condition is likely to give rise to spurious solutions which may not satisfy the unitarity condition. In other words, we have to obtain the restrictions on the constants appearing in the mass formula (3.5) such that the resultant mass formula satisfies the unitarity condition.

We substitute (3.5) into the reduced unitarity equation (2.9). In order to evaluate the S functions in (2.9), one needs the crossing matrices occurring in the problem. These may be evaluated using Eq. (2.8). We have listed them in the Appendix B. Using the relevant crossing matrix elements, reduced matrix elements given in (3.1), and the mass formula in Eq. (3.5), we can now evaluate the various S functions. For convenience, we let $I'' = I_1 = I$. Then

$$S(I,I'';I_1) = S(I,I;I) = S_1(I,I;I) + S_2(I,I;I),$$

[10] See Ref. 1. See also S. Bose, Phys. Rev. **145**, 1247 (1966).

[11] See the paper by S. Bose (Ref. 10).

where

$$S_i(I,I;I)=-\frac{I_0^2\mathfrak{M}_i(I)}{I^2(I+1)^2}-\frac{I^2-I_0^2}{I^2(2I+1)}\mathfrak{M}_i(I-1)+\frac{(I+1)^2-I_0^2}{(I+1)^2(2I+1)}\mathfrak{M}_i(I+1),\quad i=1,2$$

and

$$\mathfrak{M}_1(I)=\tfrac{1}{2}kI(I+1)/2,\quad \mathfrak{M}_2(I)=\tfrac{1}{4}k(I_0^2-C_1)[\Psi(I+1+I_0)+\Psi(I+1-I_0)];$$

the constant term C_2 in (3.5) giving a vanishing contribution to $S(I,I;I)$. Substituting for $\mathfrak{M}_1(I)$, we get

$$S_1(I,I;I)=\tfrac{1}{2}k\left(1-\frac{I_0^2}{I(I+1)}\right).$$

Further, the contribution to $S_2(I,I;I)$ from the term

$$-\gamma+\sum_{S=0}^{\infty}\left(\frac{1}{1+S}\right)$$

in $\Psi(x)$ vanishes. Hence, the only relevant term in $\Psi(x)$ is

$$-\sum_{S=0}^{\infty}\left(\frac{1}{x+S}\right).$$

This gives

$$S_2(I,I;I)=\tfrac{1}{2}k\frac{(I_0^2-C_1)}{2}\left\{-\frac{I_0^2}{I^2(I+1)^2}\left[-\frac{2(I+1)}{(I+1)^2-I_0^2}-X\right]\right.$$
$$\left.-\frac{I^2-I_0^2}{I^2(2I+1)}\left[-\frac{2I}{I^2-I_0^2}-\frac{2(I+1)}{(I+1)^2-I_0^2}-X\right]-\frac{(I+1)^2-I_0^2}{(I+1)^2(2I+1)}X\right\},$$

where

$$X=\sum_{S=2}^{\infty}\left(\frac{1}{I+I_0+S}+\frac{1}{I-I_0+S}\right).$$

Collecting the terms in X, we find that its coefficient vanishes. On simplification one finds

$$S_2(I,I;I)=\tfrac{1}{2}k(I_0^2-C_1)\frac{1}{I(I+1)}.$$

Hence,

$$S(I,I;I)=\tfrac{1}{2}k\left(1-\frac{C_1}{I(I+1)}\right).\tag{3.6}$$

Similarly, one obtains

$$S(I,I-1;I)=\frac{iI_0}{I^2}\left(\frac{I^2-I_0^2}{(2I+1)(I+1)}\right)^{1/2}[\mathfrak{M}(I-1)-\mathfrak{M}(I)]=-S(I-1,I;I)$$
$$=-\frac{iI_0k}{2I}\frac{I^2-C_1}{[(2I+1)(I+1)(I^2-I_0^2)]^{1/2}}\tag{3.7}$$

and

$$S(I,I+1;I)=\frac{iI_0}{(I+1)^2}\left(\frac{(I+1)^2-I_0^2}{I(2I+1)}\right)^{1/2}[\mathfrak{M}(I)-\mathfrak{M}(I+1)]=-S(I+1,I;I)$$
$$=-\frac{iI_0k}{2(I+1)}\frac{(I+1)^2-C_1}{\{I(2I+1)[(I+1)^2-I_0^2]\}^{1/2}}.\tag{3.8}$$

Substituting (3.6)–(3.8) into (2.9) we obtain the condition

$$\phi(I,I_0,k,C_1)=0,$$

where

$$\phi(I,I_0,k,C_1)\equiv\frac{kI_0^2}{2I}\frac{(I^2-C_1)^2}{(2I+1)(I^2-I_0^2)}+\tfrac{1}{2}k\frac{[I(I+1)-C_1]^2}{I(I+1)}+\frac{I_0^2k}{2(I+1)}\frac{[(I+1)^2-C_1]^2}{(2I+1)[(I+1)^2-I_0^2]}-I(I+1)+C_1.$$

Making the substitution $D_1=C_1-I_0^2$ in the above equation, we find

$$\phi(I,I_0,k,D_1)\equiv r(\tfrac{1}{2}k-1)+D_1(1-k)+\frac{r}{r^2-I_0^2}D_1^2=0\,,\quad k=1,2,3\,,\tag{3.9}$$

where

$$r\equiv r(I,I_0)=I(I+1)-I_0^2\,.$$

We wish to find all the roots of this quadratic equation in D_1 which are independent of I. Solving (3.9) for D_1, one obtains

$$D_1=\left\{k-1\pm\left[(k-1)^2+\frac{2r^2(k-2)}{r^2-I_0^2}\right]^{1/2}\right\}\Big/\left(\frac{2r}{r^2-I_0^2}\right),\quad k=1,2,3\,.$$

From the above equation it is clear that D_1 is a constant only when $k=2$ and its value then is zero. This implies $C_1=I_0^2$, which reduces the mass formula (3.5) to

$$\mathfrak{M}(I)=C_2+\tfrac{1}{2}I(I+1)\,,\tag{3.10}$$

which is, as noted previously, the one employed by CGS. We have shown, therefore, that the most general mass operator as a function of the Casimir operator I^2 of the subgroup $SU(2)$ of G which satisfies the unitarity equation is the one given in Eq. (3.10). We would like to emphasize that the mass operator in Eq. (3.10) gives all the representations of G, and conversely the only mass operator allowed by the unitarity condition for the group G is $\mathfrak{M}\propto I^2$.

IV. CHARGE-SYMMETRIC PSEUDOSCALAR-MESON THEORY

We consider the p-wave scattering of a pseudoscalar pion by an isobar. The appropriate group for this case is $\mathcal{G}\equiv[SU(2)_I\otimes SU(2)_J]\times T_9$.[1] Even though several unitary irreducible representations of this group are known,[12] there are only the following two irreducible representations which are physically interesting: one relating to the nucleon isobar series,[13] and the other to the hyperon isobar series.[3] Both these representations are infinite dimensional; the isospin-spin content of the nucleon isobar series representation is $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, $\cdots$, that is $I=J=\frac{1}{2},\frac{3}{2},\cdots,\infty$, whereas that of the hyperon isobar series is $(0,\frac{1}{2})$, $(1,\frac{1}{2})$, $(1,\frac{3}{2})$, $\cdots$, which can be characterized by $J=I\pm\frac{1}{2}=\frac{1}{2},\frac{3}{2},\cdots,\infty$. In what follows we shall restrict ourselves to these representations only.

A. The Nucleon Isobar Series

Since every isobar state in this representation has $I=J$, only one label is sufficient and the isobar state may be represented by $|J;\tau,m\rangle$. Therefore, Eqs. (2.18), (2.19), and (2.20) involve only one summation. The reduced matrix elements $A_{J'}{}^{J}$ are given by[13]

$$A_{J-1}{}^{J}=G_0\left(\frac{2J+1}{2J-1}\right)^{1/2},$$
$$A_J{}^J=G_0\,,$$
$$A_{J+1}{}^{J}=G_0\left(\frac{2J+1}{2J+3}\right)^{1/2}.\tag{4.1}$$

The vertex-symmetry relation (2.16) (i.e., unitary of the representation) demands that G_0 be real and the normalization of $\mathcal{P}_2$ to unity yields $G_0^2=\frac{1}{3}$. Substituting the reduced matrix elements from Eq. (4.1) into Eq. (2.20), we obtain the following difference equation:

$$F(J-1)\left(\frac{2J-1}{2J+1}\right)-2F(J)+F(J+1)\left(\frac{2J+3}{2J+1}\right)=1,\tag{4.2}$$

where $F(J)=f(J)/3k$, $k=1,2,\cdots,9$. Following the procedure used to solve the difference equation in the last section, we obtain the solution

$$f(J)=k\left\{\tfrac{1}{2}J(J+1)+\frac{C_1}{2J+1}+C_2\right\},\quad k=1,2,\cdots,9.\tag{4.3}$$

The corresponding mass formula is

$$\mathfrak{M}(J)=\tfrac{1}{2}k\left\{\tfrac{1}{2}J(J+1)+\frac{C_1}{2J+1}+C_2\right\},\quad k=1,2,\cdots,9.\tag{4.4}$$

As before, we have to obtain the restrictions on the constants k, C_1, C_2 in Eq. (4.4) so as to satisfy the reduced unitarity equation (2.18). Restricting ourselves again to $J''=J=J_1$ in Eq. (2.18), we obtain the following condition on C_1:

$$4k(J^2+J+6)C_1^2-2(k+3)C_1\times[J(J+1)(2J-1)(2J+1)(2J+3)]+(k-3)J^2\times(J+1)^2(2J-1)(2J+1)^2(2J+3)=0,\tag{4.5}$$

[12] C. J. Goebel, in *Non-Compact Groups in Particle Physics*, edited by Y. Chow, (W. A. Benjamin, Inc., New York, 1966); T. Cook and B. Sakita, Argonne National Laboratory Report (unpublished). See also for an alternative treatment P. Babu, A. Rangwala, and V. Singh, Tata Institute of Fundamental Research Report (unpublished).

[13] See Refs. 1 and 12. See also V. Singh, Phys. Rev. 144, 1275 (1966).

where $J=\frac{1}{2},\frac{3}{2},\cdots,\infty$. The dependence of $\mathfrak{M}(J)$ on C_2 is trivial; the contribution of the C_2 term to the matrix elements $S(J,J';J_1)$ vanishes identically. To obtain the solutions C_1 of the above quadratic equation which are independent of J, we note that since Eq. (4.5) is to hold for arbitrary value of J the coefficient of each power of J must vanish separately. Equating to zero the coefficient $16(k-3)$ of the highest power J^8, yields $k=3$. For $k=3$, Eq. (4.5) reduces to

$$C_1[(J^2+J+6)C_1-J(J+1)(2J-1)(2J+1)(2J+3)]=0,$$

which has $C_1=0$ as the solution independent of J. Thus for $k=3$ only does there exist a constant root C_1 of Eq. (4.5), its value being zero.

The mass formula (4 4) then becomes

$$\mathfrak{M}(J)=\tfrac{3}{4}J(J+1)+C;\quad(k=3). \tag{4.6}$$

For the nucleon isobar series the only nontrivial mass formula which satisfies the unitarity condition is that given by Eq. (4.6). Mass formula (4.6) is the same as that given by CGS, namely,

$$\mathfrak{M}=[aI^2+(1-a)J^2](3/4\mathcal{P}_2),$$

where a is an arbitrary constant and which in the present case ($I=J$) reduces to $\mathfrak{M}=3J^2/4\mathcal{P}_2$.

B. Hyperon Isobar Series

For this representation both I and J labels are necessary. Since we restrict ourselves to $J=I\pm\frac{1}{2}$, there are only five independent final states that are connected to a given initial state, $(I=n,J=n+\frac{1}{2})$ say, via the meson-current operators. The reduced matrix elements are[3]:

$$\begin{aligned}
&A_{n,n-\frac{1}{2}}{}^{n,n+\frac{1}{2}}=\frac{G_0}{[n(2n+1)]^{1/2}}; && A_{n,n+\frac{1}{2}}{}^{n,n-\frac{1}{2}}=-\frac{G_0}{[(n+1)(2n+1)]^{1/2}};\\
&A_{n-1,n-\frac{1}{2}}{}^{n,n+\frac{1}{2}}=\left(\frac{n+1}{n}\right)^{1/2}G_0; && A_{n-1,n-\frac{3}{2}}{}^{n,n-\frac{1}{2}}=\left(\frac{2n+1}{2n-1}\right)^{1/2}G_0;\\
&A_{n,n+\frac{1}{2}}{}^{n,n+\frac{1}{2}}=\left(\frac{n(2n+3)}{(n+1)(2n+1)}\right)^{1/2}G_0; && A_{n+1,n+\frac{1}{2}}{}^{n,n-\frac{1}{2}}=\left(\frac{n}{n+1}\right)^{1/2}G_0;\\
&A_{n+1,n+\frac{1}{2}}{}^{n,n+\frac{1}{2}}=\frac{G_0}{[(n+1)(2n+3)]^{1/2}}; && A_{n-1,n-\frac{1}{2}}{}^{n,n-\frac{1}{2}}=-\frac{G_0}{[n(2n-1)]^{1/2}};\\
&A_{n+1,n+\frac{3}{2}}{}^{n,n+\frac{1}{2}}=\left(\frac{2n+1}{2n+3}\right)^{1/2}G_0; && A_{n,n-\frac{1}{2}}{}^{n,n-\frac{1}{2}}=\left(\frac{(n+1)(2n-1)}{n(2n+1)}\right)^{1/2}G_0,
\end{aligned} \tag{4.7}$$

where the first and the second columns give, respectively, the reduced matrix elements when the initial states are $(I=n,\ J=n+\frac{1}{2})$ and $(I=n,J=n-\frac{1}{2})$. Unitarity demands that G_0 be real and the normalization of $\mathcal{P}_2$ to unity fixes it to $G_0{}^2=\frac{1}{3}$.

The reduced matrix elements (4.7) can be substituted into (2.20) to obtain the necessary condition on $f(I,J)$. However, in this particular case we obtain two independent difference equations depending upon whether we choose $I=n,\ J=n+\frac{1}{2}$ or $I=n,\ J=n-\frac{1}{2}$ for the initial state. The difference equations obtained are

$$\begin{aligned}
\left(\frac{2n-1}{2n+1}\right)f(n-1,n-\tfrac{1}{2})+\frac{1}{(n+1)(2n+1)}f(n,n-\tfrac{1}{2})-\frac{4n^2+6n+3}{(n+1)(2n+1)}f(n,n+\tfrac{1}{2})&\\
+\frac{1}{(n+1)(2n+1)}f(n+1,n+\tfrac{1}{2})+\frac{n+2}{n+1}f(n+1,n+\tfrac{3}{2})=3k&
\end{aligned} \tag{4.8a}$$

and

$$\begin{aligned}
\left(\frac{n-1}{n}\right)f(n-1,n-\tfrac{3}{2})+\frac{1}{n(2n+1)}f(n-1,n-\tfrac{1}{2})-\frac{4n^2+2n+1}{n(2n+1)}f(n,n-\tfrac{1}{2})&\\
+\frac{1}{n(2n+1)}f(n,n+\tfrac{1}{2})+\frac{2n+3}{2n+1}f(n+1,n+\tfrac{1}{2})=3k.&
\end{aligned} \tag{4.8b}$$

These difference equations can be solved (see Appendix A part II). The solution is

$$f(n,n+\tfrac{1}{2})=\frac{\zeta_1}{n+1}+\frac{\zeta_2}{2n+1}+\zeta_3 n(n+1)+\zeta_4(n+1)+\zeta_5;$$

$$f(n,n-\tfrac{1}{2})=\frac{\zeta_1}{n}+\frac{\zeta_2}{2n+1}+\zeta_3 n(n+1)-\zeta_4 n+\zeta_5, \tag{4.9}$$

which can be combined into one single equation as

$$f(I,J)=\frac{2\zeta_1}{2J+1}+\frac{\zeta_2}{2I+1}+\zeta_3' I(I+1)+\zeta_4' J(J+1)+\zeta_5', \tag{4.10}$$

where

$$\zeta_3'=\zeta_3-\zeta_4,\quad \zeta_4'=\zeta_4,\quad \zeta_5'=\tfrac{1}{4}\zeta_4+\zeta_5,$$

and $\zeta_1,\cdots,\zeta_5$ are the constants of integration. Substituting (4.9) into (4.8), one obtains the condition $2\zeta_3=k$.

Having obtained the expression for $\mathfrak{M}(I,J)$ $(=f(I,J)/2)$, one may now substitute this into the reduced unitarity equation (2.18). We take $I=I''=I_1=n$ and $J=J''=J_1=n+\frac{1}{2}$. After a straightforward but long and tedious calculation, one can simplify the unitarity equation to obtain

$$\begin{aligned}&(n+1)(2n+3)^4(n+2)[(2n-1)\zeta_1+(n+1)\zeta_2-n(n+1)(2n+1)(2n-1)\zeta_3]^2\\&\quad+(2n-1)(2n+3)^2(n+2)[(2n^2+n-3)\zeta_1+(n^2+2n+3)\zeta_2-n(n+1)(2n+1)(n-1)(2n+3)\zeta_3]^2\\&\quad+9n(2n-1)(2n+3)(n+2)[(2n-1)\zeta_1+(n+2)\zeta_2+n(n+1)(2n+1)(2n+3)\zeta_3]^2\\&\quad+n^2(2n-1)(n+2)[(4n^2+4n+9)\zeta_1+(2n^2+5n)\zeta_2-n(n+1)(2n+1)(2n+3)(2n+5)\zeta_3]^2\\&\quad+4n^4(2n+1)(2n-1)[(2n-1)\zeta_1+(n+2)\zeta_2-(n+1)(n+2)(2n+1)(2n+3)\zeta_3]^2\\&\qquad-9n^2(n+1)^2(2n+1)^2(2n-1)(2n+3)^2(n+2)[(2n-1)\zeta_1+(n+2)\zeta_2+n(n+1)(2n+1)(2n+3)\zeta_3]\equiv 0.\end{aligned} \tag{4.11}$$

This is a fourteenth-degree polynomial in n, the coefficients of whose terms are polynomials quadratic in $\zeta_1, \zeta_2, \zeta_3$. Note that ζ_4 and ζ_5 do not occur in Eq. (4.11). Hence the unitarity equation does not place any restriction on ζ_4 and ζ_5. We have to determine *all* possible sets of values of $\zeta_1, \zeta_2, \zeta_3$ such that (4.11) holds for arbitrary non-negative integral values of n. This implies that the coefficient of each power of n must vanish separately. From these sets of equations we can obtain the desired sets of values of $\zeta_1, \zeta_2, \zeta_3$. One such set of minimum number of equations is obtained by equating separately to zero the coefficients of n^{14}, n^{11}, and n^{10} terms. These yield the following sets of three equations:

$$\zeta_3(3-2\zeta_3)=0;$$

$$2(4\zeta_3+3)(2\zeta_1+\zeta_2)-981\zeta_3(2\zeta_3-3)=0;$$

$$\begin{aligned}-8\zeta_1(86\zeta_3+63)-2\zeta_2(216\zeta_3+171)&\\+9279\zeta_3(2\zeta_3-3)=0.&\end{aligned} \tag{4.12}$$

Two sets of values of $\zeta_1, \zeta_2, \zeta_3$ obtained from (4.12) are $(\zeta_1,\zeta_2,\zeta_3)=(0,0,0)$ and $(\zeta_1,\zeta_2,\zeta_3)=(0,0,\frac{3}{2})$. One checks that both these solutions satisfy Eq. (4.11) identically for all n. Hence, these are the only two possible sets of values of $\zeta_1, \zeta_2, \zeta_3$.

For $\zeta_1=\zeta_2=\zeta_3=0$, the mass formula becomes

$$\begin{aligned}\mathfrak{M}(I,J)&=\tfrac{1}{2}f(I,J)=\tfrac{1}{2}\zeta_3' I(I+1)+\tfrac{1}{2}\zeta_4' J(J+1)+\tfrac{1}{2}\zeta_5'\\&=\tfrac{1}{2}\zeta_3'[I(I+1)-J(J+1)]+\tfrac{1}{2}\zeta_5'.\end{aligned}$$

In this case $k=2\zeta_3=0$. This mass operator, therefore, gives a vanishing scattering amplitude and hence is trivial. However, the solution $\zeta_1=\zeta_2=0$, $\zeta_3=\frac{3}{2}$ gives a nontrivial scattering amplitude, the mass formula in this case being

$$\mathfrak{M}(I,J)=\tfrac{3}{4}[aI(I+1)+bJ(J+1)]+C, \tag{4.13}$$

where $a=\frac{2}{3}\zeta_3'$, $b=\frac{2}{3}\zeta_4'$ and $a+b=\frac{2}{3}(\zeta_3'+\zeta_4')=1$. Here $C=\frac{1}{2}\zeta_5'$. The value of k is 3. Hence the most general mass operator, for the hyperon isobar series, as a function of Casimir operator is the one given in Eq. (4.13). This mass operator is identical to the one obtained for the nucleon isobar series.

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APPENDIX A

I. Solution of the Difference Equation (3.2)

The difference equation (3.2) is

$$\frac{(I^2-I_0^2)}{I(2I+1)}F(I-1)-\frac{[I(I+1)-I_0^2]}{I(I+1)}F(I)+\frac{[(I+1)^2-I_0^2]}{(I+1)(2I+1)}F(I+1)=1. \tag{A1}$$

Let

$$H(I)=\frac{I^2-I_0^2}{I}[F(I-1)-F(I)];$$

then Eq. (A1) reduces to

$$H(I+1)-H(I)=-2I-1.$$

Therefore, we have

$$H(I)=C_1-I^2,$$

where C_1 is a constant of integration. Hence

$$F(I+1)-F(I)=(I+1)+\tfrac{1}{2}(I_0^2-C_1)\left[\frac{1}{I+1+I_0}+\frac{1}{I+1-I_0}\right].$$

Let $\Delta F(I)\equiv F(I+1)-F(I)$. Then operating on both sides of the above equation by Δ^{-1}, we get

$$F(I)=\Delta^{-1}(I+1)+\tfrac{1}{2}(I_0^2-C_1)\left[\Delta^{-1}\frac{1}{I+1+I_0}+\Delta^{-1}\frac{1}{I+1-I_0}\right]+C_2,$$

where C_2 is another constant of integration.[14] Now we have

$$\Delta^{-1}(I+1)=\tfrac{1}{2}I(I+1)$$

and[15]

$$\Delta^{-1}\left(\frac{1}{I+1\pm I_0}\right)=\Psi(I+1\pm I_0),$$

where $\Psi(x)$ is the logarithmic derivative of the gamma function:

$$\Psi(x)=\frac{1}{\Gamma(x)}\frac{d}{dx}\Gamma(x).$$

Explicit expression for $\Psi(x)$ is given by

$$\Psi(x)=-\gamma-\sum_{S=0}^{\infty}\left(\frac{1}{x+S}-\frac{1}{1+S}\right),$$

where γ = Euler's constant $=0.5772\cdots$. Hence, the complete solution of the difference equation (A1) is

$$F(I)=C_2+\tfrac{1}{2}I(I+1)+\tfrac{1}{2}(I_0^2-C_1)[\Psi(I+1+I_0)+\Psi(I+1-I_0)]. \tag{A2}$$

II. Solution of the Difference Equations (4.8)

We have the difference equations (4.8):

$$\left(\frac{2n-1}{2n+1}\right)f(n-1,n-\tfrac{1}{2})+\frac{1}{(n+1)(2n+1)}f(n,n-\tfrac{1}{2})-\frac{4n^2+6n+3}{(n+1)(2n+1)}f(n,n+\tfrac{1}{2})$$
$$+\frac{1}{(n+1)(2n+1)}f(n+1,n+\tfrac{1}{2})+\frac{n+2}{n+1}f(n+1,n+\tfrac{3}{2})=3k \tag{A3}$$

[14] For the definitions of Δ and Δ^{-1} operators see C. H. Richardson, *An Introduction to the Calculus of Finite Differences* (D. Van Nostrand and Company, Inc., New York, 1954); or L. M. Milne-Thomson, *The Calculus of Finite Differences* (MacMillan and Company Ltd., London, 1933).

[15] See, for instance, L. M. Milne-Thomson, Ref. 14, Chap. VIII.

and

$$\left(\frac{n-1}{n}\right)f(n-1,n-\tfrac{3}{2})+\frac{1}{n(2n+1)}f(n-1,n-\tfrac{1}{2})-\frac{4n^2+2n+1}{n(2n+1)}f(n,n-\tfrac{1}{2})$$
$$+\frac{1}{n(2n+1)}f(n,n+\tfrac{1}{2})+\frac{2n+3}{2n+1}f(n+1,n+\tfrac{1}{2})=3k, \quad (A4)$$

where n is a non-negative integer.

Let $n=m-\frac{1}{2}$ in Eq. (A3) and $n=m+\frac{1}{2}$ in Eq. (A4), where m is a positive half odd integer. We then obtain

$$\frac{m-1}{m}f(m-\tfrac{3}{2},m-1)+\frac{1}{m(2m+1)}f(m-\tfrac{1}{2},m-1)-\frac{4m^2+2m+1}{m(2m+1)}f(m-\tfrac{1}{2},m)$$
$$+\frac{1}{m(2m+1)}f(m+\tfrac{1}{2},m)+\frac{2m+3}{2m+1}f(m+\tfrac{1}{2},m+1)=3k \quad (A5)$$

and

$$\frac{2m-1}{2m+1}f(m-\tfrac{1}{2},m-1)+\frac{1}{(m+1)(2m+1)}f(m-\tfrac{1}{2},m)-\frac{4m^2+6m+3}{(m+1)(2m+1)}f(m+\tfrac{1}{2},m)$$
$$+\frac{1}{(m+1)(2m+1)}f(m+\tfrac{1}{2},m+1)+\frac{m+2}{m+1}f(m+\tfrac{3}{2},m+1)=3k. \quad (A6)$$

Subtracting Eq. (A4) from Eq. (A3) and Eq. (A5) from Eq. (A6), we obtain

$$\frac{n-1}{n}[f(n-1,n-\tfrac{1}{2})-f(n-1,n-\tfrac{3}{2})]-\frac{2n^2+2n+1}{n(n+1)}[f(n,n+\tfrac{1}{2})-f(n,n-\tfrac{1}{2})]$$
$$+\frac{n+2}{n+1}[f(n+1,n+\tfrac{3}{2})-f(n+1,n+\tfrac{1}{2})]=0 \quad (A7)$$

and

$$\frac{m-1}{m}[f(m-\tfrac{1}{2},m-1)-f(m-\tfrac{3}{2},m-1)]-\frac{2m^2+2m+1}{m(m+1)}[f(m+\tfrac{1}{2},m)-f(m-\tfrac{1}{2},m)]$$
$$+\frac{m+2}{m+1}[f(m+\tfrac{3}{2},m+1)-f(m+\tfrac{1}{2},m+1)]=0. \quad (A8)$$

Equation (A7) is a difference equation for $F(I)=f(I,J)-f(I,J-1)$, whereas Eq. (A8) is in $G(J)=f(I,J)-f(I-1,J)$. The coefficients of $G(I)$ in Eq. (A8) are same as those for $F(I)$ in (A7). It is, therefore, necessary to solve only one of them. Let $F(n)\equiv f(n,n+\frac{1}{2})-f(n,n-\frac{1}{2})=F_1(n)H(n)$, where $F_1(n)$ is a particular solution of Eq. (A7). One can check that $F_1(n)=2n+1$ satisfies the Eq. (A7). To reduce the order of the Eq. (A7), define

$$D(n)\equiv\Delta H(n)\equiv H(n+1)-H(n);$$

then Eq. (A7) becomes

$$n(n+2)(2n+3)D(n)-(n-1)(n+1)(2n-1)D(n-1)=0.$$

Multiply through by $(2n+1)$ to obtain

$$R(n)-R(n-1)=0,$$

where $R(n)=n(n+2)(2n+1)(2n+3)D(n)$. Solution of the above difference equation is

$$R(n)=K_1=\text{constant}.$$

Hence

$$D(n)=H(n+1)-H(n)=K_1/n(n+2)(2n+1)(2n+3).$$

The solution of this equation is

$$H(n)=-\tfrac{1}{6}K_1[n(n+1)(2n+1)]^{-1}+K_2.$$

Hence

$$F(n)=f(n,n+\tfrac{1}{2})-f(n,n-\tfrac{1}{2})=F_1(n)H(n)=[C_1/n(n+1)]+C_2(2n+1), \quad (A9)$$

where $C_1=-\frac{1}{6}K_1$, $C_2=K_2$. Exactly in the same way, we have from Eq. (A8)

$$f(m+\tfrac{1}{2},m)-f(m-\tfrac{1}{2},m)=[\gamma_1'/m(m+1)]+\gamma_2'(2m+1). \tag{A10}$$

Let $m=n-\frac{1}{2}$ in Eq. (A10); then

$$f(n,n-\tfrac{1}{2})-f(n-1,n-\tfrac{1}{2})=[\gamma_1/(2n-1)(2n+1)]+\gamma_2 n. \tag{A11}$$

Adding Eqs. (A9) and (A11), we get

$$f(n,n+\tfrac{1}{2})-f(n-1,n-\tfrac{1}{2})=\frac{C_1}{n(n+1)}+\frac{\gamma_1}{(2n-1)(2n+1)}+(\gamma_2+2C_2)n+C_2. \tag{A12}$$

The solution of (A12) is

$$f(n,n+\tfrac{1}{2})=\frac{\zeta_1}{n+1}+\frac{\zeta_2}{2n+1}+\zeta_3 n(n+1)+\zeta_4(n+1)+\zeta_5, \tag{A13}$$

where

$$\zeta_1=-C_1,\quad \zeta_2=-\tfrac{1}{2}\gamma_1,\quad \zeta_3=\tfrac{1}{2}(\gamma_2+2C_2),\quad \zeta_4=C_2.$$

From Eqs. (A13) and (A9), we obtain

$$f(n,n-\tfrac{1}{2})=\frac{\zeta_1}{n}+\frac{\zeta_2}{2n+1}+\zeta_3 n(n+1)-\zeta_4 n+\zeta_5. \tag{A14}$$

APPENDIX B

We list here the relevant crossing matrices:

$$C^{I-1,I}=C^{I,I-1}=\begin{array}{cc} I_2=I-1 & I_2=I \\ \left(\begin{array}{cc} \dfrac{1}{I} & -\dfrac{2I+1}{I}\left[\dfrac{(I+1)(I-1)}{(2I+1)(2I-1)}\right]^{1/2} \\ -\dfrac{2I-1}{I}\left[\dfrac{(I+1)(I-1)}{(2I+1)(2I-1)}\right]^{1/2} & -\dfrac{1}{I} \end{array}\right) \end{array}\begin{array}{l} \\ I_1=I-1 \\ I_1=I \end{array}$$

$$C^{I,I}=\begin{array}{ccc} I_2=I-1 & I_2=I & I_2=I+1 \\ \left(\begin{array}{ccc} \dfrac{1}{I(2I+1)} & -\dfrac{1}{I} & \dfrac{2I+3}{2I+1} \\ -\dfrac{2I-1}{I(2I+1)} & \dfrac{I^2+I-1}{I(I+1)} & \dfrac{2I+3}{(2I+1)(I+1)} \\ \dfrac{2I-1}{2I+1} & \dfrac{1}{I+1} & \dfrac{1}{(2I+1)(I+1)} \end{array}\right) \end{array}\begin{array}{l} \\ I_1=I-1 \\ I_1=I \\ I_1=I+1 \end{array}$$

$$C^{I+1,I}=C^{I,I+1}=\begin{array}{cc} I_2=I & I_2=I+1 \\ \left(\begin{array}{cc} \dfrac{1}{I+1} & -\dfrac{2I+3}{I+1}\left[\dfrac{I(I+2)}{(2I+1)(2I+3)}\right]^{1/2} \\ -\dfrac{2I+1}{I+1}\left[\dfrac{I(I+2)}{(2I+1)(2I+3)}\right]^{1/2} & -\dfrac{1}{I+1} \end{array}\right) \end{array}\begin{array}{l} \\ I_1=I \\ I_1=I+1 \end{array}$$

$$C^{I-2,I}=C^{I,I-2}=C_{I-1,I-1}{}^{I,I-2}=1$$

and

$$C^{I+2,I}=C^{I,I+2}=C_{I+1,I+1}{}^{I,I+2}=1.$$