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Crossing Symmetry and Hadron Dynamics*

R. E. CUTKOSKY
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
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A method of carrying out bootstrap calculations is described in which analyticity properties enter indirectly. There are two main features: the use of a nonlinear Bethe-Salpeter equation which automatically maintains the symmetry properties of amplitudes, and the use of crossing symmetry in the construction of effective propagators. Two illustrative examples are considered. One is a simplified nonlinear equation for the familiar static model of baryon-meson interactions. The other is a model for interactions mediated by vector mesons.

I. INTRODUCTION

THE idea that the low-energy dynamics of hadrons is primarily governed by crossing symmetry is based on pioneering work by Chew, Low, and Mandelstam (CLM).[1,2] In applying this principle, they developed an approximation scheme based on analyticity properties of the S matrix, practical exploitation of which requires a drastic truncation of the number of channels which are considered explicitly ("elastic unitarity," or various refinements). Since fluctuation in the number of particles is also an intrinsic feature of quantum field theory, it is not surprising that difficulties often appear when this feature is artificially suppressed.[3,4] For this reason, there is interest in finding ways to implement the CLM principle which do not involve the CLM approximation. It is natural to consider, for this purpose, more old-fashioned methods of field theory—so called "off-mass-shell" techniques—which, by incorporating more detailed information about the variation of fields at nearby space-time points, automatically take better account of fluctuations in the number of quanta.[5]

The CLM principle involves the crossing symmetry of four-line diagrams; the effective potential acting between two particles is associated with exchange of particles which arise as bound or resonant states in crossed channels, thereby giving rise to the cooperative features of hadron dynamics. Another aspect of crossing symmetry appears in three-line diagrams (which may also be considered as subsections of more complicated processes): The value of the vertex part must not depend on whether a given particle is outgoing or incoming. While this "vertex symmetry" does not directly influence the cooperative phenomena, it must be intrinsic to any valid calculational scheme. It is obvious that vertex symmetry is not compatible with any scheme in which the total number of particles which are considered to exist at any given time is arbitrarily limited.

It has been suggested that a calculational method with manifest vertex symmetry can be based on the Bethe-Salpeter (B-S) equation.[6] So far, this method has been used only in a rather simplified approximation,[7,8] and there may be some doubts as to how the method could be extended to higher orders of approximation. The purpose of this paper is to discuss such an extension, but for the sake of readability, we shall concentrate on the next order of approximation. The main problem concerning still higher orders of approximation appears to be how to decide which of several possible routes it would be most expeditious to follow.

The assumptions on which the procedure rests are as follows: We assume that some underlying local field theory exists and has the analyticity properties suggested by the regularized perturbation expansion.

* Supported, in part, by the U. S. Atomic Energy Commission.

[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958). G. F. Chew and S. Mandelstam, *ibid.* **119**, 476 (1960).

[3] R. F. Sawyer, Phys. Rev. **142**, 991 (1966).

[4] R. E. Cutkosky, in *Particle Symmetries* (1965 Brandeis University Summer Institute in Theoretical Physics, Vol. II), edited by M. Chreticn and S. Deser, [Gordon and Breach, Science Publishers, Inc., New York, (1966)], p. 97.

[5] The usefulness of the field idea in self-consistent models has been emphasized by J. Schwinger, Phys. Today **19**, No. 6, 27 (1966).

[6] R. E. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964).

[7] R. E. Cutkosky and M. Leon, Phys. Rev. **138**, B667 (1965).

[8] K. Y. Lin and R. E. Cutkosky, Phys. Rev. **140**, B205 (1965); K. Y. Lin, Carnegie Institute of Technology dissertation, 1966 (unpublished).

However, the "fundamental" fields are assumed to be so removed from low-energy hadronic phenomena as to be useless for calculational purposes. Instead, we use phenomenological constructs which are derived from the scattering matrix, with field theory being used to justify the existence, and suggest the form, of off-mass-shell quantities. Dynamics is put in through the CLM principle: Long-range effects, i.e., those associated with nearby singularities in the space of complex momentum variables, dominate the structure of the low-lying hadronic states. Short-range effects are assumed to be adequately representable through a small number of adjustable parameters (which are to be introduced in a symmetrical way, so that, in accordance with Chew's idea of "nuclear democracy," the covariance properties[4,9] of the bootstrap equations are not spoiled). In formulating the approximation scheme, we rely on the empirically demonstrated dominance of "quasitwo-body" states.

II. OUTLINE OF PROCEDURE

Before discussing any specific details of the method, we give a list of the main steps: (1) Assume a set of particles (that is, assume the quantum numbers; the masses and coupling constants are to be adjustable). (2) Calculate the propagators and vertex functions. In the lowest approximation, we might use a regularized pole approximation for the propagators. In order to maintain vertex symmetry, the vertices (with all particles off of the mass shell) will have to be approximated not by constants nor even by form factors, but by functionals of the Bethe-Salpeter amplitudes (which are vertices with one particle on the mass shell). (3) Solve (in some approximation) the "homogeneous" Bethe-Salpeter equation, and normalize the amplitudes. Since the vertices are functionals of the amplitudes, this is really a nonlinear problem. At this step the masses and coupling constants are determined, as well as the explicit form of the vertices. (4) Solve the inhomogeneous Bethe-Salpeter equation to determine the scattering amplitudes; this is a linear problem. The self-consistency of the quantum numbers assumed for the bound states and resonances is checked in this step. (5) Using crossing symmetry, calculate improved approximations to the propagation functions and the vertex functionals.

Steps (2) and (5), which give the scheme a recursive structure, are the novel points of this method. However, most of the ideas are already contained in some earlier papers by Balázs[10] and the author.[11] To aid in picturing the approximation scheme, we shall introduce the notion of "quasicurrents," which resemble the field-theoretical currents but are constructed from the S matrix. However, these quasicurrents are only defined recursively, as one successively identifies and eliminates various nonlocal effects.

A quasicurrent with given quantum numbers ξ (charge, etc.) is to be constructed by taking a pair of particles (a_ξ,b_ξ) (or possibly more) whose quantum numbers add up to the right value, and which are considered to be on their respective mass shells. The total angular momentum of this set in its center-of-mass frame is projected into the desired value. The squared center of mass energy s_ξ of the set is to be varied and represents the momentum transferred to the quasicurrent J_ξ. Then we consider a scattering amplitude S_{2n} with a total of $2n$ particles either entering or leaving. The "matrix element" of n quasicurrents will be identified by means of the poles and normal-threshold absorptive parts of S_{2n} in the n variables s_ξ, after the contribution of simple Feynman graphs to the same absorptive parts is eliminated; we shall speak of "extracting the quasicurrent interaction" from S_{2n}. It will usually be assumed that subtraction are unnecessary. For given quantum numbers ξ, there are obviously infinitely many candidates for J_ξ; of course, Lagrangian field theory has a similar ambiguity. It is not clear how this ambiguity and the question of subtractions are related.

In order to show that the nonlinearity of the vertex equation used in step (3) is not necessarily a source of grave difficulties, we shall consider a simple example in Sec. V. Kinematical questions involving spin will be ignored in this paper. However, in Sec. VI we shall consider, as an illustrative application, properties of vector-particle propagators which seem to be useful in some current models.

As a final remark, we wish to emphasize that this is an approximation scheme which is directed towards certain specific cooperative phenomena and which is useful in a relatively low order. We do not wish to suggest its *ad infinitum* extension, because we think that at very small distances it will no longer be possible to maintain the phenomenological distinctions between the strong, electromagnetic, and weak interactions.

III. CONSTRUCTION OF PROPAGATORS

Suppose we wish to construct a propagator for the particle c. For simplicity, let us assume first that with the given quantum numbers ξ there is a single pole (possibly a resonance pole) in appropriate two-particle scattering amplitudes. Let us suppose that we have calculated these scattering amplitudes from the Bethe-Salpeter equation, perhaps in the ladder approximation. The quasicurrent J_ξ is associated with a particular pair of particles (a,b) (or perhaps with a linear combination); we focus our attention upon the partial-wave amplitude $T_{\xi'\xi}(c)$, which contains the effective interaction between the quasicurrents $J_{\xi'}$ and J_ξ. This amplitude $T_{\xi'\xi}(c)$ is a function of the center-of-mass energy, which we

[9] J. G. Belinfante, R. E. Cutkosky, and G. H. Renninger, *Seminar on High Energy Physics and Elementary Particles Trieste 1965* (International Atomic Energy Agency, Vienna, 1965), p. 865.

[10] L. P. Balázs, Phys. Rev. **141**, 1532 (1966).

[11] R. E. Cutkosky, Rev. Mod. Phys. **33**, 448 (1961); Phys. Rev. **125**, 745 (1962); Nucl. Phys. **37**, 57 (1962).

denote by c; it will possess both right- and left-hand branch cuts in this variable. We represent $T_{\xi'\xi}(c)$ as follows:

$$T_{\xi'\xi}(c)=\Gamma_{\xi'}(c)G_{\xi'\xi}(c)\Gamma_{\xi}(c)+T_L(c)+T'(c). \quad (1)$$

Here $\Gamma_{\xi}(c)$ is a vertex function $\Gamma_{ab}(c)$ with both a and b on their mass shells. The contribution of all left-hand branch cuts is included in $T_L(c)$, and $T'(c)$ is a somewhat more complicated term which is to be subtracted in order to avoid double counting. The term T_L is eliminated by considering the absorptive part of $T_{\xi'\xi}$ which is associated with the right-hand cut. Since $\Gamma_{\xi}(c)$ is known through the calculation in step (3), we obtain $G_{\xi'\xi}(c)$. Note that the pole in G is guaranteed to have the correct position and residue through the calculation in step (3). The first term on the right-hand-side of (1) is what we call the quasicurrent interaction term.

We determine T' by considering a use to which $G_{\xi'\xi}(c)$ will be put: We are going to calculate the scattering $\bar{b}_1+b_2\rightarrow a_1+\bar{a}_2$, perhaps again in the ladder approximation. The potential will be given by

$$I(a_1,\bar{a}_2;\bar{b}_1,b_2)=\Gamma(\bar{a}_1,\bar{b}_1,c)G_{\xi'\xi}(c)\Gamma(a_2,b_2,c), \quad (2)$$

where now none of the particles is restricted to the mass shell. When we iterate this term by means of the B-S equation, we generate the graph B in Fig. 1. By crossing symmetry, if we take the solution of the B-S equation which uses the potential of Eq. (2), and project out the cross-channel partial wave which contains the quasicurrent interaction, we should obtain $T_{\xi'\xi}(c)$ again. For consistency, therefore, $T'(c)$ must contain the right-hand absorptive part of graph 1(b). If we go beyond the ladder approximation, and include graph (c) of Fig. 1 in the interaction $I(a_1,\bar{a}_2;\bar{b}_1,b_2)$, then T' must also include the right-hand absorptive part of this graph.

Note that if $T_{\xi'\xi}(c)$ contains anomalous thresholds, these are contained either in T' or in the vertex factors, so that G will have only the normal thresholds.

IV. SYMMETRY VERTEX EQUATIONS

Symmetrical vertex equations can be constructed according to the following method: We form a functional $\Gamma^0(\Gamma_x;a,b,c)$ of the (truncated) Bethe-Salpeter amplitudes Γ_x which has the property that it automatically reduces to the appropriate amplitude when any of the three particles is on the mass shell [$\Gamma_a{}^0(\Gamma_x;b,c)\equiv\Gamma_a(b,c)$]. We shall use our knowledge of the singularity structure of graphs which contain vertex parts to improve the extrapolation of Γ away from the mass shell. We use the approximate functional Γ^0 and appropriately constructed propagators in forming an effective interaction $I(\Gamma_x)$, which is to contain all irreducible graphs up to some given order. The "homogeneous" Bethe-Salpeter equation will then take the form

$$\Gamma_a=\Lambda_a\Gamma^0(\Gamma_x), \quad (3)$$

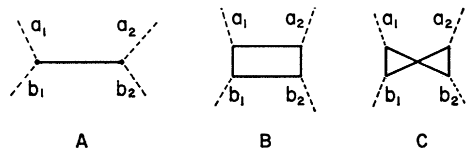

FIG. 1. Graph (a) represents the quasicurrent interaction; (b) and (c) describe two subtraction graphs. The dashed lines represent particles which are on the mass shell.

where Λ is a nonlinear integral operator on the Γ which corresponds to a set of irreducible vertex parts. (We shall use consistently, in this section, a subscript letter on a function to indicate that its value on the corresponding mass shell has been taken.) Since Γ^0 is expressed in terms of the Γ_x, $\Lambda\Gamma_0$ can be expressed as a somewhat more complicated operator R acting directly on the Γ_x:

$$\Gamma_a=R_a\Gamma_x. \quad (4)$$

Equations (3) and (4) automatically possess vertex symmetry, provided that Γ^0 involves the particles (a,b,c) symmetrically; if a second particle is also put on the mass shell, the form of the resulting equation

$$\Gamma_{ab}=R_{ab}\Gamma_x \quad (5)$$

is independent of which of the two of Eq. (3) we start with. Moreover, for given Γ_x, $I(\Gamma_x)$ is self-adjoint, so that (in contrast with the CLM approximation scheme) there is no difficulty in maintaining simultaneously vertex symmetry and the symmetry of the scattering matrix.

We might use for the functional $\Gamma^0(\Gamma_x;a,b,c)$ any of the following expressions[12]:

$$\begin{aligned}
A&=\Gamma_a+\Gamma_b+\Gamma_c-\Gamma_{ab}-\Gamma_{bc}-\Gamma_{ca}+\Gamma_{abc},\\
B&=\Gamma_{abc}{}^{-1}[(\Gamma_a\Gamma_{bc}+\Gamma_b\Gamma_{ca}+\Gamma_c\Gamma_{ab})\\
&\qquad-2(\Gamma_{ab}\Gamma_{ac}+\Gamma_{bc}\Gamma_{ba}+\Gamma_{ca}\Gamma_{cb})]\\
&\qquad\qquad+2(\Gamma_{ab}+\Gamma_{bc}+\Gamma_{ca}-\Gamma_{abc}),\\
C&=\tfrac{2}{3}(\Gamma_a\Gamma_b/\Gamma_{ab}+\Gamma_b\Gamma_c/\Gamma_{bc}+\Gamma_c\Gamma_a/\Gamma_{ca})\\
&\qquad-\tfrac{1}{3}(\Gamma_a\Gamma_{bc}+\Gamma_b\Gamma_{ca}+\Gamma_c\Gamma_{ab})/\Gamma_{abc},\\
D&=\Gamma_{abc}\Gamma_a\Gamma_b\Gamma_c/\Gamma_{ab}\Gamma_{bc}\Gamma_{ca}.
\end{aligned} \quad (6)$$

We could also use linear combinations of these or of still more complicated expressions. Through use of an arbitrary modulating function $F(a,b,c)$, we can further generalize any Γ^0 as follows:

$$\Gamma^{0\prime}(\Gamma_x;a,b,c)=F(a,b,c)\Gamma^0(\Gamma_x/F_x;a,b,c). \quad (7)$$

Our rule for improving the approximate functional is an extension of that used in the last section. We consider a six-line amplitude and project out the part which contains an interaction between quasicurrents J_α, J_β, and J_γ:

$$T_{\alpha\beta\gamma}(a,b,c)=\Gamma_\alpha(a)G_{\alpha\alpha'}(a)\Gamma_\beta(b)G_{\beta\beta'}(b)\Gamma_\gamma(c)G_{\gamma\gamma'}(c)\times\Gamma(a,b,c)+T'(a,b,c). \quad (8)$$

[12] These formulas can be used literally if the particles are spinless. Otherwise Γ has several components, which must be treated separately.

Here $T'(a,b,c)$ is to contain all the terms which would give Γ an incorrect analytic structure, i.e., left-hand branch cuts in any of the energy variables a, b, or c, as well as terms which correspond to graphs not sufficiently connected to possess, simultaneously, poles or normal thresholds in a, b, and c, etc. In other words, the rule is: calculate $T_{\alpha\beta\gamma}(a,b,c)$ using the G and Γ and all appropriate Feynman graphs; then choose the Γ so that the answer will have the analytic structure that would be required by a Lagrangian perturbation theory in which all the particles were "elementary." (The rule of Sec. III can be expressed in the same way.) For example, $\Gamma(a,b,c)$ should have the appropriate anomalous threshold corresponding to the graph shown in Fig. 2.

Now, Γ^0 will have anomalous thresholds, but they will be in the wrong places, in general, because of replacing some of the variable energies by fixed masses. One way to fix this would be through use of a cleverly chosen modulating function in Eq. (7). In order to formulate concisely a somewhat more systematic procedure, we introduce a projection operator P, which means, "take the mass-shell value"; $P\Gamma\equiv\Gamma_{abc}$. We also select a suitable function $K(a,b,c)$, which is chosen to suit convenience, but must satisfy $PK=1$, not have any (incorrect) singularities too close to the mass shell, and lead to convergent integrals. Then, a zero-order approximation to Γ is provided by $\gamma=KP\Gamma$; another approximation is given by $\Lambda\gamma$, but in general, $P\Lambda\gamma\neq\Gamma_{abc}$. We may use, as a corrected vertex functional, the following:

$$\Gamma^{0\prime}(\Gamma_x)=\Gamma^0(\Gamma_x)+\Lambda\gamma-\Gamma^0(\Lambda_x\gamma). \tag{9}$$

The expression (9) retains all the desirable features of the original Γ^0, and in addition we have added in a function which contains some of the correct anomalous branch cuts, at the same time subtracting the corresponding incorrect branch cuts. Unfortunately, not all singularities correspond to irreducible diagrams, so we have to go further, and use in (9) a γ' which might be obtained by iteration:

$$\gamma'=\gamma+\Lambda\gamma-KP\Lambda\gamma. \tag{10}$$

This procedure can be continued, and will enable us to introduce into the operator R the anomalous threshold discontinuities of Γ which lie in some region of the variables a, b, and c which is near to the mass shell. The normal threshold discontinuities have to be incorporated into Γ indirectly, by having the right anomalous branch cuts, and through the solution of Eq. (4); the normal thresholds are, of course, always in the right place.

The above discussion pertains to the problem of constructing, in a certain approximation, the nonlinear operator R which appears in Eq. (4). The problem of solving (4) is much harder, and we have to consider whether it is feasible. A simple iterative scheme constructed along the following lines will often be efficacious. It should be noted that the most straightforward iteration of (4) will usually diverge, but the iteration can often be controlled through a separate treatment of the coupling constants. We write the nth approximation in the form $\Gamma_a{}^n(b,c)=g_{abc}{}^nH_a{}^n(b,c)$, where $PH=1$. The equation

$$g^n=PR(g^nH^n) \tag{11}$$

is a nonlinear algebraic equation for the $g_{abc}{}^n$, if the H^n are considered as given functions; this equation is solved for the g's. Then we write

$$X_a{}^n(b,c)=(g_{abc}{}^n)^{-1}R_a(g^nH^n)(b,c). \tag{12}$$

A linear combination of the form $H^{n+1}=tX^n+(1-t)H^n$ can be used to begin the next step. (For the model considered in the next section, it is convenient to take $t=\frac{1}{2}$.) Note that if R is a linear operator, and we have to find the first eigenvalue instead of a scale factor for the solution, the procedure given reduces to a standard method. In the case of our nonlinear equation, we are also looking for the solution in which the effect of the attraction is greatest, because we do not want the same potential $I(\Gamma_x)$ to have any states which are more deeply bound; this fact helps convergence.

In the ladder approximation, Eq. (11) can be written in the form

$$g_{abc}=\sum_{efg} D_{abc}{}^{efg}g_{afg}g_{ebg}g_{efc}, \tag{13}$$

where the factor $D_{abc}{}^{efg}$ depends only on the G's and on the H^n, and not on the form of the functional Γ^0. The form of Γ^0 enters only indirectly; it affects the form of the solution H^∞. In previous work, the approximation made was that the form of H was chosen on the basis of intuition and general arguments, and an iteration was not carried out.[4,7,8]

V. A SIMPLE MODEL

We shall consider here, as an illustration, the so-called static model of the interaction of baryons and mesons, which has been treated according to the linear (nonsymmetric) Bethe-Salpeter equation by Belinfante and Renninger,[13] whose work we shall follow. Before we proceed, however, we must emphasize that there is actually a fundamental inconsistency between this model and complete vertex symmetry, because the mesons are not ordinarily considered as bound states of baryon-antibaryon pairs, and in fact, virtual mesons can be treated as remaining on the mass shell. Moreover, the prescription given in the last section for improving the form of the vertex functional becomes ambiguous, because the anomalous thresholds all coincide with normal ones. In our model, we shall only try to maintain symmetry between the two baryon lines, and for the improvement term $\Lambda\gamma$ shall take a function suggested by the form of the amplitude for a deeply bound state of two equally massive heavy particles. As usual, we

[13] J. G. Belinfante and G. H. Renninger, Phys. Rev. 148, 1573 (1966).

cut off the meson momentum integration at large values, and, as in Ref. 13, we approximate the form factor by a delta function centered at a meson energy which we take as the unit.

The interaction $I(a,b,e)$ depends on the total energy e (measured from the baryon mass) and also on the energies a and b of the final and initial mesons, respectively. In the ladder approximation I takes the form

$$I(a,b,e)=r\Gamma(a-e,a+b-e)G(a+b-e)\times\Gamma(a+b-e,b-e), \quad (14)$$

where $\Gamma(a,b)$ is the vertex function, $G(a)$ is the baryon propagator, and r is a suitably normalized Racah coefficient for the representations to which the multiplets of degenerate baryons and mesons are assumed to belong. [In the $SU(6)$ model, $r=11/15$.] With the notation $\phi(a)=\Gamma(a,0)$, the vertex equation takes the form

$$\phi(a)=Cr\Gamma(a,a+1)G(a+1)\Gamma(a+1,1)G(1)\phi(1), \quad (15)$$

where C is an uninteresting numerical factor that enters via the suppressed integration. The normalization condition is[6,13]:

$$1=C\phi(1)^2G'(1)+C^2r\phi(1)^2G(1)^2\times[\Gamma(1,2)^2G'(2)+2\Gamma(1,2)\Gamma'(1,2)G(2)], \quad (16)$$

where

$$G'(a)=(dG(a-e)/de)_{e=0}$$

and

$$\Gamma'(a,b)=(d\Gamma(a-e,b-e)/de)_{e=0}.$$

With use of (15), the normalization condition simplifies to

$$1=C\phi(1)^2G(1)[G'(1)/G(1)+G'(2)/G(2)+2\Gamma'(1,2)/\Gamma(1,2)]. \quad (17)$$

We shall choose the improvement term $\Lambda\gamma$ to be a function of the squared sum of the four-momenta of the incoming and outgoing baryon lines, as is suggested by the form of the B-S amplitude for a deeply bound state. It may be parametrized as follows:

$$\Lambda\gamma(a,b)=\phi(0)B(A+a+b)^{-1}. \quad (18)$$

A family of suitable vertex functions is given by

$$\Gamma^0(a,b)=\frac{\phi(0)B}{A+a+b}+\alpha\left\{\frac{\phi(a)\phi(b)}{\phi(0)}-\frac{\phi(0)AB}{(A+a)(A+b)}\right\}+(1-\alpha)\left\{\phi(a)+\phi(b)-\phi(0)\times\left[\frac{B}{A+a}+\frac{B}{A+b}+1-\frac{B}{A}\right]\right\}. \quad (19)$$

We use the following approximation for the baryon propagator:

$$G(a)=a^{-1}+\beta. \quad (20)$$

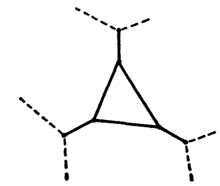

FIG. 2. A graph which generates a singularity of the vertex function.

The term β is supposed to represent the effect of the branch cut of $G(a)$; it includes contributions from very high energies which lie outside the region of validity of our model, so we treat it as an adjustable parameter. In fact, such an adjustable parameter is necessary, in order that (17) can be satisfied by the solution to (15). An adjustable factor was used in Refs. 7 and 8 to allow normalizability; the present model allows us to understand that factor in a more physical way. When the static model is treated according to the CLM scheme, a similar adjustable parameter referring to high-energy phenomena outside the model is also needed, if a solution is to exist, as shown by Huang and Low[14] and Huang and Mueller.[15]

The cutoff energy fixes the energy scale of the model, and also the value of C, which determines a multiplicative factor in $\phi(a)$. The properties of the solution do not seem to depend very much on the parameters A, B, and α, at least if $\alpha\approx 2$, in which case Eq. (15) has the property that $\phi(a)$ for small values of a is not very sensitive to the asymptotic value of ϕ for large a. At least in this case, the iteration method described in the last section seems to converge quite rapidly, if one starts, for example, with a trial solution which is constant.

If α is chosen to have the special value

$$\alpha=\phi(0)/[\phi(0)-\phi(1)]=(1-h)^{-1}, \quad (21)$$

the values $g=\phi(0)$ and $gh=\phi(1)$ are given by equations in which higher meson energies do not enter:

$$g=Crg^3h^2(\beta+1)^2\times\left\{h+B\left[\frac{1}{A+2}-\frac{\alpha A}{(A+1)^2}-(1-\alpha)\left(\frac{2}{A+1}-\frac{1}{A}\right)\right]\right\}, \quad (22)$$

$$gh=Crg^3h(\beta+1)(\beta+\tfrac{1}{2})\times\left\{h+B\left[\frac{1}{A+3}-\frac{\alpha A}{(A+1)(A+2)}-(1-\alpha)\left(\frac{1}{A+1}+\frac{1}{A+2}-\frac{1}{A}\right)\right]\right\}. \quad (23)$$

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[14] K. Huang and F. E. Low, Phys. Rev. Letters 13, 596 (1964); J. Math. Phys. 6, 795 (1965).

[15] K. Huang and A. H. Mueller, Phys. Rev. 140, B365 (1965).

If $B=0$, there is one solution with $h=(\beta+\frac{1}{2})/(\beta+1)$ (the question of uniqueness has not been looked into in the general case). A linear equation with the same $G(a)$ would also lead to this value for h. In order to discuss the normalization, it is necessary to construct the entire solution, because the derivative of $\phi(a)$ is needed. It appears that β usually lies in the range $\frac{1}{2}\lesssim\beta\lesssim 1$. The value of Cg^2 remains close to the value obtained from the linear B-S equation, showing that the vertex modifications tend to damp out the high-energy contributions, and thereby compensate for the use of a propagator which remains large.

A detailed discussion of the results is omitted. The main point of this section is that it is possible to construct reasonable looking nonlinear equations which are not only easy to solve, but whose solutions appear to have a reasonable behavior. Although this example is rather academic, it is possible that it would give interesting results if applied to baryons which were not assumed to be degenerate.

The off-mass-shell scattering amplitude $T(a,b,e)$ is the solution of the equation

$$T(a,b,e)=I(a,b,e)+\int d\rho(c)I(a,c,e)G(c-e)T(c,b,e), \quad (24)$$

where $I(a,b,e)$ is given by Eq. (14). The mass-shell amplitude $T(e)$ is obtained by setting $a=b=e$. A simple approximate way to solve (24) in the neighborhood of the pole at $e=0$ is to use Schwinger's variational method with a trial function proportional to $\phi(a)$. The amplitude $T(e)$ has both right- and left-hand branch cuts, which are related by crossing symmetry. The integral on the right side of (24) does not have a left-hand branch cut, so this must be provided by $I(e,e,e)=r\phi(e)^2G(e)$. Therefore, when we solve the scattering problem, we learn something about the nearby singularities of G, and have the possibility redoing the calculations with a more exact form for $G(e)$. However, in the present model we would still need to use a parameter like β to represent more distant parts of the branch cut of $G(e)$. If the baryons were treated relativistically, or if meson-exchange forces were included in $I(a,b,e)$, then the integral in (24) would also contribute to the discontinuity across the left-hand branch cut.

VI. PHENOMENOLOGICAL VECTOR-MESON PROPAGATORS

The ideas presented above can be applied to a feature of phenomenological vector-meson interactions which has been used in some recent work by Capps[16] and also by Jacobs and the author.[17] In the treatment by Capps, this feature was introduced as a "subtraction" term, in order to maintain $SU(6)$ symmetry. We shall show that although this feature is somewhat model-dependent, it is associated naturally with a composite-particle picture of the vector mesons.

For the quasicurrent J we consider a pair of particles which we may take to have a nonzero spin; then we must specify the combination of helicity states that enters into the definition of J and of the partial-wave amplitude T. In particular, the spins may be chosen to enter in manner corresponding to a "convective" current J_1 or to a "spin" current J_2, that is, to vertices having roughly the form

$$\Gamma_{\mu 1}\propto P_\mu-P_\mu{}',$$
$$\Gamma_{\mu 2}\propto S_{\mu\nu}k^\nu, \quad (25)$$

where P_μ and $P_\mu{}'$ are the momenta of the two particles, S is an antisymmetric spin tensor, and $k_\mu=P_\mu+P_\mu{}'$. Let us consider the absorptive parts of the amplitudes $T_{11}(k^2)$ and $T_{22}(k^2)$, assuming, for the sake of argument, that the vector mesons are stable against decay into pseudoscalar mesons. In addition to the poles, these absorptive parts contain contributions from two-meson states and baryon-antibaryon states. We may expect the two-meson contributions to be similar, and in fact, they both have a P-wave threshold behavior. In Capp's model,[16] the coefficients are proportional to the residues of the poles. The threshold behavior of the baryon-antibaryon states is different for the two amplitudes, but it can be shown to be approximately self-consistent to assume that T_{11} and T_{22} are nearly proportional outside the threshold region.[17] Since the threshold region is small compared with the distance from the threshold to the pole, we may neglect the difference. The same remarks apply to the subtraction terms T_L and T'. The natural expectation, therefore, is that the two expressions $I_{11}=\Gamma_{\mu 1}G_{11}{}^{\mu\nu}\Gamma_{\nu 1}$ and $I_{22}=\Gamma_{\mu 2}G_{22}{}^{\mu\nu}\Gamma_{\nu 2}$ should have a similar dependence on k^2, provided that we don't need a subtraction when we try to reconstruct the entire propagator by integrating over the absorptive part.

On the other hand, the usual pole approximation for the propagators,

$$G_{\mu\nu}(k)=(g_{\mu\nu}-k_\mu k_\nu/m^2)(k^2-m^2)^{-1}, \quad (26)$$

when used with (25), leads to expressions for I_{11} and I_{22} which have entirely different behavior when $k^2\sim 0$. This suggests that a more exact phenomenology is obtained if the form (26) is given up and the effective propagator is constructed in such a way that both I_{11} and I_{22} will vanish when k^2 is close to zero. In order to do this, we define several propagators G_{11}, G_{22}, G_{12}, etc., by using the independent quasicurrents J_1 and J_2, and use them with the corresponding vertex functions. (Vertex terms corresponding to higher multipoles could also use G_{22}; the essential point is that the convective current ought to be treated separately.) Thus, we exploit the non-

[16] R. H. Capps, Phys. Rev. **148**, 1332 (1966); **150**, 1263 (1966); Phys. Rev. Letters **16**, 1066 (1966).

[17] R. E. Cutkosky and M. Jacobs (unpublished).

uniqueness of the quasicurrents in order to satisfy crossing symmetry more exactly.

The vertex equation has such a structure that a solution with

$$k^\mu\Gamma_\mu=0 \qquad (27)$$

is self-consistent. That is, if we assume (27), the B-S amplitudes will also possess this property, and it can be retained in the procedure given in Sec. IV for constructing the general vertex from the B-S amplitudes. This suggests that a more accurate approximation than (25) is

$$\Gamma_{\mu1}\propto k^2(P_\mu-P_\mu')-k_\mu(P^2-P'^2). \qquad (28)$$

[$\Gamma_{\mu2}$ already has the property (27).] A factor k^{-2} must then be included in G_{11} so that I_{11} will not have an extra zero for small k^2. If we use the following expression for I_{22},

$$I_{22}=g_2g_2'S_{\mu\nu}S'^{\mu}{}_{\rho}k^\nu k^\rho F(k^2)(k^2-m^2)^{-1}, \qquad (29)$$

where $F(k^2)$ is a phenomenological form factor, we should use the following for I_{11}:

$$I_{11}=g_1g_1'[k^2(Q_\mu-Q_\mu')(P^\mu-P'^\mu)+(P^2-P'^2)(Q^2-Q'^2)] \times F(k^2)(k^2-m^2)^{-1}, \qquad (30)$$

where Q_μ and Q_μ' are the momenta at the second vertex, satisfying $Q_\mu+Q_\mu'=-k_\mu$, and g_i, g_i' are coupling constants. Equations (29) and (30) are approximations, in that we assume there that P, P', Q, and Q' are nearly on the mass shell. For the full interaction, we also need I_{12}, etc.

Let us apply (29) and (30) to an example in which the two vertices lie on the same baryon line, and assume that the baryon mass M is very large. To the leading order in M, we find in the baryon rest frame that

$$P^2-P'^2\approx Q^2-Q'^2\approx 2Mk_0 \qquad (31)$$

and

$$(Q-Q')_\mu(P-P')^\mu\approx -4M^2,$$

so that the bracketed expression in (30) reduces to $4M^2\mathbf{k}^2$. Both I_{11} and I_{22}, in this case, depend on the virtual momentum k_μ in the same way, as is assumed in treatments of the static model.[13] (The contribution of I_{12} vanishes.) In other words, our treatment of the vector-meson vertices and propagators, which is both relativistic and crossing-symmetric, provides a natural generalization of the static model. Further implications of (30) are discussed elsewhere.[16,17]

Our conclusion, therefore, is that the form of (30) does not need to be imposed through choice of an arbitrary subtraction constant but is a consequence of crossing symmetry provided that in the dispersion relation no subtractions are necessary. There are some cases in which the nature of the interaction at small distances may be such that we do need a subtraction, and then the usual pole-approximation treatment of vector mesons might be more accurate than (30). For example, this might apply to analysis of electromagnetic form factors in terms of vector mesons. As another example we might consider the use of a gauge-invariant theory involving the conserved baryon-number current,[18] in which we would have to replace (27) by Ward's identity in dealing with one of the vector mesons.

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[18] J. Schwinger, *Theoretical Physics* (Trieste Seminar, IAEA, 1962), p. 89.