General Solution of the One-Channel-Scattering Singular Integral Equations

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Integral equations for one-channel scattering are written and solved, starting from dispersion relations for the generalized Jost function in the momentum k plane. This method is an alternative to the conventional N/D method, but it allows a simple, physically meaningful generalization to the many-channel case, where dispersion relations and integral equations can be written for a unique generalized Jost function in the complex plane of a suitable variable which uniformizes all the right-hand cuts of the scattering amplitude. Even in the pure elastic-scattering case, a unified treatment is possible, whether the phase shift at infinity is or is not an integral multiple of π. In all cases, our singular integral equations are reduced to a Fredholm-type i ntegral equation with a Hilbert-Schmidt kernel.

I. INTRODUCTION

In recent years many authors[1-3] have made interesting investigations devoted to the effect of the poles situated on different Riemann sheets of the amplitude of a multichannel reaction. They were concerned especially with the simplest case, the two-channel reaction. An interesting physical result was obtained by Frazer.[2] If the resonance is situated on a suitable, remote Riemann sheet, then the maximum of the cross section is always shifted to the same energy value, namely the threshold of the second channel. On the other hand, all elements of the S matrix can be written as quotients of the values of a unique Jost function on various Riemann sheets.[3,4] The above-mentioned facts justify a more careful investigation of the analytical properties of this unique Jost function on the whole Riemann manifold.

To this end, one can write a dispersion relation for the unique Jost function in the complex plane of a suitable variable which uniformizes all right-hand-cut Riemann sheets, and then derive integral equations from the conditions imposed by unitarity. Of course, the integral equation for the unique Jost function will be an alternative to the many-channel N/D method,[5,6] and at the moment it is difficult to say what would be the real advantages of this method.

For the one-channel case, such an integral equation has already been written.[7,8] The main difficulty of this "f/f" method was considered to be the fact that the integral equation is not of the Fredholm type. For instance, only the case when the iterative series of the integral equation converges is considered in Ref. 7.

Since such problems are of capital importance for the general multichannel case, in the present paper we focus our attention on the construction and solution of the integral equation for the one-channel elastic scattering, starting from dispersion relations for the generalized Jost function in the complex momentum plane.[9] We emphasize that the singular integral equations are brought to a Fredholm-type form whether the phase shift at infinity $\delta(\infty)$ is, or is not, an integral multiple of π.[10]

As in Ref. 8, we uniformize the right-hand cut of the scattering amplitude from the s plane, passing to the k plane, where the S matrix satisfies the well-known reality and unitarity relations (Fig. 1):

$$S^*(-k^*)=S(k), \tag{1.1}$$

$$S^*(+k^*)=S^{-1}(k). \tag{1.2}$$

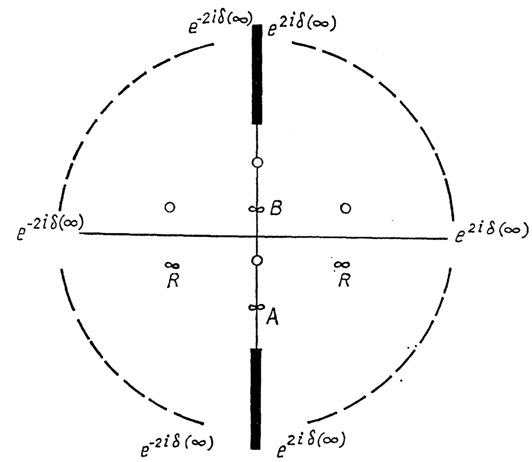

FIG. 1. The asymptotic behavior of the S matrix. B, A, R—bound-, antibound-, and resonant-state poles. ○—zeros of the S matrix.

[1] R. E. Peierls, Proc. Roy. Soc. (London) **A253**, 16 (1959).
[2] W. R. Frazer and A. W. Hendry, Phys. Rev. **134**, B1307 (1964).
[3] M. Kato, Ann. Phys. (N. Y.) **31**, 130 (1965).
[4] K. J. LeCouteur, Proc. Roy. Soc. (London) **A256**, 115 (1960).
[5] J. B. Hartle and J. R. Taylor, Princeton University (unpublished).
[6] R. L. Warnock, paper submitted to the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966 (unpublished).
[7] V. de Alfaro and T. Regge, Nuovo Cimento **20**, 957 (1961).
[8] R. Omnès, Nuovo Cimento **21**, 524 (1961).

[9] Here we deal with the simplest case of the scattering of two spinless particles of equal mass. We take the system of units in which $\mu=\hbar=c=1$; k denotes the c.m. momentum and s is the square of the c.m. energy; $s=4(k^2+1)=4(\nu+1)$, $\nu\equiv k^2$.
[10] $\delta(\infty)$ may differ from the limit of the physical phase shift at infinity by a multiple of π. A discussion about the choice of the parameter $\delta(\infty)$ is given in Sec. IV after Eq. (4.34c).

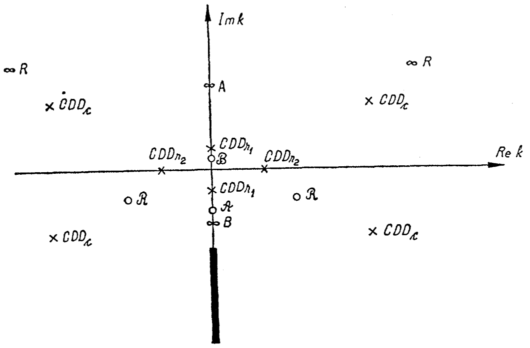

FIG. 2. The singularities and the zeros of the Jost function in the k plane. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{R}$—output antibound, bound, and resonant zeros of the Jost function. A, B, R—input antibound, bound, and resonant poles. CDD_r—"real" CDD poles. CDD_c—"complex" CDD poles.

Combining these two relations we obtain

$$S(-k)=S^{-1}(k) \tag{1.3}$$

which allows the factorization

$$S(k)=F(-k)/F(k)\,, \tag{1.4}$$

where $F(k)$ will be called the generalized Jost function. As will emerge clearly from Sec. III, the key to our method consists in dividing the S matrix into two parts

$$S(k)=S_0(k)S'(k)\,. \tag{1.5}$$

The first factor $S_0(k)$ is a rather arbitrary function of k, having two cuts, and is chosen to have an $\exp[\pm 2i\delta(\infty)]$ asymptotic behavior,[11] i.e., the same asymptotic behavior as the whole S matrix. If we now impose the properties (1.1)–(1.3) on $S_0(k)$, both S_0 and S' can be factorized in a way similar to (1.4), i.e.,

$$S_0(k)=f_0(-k)/f_0(k)\,, \tag{1.6a}$$

$$S'(k)=f(-k)/f(k)\,, \tag{1.6b}$$

where $f_0(k)$ is a *known function* [see Sec. III, Eq. (3.8a)] having *known discontinuities* along its *two cuts*, while the second Jost factor $f(k)$, related to $S'(k)$, has only one cut (see Fig. 2). There are no inconveniences related to the fact that $F(k)$ has two cuts; two cuts are produced by the factor $f_0(k)$. If one wishes to define the usual $D(s)$ function on its first (second) Riemann sheet through the values of $F(k)$ in the upper (lower) k half plane, this $D(s)$ function will exhibit a left-hand cut even on its first Riemann sheet (in the s plane), owing to the analytical properties of the known function f_0. Of course, $D(s)$ could be defined without this left-hand cut [by taking another $f_0(k)$ with only one cut, in the lower half plane], but in that case it would be divergent at infinity, as actually happens in the paper of Atkinson and Contogouris.[12]

[11] The condition that the phase shift at infinity must have a finite limit will be relaxed in a subsequent paper in which the unequal-mass case will be treated also.

[12] D. Atkinson and A. Contogouris, Nuovo Cimento **39**, 1082 (1965).

In order to establish the physical content of the inhomogeneous term of our integral equations (the "physical input"), in Sec. II the poles [Castillejo–Dalitz–Dyson (CDD) and elementary] of the Jost function in the k plane are studied. As will be emphasized in that section, the CDD poles can be replaced by some "input" poles—elementary particles—situated either on the positive imaginary axis or in the lower half k plane. Therefore, even a resonance can be treated as elementary, i.e., as an "input force" for our problem.

In Sec. III a Riemann problem is stated for the second factor $f(k)$ of the generalized Jost function, which leads to a singular integral equation for this function.

Section IV is devoted to the study of the solution of this equation. The kernel of this integral equation has a marginal singularity at infinity and is split into a regular part and a Dixon kernel. The Dixon equation is then solved by means of a Mehler transformation.[13] Although the context is different, we use the same methods as Atkinson and Contogouris[12] used for their nonrelativistic $N(s)$ equation, and we obtain finally a Fredholm equation for the Jost factor $f(k)$. This equation is then solved by the Hilbert–Schmidt method. To ease the task of the reader, in Appendix A the Phragmen–Lindelöf theorem is stated, while Appendix B is devoted to a short review of the theory of the singular integral equations, which is necessary for Sec. III.

II. CDD POLES AND INPUT AND OUTPUT PARTICLES

The poles of the S matrix can be produced either by the poles of the numerator or by the zeros of the denominator of Eq. (1.4). Particles related to the first ones will be called input particles, because they correspond to input poles [see Eq. (2.3)] for the equations we shall derive. In contradistinction to the input poles, the zeros of the Jost function are known only after the integral equations are solved; therefore, these poles of the S matrix related to these zeros will be called output particles. Of course both input and output particles may be bound, antibound, or resonant states. For instance, the composite bound states of an electron and a proton (the hydrogen atom) are "output" particles.

We start from the case in which the imaginary part of the scattering amplitude vanishes quickly enough at infinity in order to obtain Fredholm N/D equations. (Of course, $S(\infty)=1$.) The Jost function has in this case only a negative imaginary cut[14] and poles related to elementary particles and CDD poles. $S_0(k)$ is taken identical to 1, i.e., $F(k)=f(k)$. Owing to the reflection

[13] V. Ditkin and A. Prudnicov, *Integral Transforms and Operational Calculus* (Pergamon Press, Inc., New York, 1962).

[14] The usual convention is $S(k)=F(k)/F(-k)$, i.e., $F(k)$ is defined usually with a positive imaginary cut, but in the present paper *we adopt the opposite convention* (1.4) which allows a smooth generalization towards the LeCouteur-Newton function $d(s)=\det\{D(s)\}$ from the many-channel case (see Ref. 3).

property

$$f^*(-k^*)=f(k)\,, \tag{2.1}$$

we put

$$f(k)=1+\phi(k) \tag{2.2}$$

with

$$\phi(k)=\pi^{-1}\int_{-i\infty}^{-i}\frac{\sigma(k')dk'}{k'-k}+\sum_j\frac{ir_j}{k-im_j}+\sum\frac{a_j+ikb_j}{(k-c_j)(k+c_j^*)}\,, \tag{2.3}$$

where the quantities $\sigma(k')$, m_j, r_j, a_j, and b_j are real.

A CDD pole on the real axis of the energy plane is given either by a pair of poles situated on the real k axis:

$$\frac{a_j+ikb_j}{(k-d_j)(k+d_j)}=\frac{a_j+ikb_j}{k^2-d_j^2} \tag{2.4}$$

(corresponding to a CDD pole on the right real $\nu=k^2$ axis), or by a pair of poles located symmetrically on the imaginary k axis

$$\frac{ir_j^{(1)}}{k-in_j}+\frac{ir_j^{(2)}}{k+in_j}=\frac{-n_j(r_j^{(1)}-r_j^{(2)})+ik(r_j^{(1)}+r_j^{(2)})}{k^2+n_j^2} \tag{2.5}$$

(corresponding to a CDD pole on the left real ν axis).

Since

$$S(\nu)=1+2i\rho(\nu)[N(\nu)/D(\nu)]\,, \tag{2.6}$$

it follows that

$$N(k^2)=[f(-k)-f(k)]/2i\rho(k)\,. \tag{2.7}$$

If a parameter b_j from (2.4) is zero, a pole of the type in Eq. (2.4) appears only in the D function; if not, the pole appears in both N and D, and therefore is reducible. The unitarity condition provides a relation between the residues of N and D:

(*s<4 case*): For a pole of the type in Eq. (2.5), the residues of N and D are both real and independent:

$$\begin{aligned}&\mathrm{Res}D|_{\nu=-n_j^2}=-2n_jr_j^{(1)}\,,\\&\mathrm{Res}N|_{\nu=-n_j^2}=-(r_j^{(1)}+r_j^{(2)})(-n_j^2+1)^{1/2}\,.\end{aligned} \tag{2.8a}$$

(*4<s case*): If the CDD is placed on the right-hand cut [the (2.4) case] $\mathrm{Res}N=-(1/\rho)$ Im $\mathrm{Res}D$ with

$$\mathrm{Res}D(s\pm i\epsilon)=a_j\pm id_jb_j\,. \tag{2.8b}$$

Complex CDD poles may also exist, represented by a quadruplet of symmetrical poles

$$\frac{a_j^{(1)}+ikb_j^{(1)}}{(k-c_j)(k+c_j^*)}+\frac{a_j^{(2)}+ikb_j^{(2)}}{(k+c_j)(k-c_j^*)}=\frac{\text{polynomial in }k}{(k^2-c_j^2)(k^2-c_j^{*2})}\,. \tag{2.9}$$

From (2.4), (2.5), and (2.9) we see that all the denominators of the CDD terms are functions of k^2, and therefore, in the expression $f(-k)/f(k)$ which defines the S matrix, they disappear.

It is worth while noting that it is always possible to replace a Jost function $f(k)$ with a complex CDD pole with a Jost function $\bar{f}(k)$ exhibiting the same asymptotic behavior, but with two real CDD poles instead of the complex one:

$$\bar{f}(k)=\kappa(k^2)f(k)\,, \tag{2.10a}$$

$$\kappa(k^2)=\prod_j\frac{(k^2-c_j^2)(k^2-c_j^{*2})}{(k^2-d_j'^2)(k^2-d_j''^2)}\,. \tag{2.10b}$$

Of course, $f(k)$ and $\bar{f}(k)$ are equivalent in the sense that they lead to the same S matrix

$$S(k)=f(-k)/f(k)=\bar{f}(-k)/\bar{f}(k)\,. \tag{2.11}$$

Indeed, as was stressed earlier by Atkinson and Morgan,[15] the positions of the CDD poles are rather arbitrary, since they can be shifted by means of a reducible function $\kappa(k^2)$; of course, their residues must be changed correspondingly.

On the other hand, one may use a function $\kappa(k^2)$ to move CDD poles into the zeros of the Jost function, in this way transforming the output particles generated by CDD poles into input particles.

For instance, if $k=im_j$ are output bound or antibound zeros, then

$$\kappa(k^2)=\prod_j(k^2-d_j^2)/(k^2+m_j^2) \tag{2.12}$$

transforms the Jost function $f(k)$ with CDD poles at $k=\pm d_j$ and zeros at $k=im_j$ into the equivalent Jost function $\bar{f}(k)=\kappa(k^2)f(k)$. Instead of output particles (zeros) and CDD poles, the new Jost function contains only the input particle (bound or antibound) poles at $k=-im_j$.

Composite resonances can be treated in a similar way using the inverse of the function $\kappa(k^2)$ defined by Eq. (2.10b), two pairs of CDD poles being necessary in order to annihilate the pair of complex zeros (output resonances) and to produce an input resonance, i.e., a pair of complex poles.

To distinguish between the various possibilities which may occur, we note first that when the cut due to the potential does not exist, it follows from d'Alembert's theorem that the number of zeros must exceed the number of CDD poles (since a rational function has an equal number of zeros and poles).

This is approximately the case of weak potentials. A new CDD pair of poles with small residues always generates a pair of zeros (new output particles) in its neighborhood. Now, if the residue increases or the po-

[15] D. Atkinson and D. Morgan, CERN preprint 65/1343/5-TH603, 1965 (unpublished).

tential becomes very strong, the zeros can move to the cut and disappear through it.

Therefore, if such a strong interaction between the potential and the CDD poles exists, a deficiency of zeros may occur and one will be forced to maintain the excess of CDD poles in the transformed Jost function.

III. SINGULAR INTEGRAL EQUATIONS FOR THE JOST FUNCTION

As has been stated in Sec. I, the S matrix as a function of k has the properties (1.1)–(1.3). Let the S matrix be bounded at infinity and approach definite, finite limits along the real axis. For $s\to+\infty+i\epsilon$ we shall write the S matrix in the form $\exp\{2i\delta(\infty)\}$, where $\delta(\infty)$ is no longer a multiple of π, i.e., it is any real number. In addition, it is an analytic function in the cut k plane with a finite number of possible poles, so that we can apply the Phragmén–Lindelöf theorem (see Appendix A) with the result that in the right half plane $\mathrm{Re}k>0$, the function $S(k)$ has the same value at infinity in any direction:

$$\lim_{k\to\infty,\,(\mathrm{Re}k>0)} S(k)=\exp[2i\delta(\infty)]. \qquad (3.1)$$

A similar result can be obtained for the left half plane $\mathrm{Re}k<0$, i.e.:

$$\lim_{k\to\infty,\,(\mathrm{Re}k<0)} S(k)=\exp[-2i\delta(\infty)]. \qquad (3.2)$$

(See Fig. 1.) These results are in accordance with the properties (1.1)–(1.3) required for $S(k)$.

The above-stated results express the fact that the discontinuities of the S matrix at infinity on the two cuts are equal.

In order to have the same asymptotic behavior for $S_0(k)$ as for $S(k)$, not only on the real axis but also in any direction at infinity, we shall take for $S_0(k)$ the same cuts as for $S(k)$, i.e., the cuts $(i,i\infty)$ and $(-i,-i\infty)$. The poles of $S(k)$ will all be left in $S'(k)$. We write

$$S_0(k)=\exp[2id_0(k)], \qquad (3.3)$$

where $d_0(k)$ has the same cuts as $S(k)$. Since $S_0(k)$ satisfies the relations (1.1)–(1.3), the following properties result for $d_0(k)$:

$$d_0(-k)=-d_0(+k) \qquad (3.4)$$

and

$$d_0^*(-k^*)=-d_0(k) \qquad (3.5)$$

with the asymptotic forms

$$\lim_{k\to\infty,\,(\mathrm{Re}k>0)} d_0(k)=+\delta(\infty), \qquad (3.6a)$$

$$\lim_{k\to\infty,\,(\mathrm{Re}k<0)} d_0(k)=-\delta(\infty). \qquad (3.6b)$$

A possible expression for $d_0(k)$ is[16]

$$d_0(k)=\delta(\infty)\{[(k^2+1)^{1/2}/k]-k^{-1}\}, \qquad (3.7)$$

where the cuts of the square root are shown in Fig. 3(a). It is important to note that this function makes the conformal mapping of the cut k plane into *the interior* of the circle of radius $\delta(\infty)$ with the center at the origin; therefore $d_0(k)$ is always finite, so that $S_0(k)$ *has no essential singularities.*

Our purpose is to establish an integral equation for the function $f(k)$ starting from the input function

$$\mathcal{A}(i\kappa)\equiv S(i\kappa+\epsilon)+S(i\kappa-\epsilon), \quad (1\leqslant\kappa<\infty) \qquad (3.8)$$

which is supposed to be explicitly given. In fact, from the partial-wave projection of the Laplace transform of the potential, or from the crossing relation in the Chew–Mandelstam theory, we get the explicit form of the discontinuity $\omega(s)$ of the scattering amplitude along the left-hand cut (in the s plane).[17] Now in the equal-mass case, if no lighter particle can be exchanged,[18] the left-hand cut of the partial amplitude coincides with that of $\rho(\nu)=[\nu/(\nu+1)]^{1/2}$. Thus

$$\mathcal{A}(i\kappa)=[1+2i\rho_+A(i\kappa+\epsilon)]+[1-2i\rho_+A(i\kappa-\epsilon)] =2-4|\kappa/(\kappa^2-1)^{1/2}|\omega(i\kappa), \qquad (3.8a)$$

where ω is a known function.

Now using (3.3), Eq. (3.8) takes the form

$$S_0(i\kappa+\epsilon)\frac{f(-i\kappa-\epsilon)}{f(i\kappa+\epsilon)} +S_0(i\kappa-\epsilon)\frac{f(-i\kappa+\epsilon)}{f(i\kappa-\epsilon)}=\mathcal{A}(i\kappa). \qquad (3.9)$$

But in the k plane, $f(k)$ has only a lower-half-plane cut, i.e., for $1\leqslant\kappa<\infty$, $f(i\kappa+\epsilon)=f(i\kappa-\epsilon)=f(i\kappa)$, so that we obtain

$$f(-i\kappa-\epsilon)=-\frac{S_0(i\kappa-\epsilon)}{S_0(i\kappa+\epsilon)}f(-i\kappa+\epsilon) +\frac{\mathcal{A}(i\kappa)}{S_0(i\kappa+\epsilon)}f(i\kappa). \qquad (3.10)$$

Since $f(k)$ can be taken to equal 1 at infinity, we put it in the form

$$f(k)=1+\phi(k), \qquad (3.11)$$

[16] Of course, this is only one of the possible choices. Another possibility would be to choose, for instance,

$$S_0(k)=\{\cos\delta(\infty)+i[(k^2+1)^{1/2}/k]\sin\delta(\infty)\} \times\{\cos\delta(\infty)-i[(k^2+1)^{1/2}/k]\sin\delta(\infty)\}^{-1}.$$

[17] A. Martin, in *Lectures on High Energy Physics* (Federal Nuclear Commission of Yugoslavia, 1961); G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[18] We shall deal with the general unequal-mass case in a subsequent paper.

where the function $\phi(k)$ has the cut $(-i\infty, -i)$ and contains poles of the type described in Sec. II:

$$\sum_j \frac{ir_j}{k-im_j}+\sum_j \frac{a_j+ikb_j}{(k-c_j)(k+c_j^*)}.$$

From (3.10) we obtain:

$$\phi(-i\kappa-\epsilon)=-\frac{S_0(i\kappa-\epsilon)}{S_0(i\kappa+\epsilon)}\phi(-i\kappa+\epsilon)$$
$$+\frac{\mathfrak{a}(i\kappa)}{S_0(i\kappa+\epsilon)}\phi(i\kappa)+\left[\frac{\mathfrak{a}(i\kappa)}{S_0(i\kappa+\epsilon)}\frac{S_0(i\kappa-\epsilon)}{S_0(i\kappa+\epsilon)}-1\right]. \quad (3.12)$$

It is now more convenient to make the change of variable:

$$k=iz$$

which rotates the cut $(-i\infty, -i)$ from the k plane into the cut $(-\infty, -1)$ in the z plane. We denote by t the values of z on this cut and use the same symbols for the functions in the new z plane. In the z variable, the function $d_0(k)$ defined by (3.7) becomes

$$d_0(z)=i\left(\frac{1}{z}+i\frac{(z^2-1)^{1/2}}{z}\right)\delta(\infty), \quad (3.13)$$

where the cuts of the square root are shown in Fig. 3(b).

Equation (3.12) is equivalent to a Riemann problem (see Appendix B) for the open contour $C\equiv[-\infty, -\frac{1}{2}]$ in the z plane:

$$\phi^+(t)=G(t)\phi^-(t)+g(t), \quad -\infty<t\leqslant-1 \quad (3.14)$$

with

$$\phi^\pm(t)\equiv\phi(t\pm i\epsilon), \quad (3.15)$$

$$G(t)\equiv-S_0^+(-t)/S_0^-(-t) \quad (3.16)$$

and

$$g(t)\equiv\frac{\mathfrak{a}(-t)}{S_0^-(-t)}\phi(-t)+\frac{\mathfrak{a}(-t)-\mathfrak{a}_0(-t)}{S_0^-(-t)}, \quad (3.17)$$

where

$$S_0^\pm(-t)\equiv S_0(-t\pm i\epsilon)$$
$$=\exp\left\{2\left(t^{-1}\pm i\frac{|(t^2-1)^{1/2}|}{t}\right)\delta(\infty)\right\}, \quad (3.18a)$$

$$\mathfrak{a}_0(-t)\equiv S_0^+(-t)+S_0^-(-t)$$
$$=2\exp\left(\frac{2\delta(\infty)}{t}\right)\cos\left(2\frac{|(t^2-1)^{1/2}|}{t}\delta(\infty)\right). \quad (3.18b)$$

[The cuts of the square root are shown in Fig. 3(b).]

This Riemann problem can be solved if the free term $g(t)$ given by Eq. (3.17) is Hölder continuous. It results, therefore, that $\mathfrak{a}(t)$ must be also Hölder continuous.

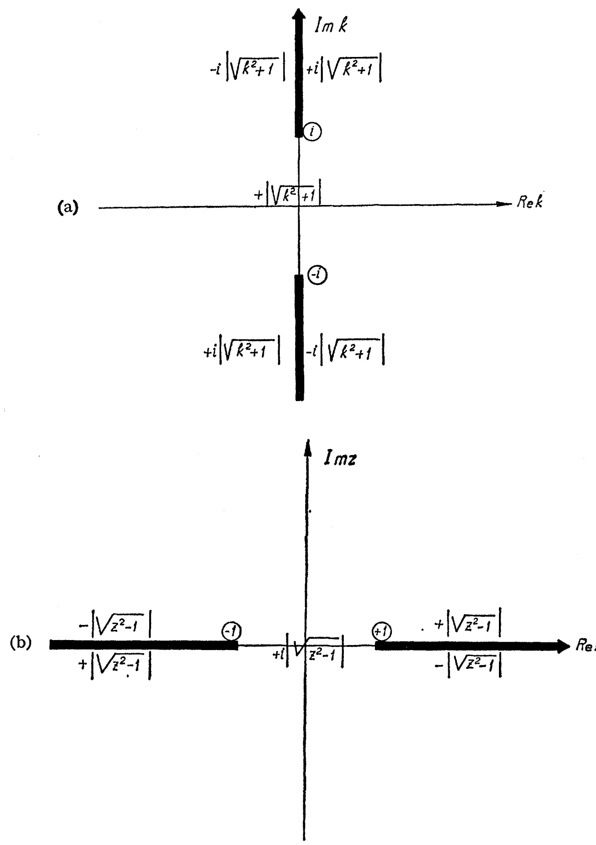

FIG. 3. (a) and (b). The determinations of the square roots $(k^2+1)^{1/2}$ and $(z^2-1)^{1/2}$.

In order to solve it, we introduce the function

$$\Gamma(z)=\frac{z}{2\pi i}\int_{-\infty}^{-1}\frac{\ln G(t)}{t(t-z)}dt=\frac{z}{2\pi i}\int_1^\infty\frac{\ln G(-\tau)}{\tau(\tau+z)}d\tau. \quad (3.19)$$

Explicit evaluation of the integral with G [Eqs. (3.16), (3.18a)] given by

$$G(-\tau)=\exp i[(2m+1)\pi-4\delta(\infty)(\tau^2-1)^{1/2}/\tau] \quad (m=\text{integer}) \quad (3.20)$$

yields

$$\Gamma(z)=(m+\tfrac{1}{2})\ln(1+z)-\frac{\delta(\infty)}{z}+\frac{2\delta(\infty)}{\pi}+\frac{4\delta(\infty)}{\pi z}(1-z^2)^{1/2}\arctan\left(\frac{1-z}{1+z}\right)^{1/2}$$
$$=(m+\tfrac{1}{2})\ln(1+z)-\frac{\delta(\infty)}{z}+\frac{2\delta(\infty)}{\pi}-\frac{2\delta(\infty)}{\pi}\frac{(z^2-1)^{1/2}}{z}\ln[z+(z^2-1)^{1/2}]. \quad (3.21)$$

The functions $\Gamma(z)$ plays an important role in our theory, since it determines (besides the discontinuity function $\mathcal{A}$, which contains the physical information) the form of the kernel of our integral equations. Therefore, further on, a more extensive study of its properties is made.

We first note that $\Gamma(z)$ has no pole at the origin. Indeed, expanding the logarithmic term around $z=0$ we obtain

$$-\frac{2\delta(\infty)}{\pi}\frac{(z^2-1)^{1/2}}{z}\ln[z+(z^2-1)^{1/2}] = \frac{\delta(\infty)}{z}-\frac{2\delta(\infty)}{\pi}+\cdots \quad (3.22)$$

and we can verify that $\Gamma(z)$ is zero at $z=0$, as expected from the subtracted form (3.19). We can further verify that $\Gamma(z)$ has no right-hand cut in the z plane. Indeed for $1\leqslant z$, $\ln[z+(z^2-1)^{1/2}]$ has opposite signs if z lies above or under the real axis, since

$$[z+(z^2-1)^{1/2}]^{-1}{}_{z=t+i\epsilon}=[z+(z^2-1)^{1/2}]_{z=t-i\epsilon} \quad (3.23)$$

which compensates the changes of sign of the factor $(z^2-1)^{1/2}$ which multiplies the logarithm, and therefore no singularities may appear for $z>1$.

Let us return now to the study of the left-hand cut. We notice that

$$y=z+(z^2-1)^{1/2}$$

maps the z cut plane into the upper half-plane of y, since the imaginary part of the square root $(z^2-1)^{1/2}$ is positive even for $\mathrm{Im}z<0$. Indeed, introducing $z=-\bar{z}$ ($\mathrm{Im}\bar{z}>0$) one has

$$\bar{y}=\bar{z}+(\bar{z}^2-1)^{1/2}=[\bar{z}-(\bar{z}^2-1)^{1/2}]^{-1}=-1/y$$

and as $\mathrm{Im}\bar{y}$ is positive, $\mathrm{Im}y$ will be positive too.

For $z\leqslant-1$, y is negative, but it is always situated above the cut of the logarithm; remembering [see Eq. (3.23)] that $y(z+i\epsilon)=y^{-1}(z-i\epsilon)$, we obtain

$$\ln y(z\pm i\epsilon)=i\pi\pm\ln|y(z\pm i\epsilon)|\,. \quad (3.24)$$

(The cut of the logarithm is taken along the negative real axis.) Hence for $z<-1$

$$\Gamma(z\pm i\epsilon)=(m+\tfrac{1}{2})\ln|1+z|\pm i\pi(m+\tfrac{1}{2}) -\frac{2\delta(\infty)}{\pi}\frac{|(z^2-1)^{1/2}|}{z}\ln(|z|-|(z^2-1)^{1/2}|) \pm 2i\delta(\infty)\frac{|(z^2-1)^{1/2}|}{z}-\frac{\delta(\infty)}{z}+\frac{2}{\pi}\delta(\infty)\,. \quad (3.25)$$

By inspection we see that the discontinuity of $\Gamma(z)$ for $z<-1$ is exactly $2\pi i$ times

$$m+\tfrac{1}{2}+[2\delta(\infty)/\pi][|(z^2-1)^{1/2}|/z]$$

as required by the definition (3.19).

The next step is to introduce the characteristic function (Appendix B)

$$X(z)=\exp[\Gamma(z)](z+1)^{-\kappa}, \quad (3.26)$$

where κ is an integer which is not yet determined. Using the function $X(z)$, we can write

$$G(t)=\frac{X(t+i\epsilon)}{X(t-i\epsilon)}\equiv\frac{X^+(t)}{X^-(t)}, \quad (-\infty<t\leqslant-1) \quad (3.27)$$

so that Eq. (3.14) becomes

$$\frac{\phi^+(t)}{X^+(t)}=\frac{\phi^-(t)}{X^-(t)}+\frac{g(t)}{X^+(t)}\,. \quad (3.28)$$

Introducing the function $\psi(z)$ (Appendix B):

$$\psi(z)=\frac{1}{2\pi i}\int_{-\infty}^{-1}\frac{g(t)}{X^+(t)}\frac{dt}{t-z} =-\frac{1}{2\pi i}\int_1^{\infty}\frac{\mathcal{A}(\tau)}{S_0^-(\tau)X^+(-\tau)}\phi(\tau)\frac{d\tau}{\tau+z} -\frac{1}{2\pi i}\int_1^{\infty}\frac{\mathcal{A}(\tau)-\mathcal{A}_0(\tau)}{S_0^-(\tau)X^+(-\tau)}\frac{d\tau}{\tau+z} \quad (3.29)$$

the solution of the Riemann problem is

$$\phi(z)=X(z)[\psi(z)+\Pi(z)], \quad (3.30)$$

where $\Pi(z)$ contains the sum of the poles of the Jost function.

Equation (3.30) is equivalent to the following marginally singular integral equation for $\phi(z)$:

$$\phi(z)=-\frac{X(z)}{2\pi i}\int_1^{\infty}\frac{\mathcal{A}(\tau)}{S_0^-(\tau)X^+(-\tau)}\phi(\tau)\frac{d\tau}{\tau+z} -\frac{X(z)}{2\pi i}\int_1^{\infty}\frac{\mathcal{A}(\tau)-\mathcal{A}_0(\tau)}{S_0^-(\tau)X^+(-\tau)}\frac{d\tau}{\tau+z}+X(z)\Pi(z)\,. \quad (3.31)$$

IV. THE SOLUTION OF THE INTEGRAL EQUATION

We now solve the integral Eq. (3.31). The first step is to bring the integral equation into a finite interval and to symmetrize its kernel. This is done by the inversion $z\to1/x$ and the introduction of a new function

$$\bar{\phi}(x)=\frac{[\mathcal{A}(1/x)]^{1/2}}{[S_0^-(1/x)X(1/x)X^+(-1/x)]^{1/2}}\frac{\phi(1/x)}{x} \quad (4.1)$$

for which the resulting integral equation with a symmetric kernel has the form:

$$\bar{\phi}(x)=\bar{h}(x)+\int_0^1\frac{M(x,y)}{x+y}\bar{\phi}(y)dy\,, \quad (4.2)$$

where

$$M(x,y)=-\frac{1}{2\pi i}\frac{[\mathfrak{A}(1/x)\mathfrak{A}(1/y)]^{1/2}[X(1/x)X(1/y)]^{1/2}}{[S_0^-(1/x)S_0^-(1/y)]^{1/2}[X^+(-1/x)X^+(-1/y)]^{1/2}} \tag{4.3}$$

and

$$\bar{h}(x)=\frac{[\mathfrak{A}(1/x)X(1/x)]^{1/2}}{[S_0^-(1/x)X^+(-1/x)]^{1/2}}\left\{\frac{-1}{2\pi i}\int_0^1\frac{\mathfrak{A}(1/y)-\mathfrak{A}_0(1/y)}{S_0^-(1/y)X^+(-1/y)}\frac{1}{y}\frac{dy}{x+y}+\frac{1}{x}\Pi\left(\frac{1}{x}\right)\right\}. \tag{4.4}$$

We separate the singular part of the integral equation (4.2)

$$\bar{\phi}(x)=\bar{h}(x)+\lambda\int_0^1\frac{\bar{\phi}(y)}{x+y}dy+\int_0^1 K(x,y)\bar{\phi}(y)dy \tag{4.5a}$$

or

$$(\mathbf{I}-\lambda\mathbf{A}-\mathbf{K})\bar{\phi}(x)=\bar{h}(x), \tag{4.5b}$$

where

$$\lambda\equiv M(0,0), \tag{4.6a}$$

$$\mathbf{A}\bar{\phi}(x)\equiv\int_0^1\frac{\bar{\phi}(y)}{x+y}dy, \tag{4.6b}$$

$$K(x,y)=[M(x,y)-M(0,0)/(x+y)], \tag{4.6c}$$

I is the unit operator,

and

$$\mathbf{K}\bar{\phi}(x)\equiv\int_0^1 K(x,y)\bar{\phi}(y)dy. \tag{4.6d}$$

We observe that $K(x,y)$ is a bounded continuous function in the origin. Indeed, for x, $y\rightarrow 0$, $M(x,y)$ tends uniformly to $M(0,0)$ [see Eqs. (4.23)–(4.28)].

In order to solve the equation (4.5b), we must examine the operator $\mathbf{R}=(\mathbf{I}-\lambda\mathbf{A})^{-1}$. The operator $\mathbf{R}$ exists and is bounded on $L^2[0,1]$ if

$$|\lambda|<\|\mathbf{A}\|^{-1}, \tag{4.7}$$

where $\|\mathbf{A}\|$ is the norm of the operator $\mathbf{A}$:

$$\|\mathbf{A}\|=\sup\|\mathbf{A}f\|/\|f\|,\quad \|f\|\neq 0,\quad f\in L^2[0,1]. \tag{4.8}$$

Now we prove, using Schur's test,[19] that $\mathbf{A}$ is a bounded operator:

$$\|\mathbf{A}\|\leqslant\pi. \tag{4.9}$$

To this end we calculate the following integral[20]:

$$\int_0^1\frac{dy}{(x+y)\sqrt{y}}=\frac{2}{\sqrt{x}}\arctan\frac{1}{\sqrt{x}}\leqslant\frac{\pi}{\sqrt{x}}\quad(0<x,\ y\leqslant 1). \tag{4.10}$$

Denoting

$$k(x,y)=(x+y)^{-1}, \tag{4.11a}$$

$$p(x)=\sqrt{x^{-1}} \tag{4.11b}$$

Eq. (4.10) can be written

$$\int_0^1 k(x,y)p(y)dy\leqslant\pi p(x). \tag{4.12}$$

The norm of $\mathbf{A}f(x)$ is then:

$$\|\mathbf{A}f(x)\|^2=\int_0^1|\mathbf{A}f(x)|^2dx=\int_0^1\left|\int_0^1 k(x,y)f(y)dy\right|^2dx$$

$$=\int_0^1\left|\int_0^1[k(x,y)]^{1/2}[p(y)]^{1/2}\times\frac{f(y)[k(x,y)]^{1/2}}{[p(y)]^{1/2}}dy\right|^2dx. \tag{4.13}$$

Using Schwarz's inequality we obtain

$$\|\mathbf{A}f(x)\|^2\leqslant\int_0^1\left(\int_0^1 k(x,y)p(y)dy\right)\times\left(\int_0^1\frac{|f(y)|^2k(x,y)}{p(y)}dy\right)dx. \tag{4.14}$$

From (4.12) we get

$$\|\mathbf{A}f(x)\|^2\leqslant\pi\int_0^1 p(x)\left(\int_0^1\frac{k(x,y)|f(y)|^2}{p(y)}dy\right)dx$$

$$\leqslant\pi^2\int_0^1\frac{|f(y)|^2}{p(y)}p(y)dy=\pi^2\|f\|^2. \tag{4.15}$$

From (4.15) we have

$$\|\mathbf{A}f\|^2/\|f\|^2\leqslant\pi^2. \tag{4.16}$$

That means [see definition (4.8)] that

$$\|\mathbf{A}\|\leqslant\pi. \tag{4.17}$$

From (4.7) and (4.17) it follows that if

$$|\lambda|<\pi^{-1}\leqslant\|\mathbf{A}\|^{-1}, \tag{4.18}$$

the operator

$$\mathbf{R}=(\mathbf{I}-\lambda\mathbf{A})^{-1}=\mathbf{I}+\lambda\mathbf{A}+\lambda^2\mathbf{A}^2+\cdots \tag{4.19}$$

exists and is bounded on $L^2[0,1]$.

Taking these facts into account, the Eq. (4.5b) can

[19] A. Brown, P. Halmos, and A. Shields, Acta Sci. Math. Szeged **26**, 125 (1965).

[20] C. Foias (private communication).

be written in the form

$$\bar{h}(x)=\{\mathbf{I}-\lambda\mathbf{A}-\mathbf{K}\}\bar{\phi}(x) =(\mathbf{I}-\lambda\mathbf{A})\{\mathbf{I}-(\mathbf{I}-\lambda\mathbf{A})^{-1}\mathbf{K}\}\bar{\phi}(x). \quad (4.20)$$

Multiplying Eq. (4.20) by $\mathbf{R}$, we obtain

$$(\mathbf{I}-\mathfrak{R})\bar{\phi}(x)=\bar{\mathfrak{f}}(x), \quad (4.21a)$$

where

$$\mathfrak{R}=\mathbf{RK} \qquad \bar{\mathfrak{f}}(x)=\mathbf{R}\bar{h}(x). \quad (4.21b)$$

Now, $\mathbf{K}$ being Hilbert–Schmidt, if $\mathbf{R}$ is bounded, $\mathfrak{R}$ will be Hilbert–Schmidt also. Moreover, Eq. (4.21a) as a whole will be Hilbert–Schmidt if $\bar{\mathfrak{f}}(x)\in L^2[0,1]$. Therefore, before solving (4.20) it is of interest to calculate λ to see whether it satisfies the boundedness condition for the resolvent $\mathbf{R}$ (i.e., to see whether $|\lambda|<1/\pi$) and to determine the index κ so that $\bar{\mathfrak{f}}(x)$ is a square integrable function on the interval $[0,1]$.

The constant λ defined by (4.6a) is

$$\lambda=\lim_{x\to 0,y\to 0}\left\{-\frac{1}{2\pi i}\frac{[\alpha(1/x)\alpha(1/y)]^{1/2}[X(1/x)X(1/y)]^{1/2}}{[S_0^-(1/x)S_0^-(1/y)]^{1/2}[X^+(-1/x)X^+(-1/y)]^{1/2}}\right\} =-\frac{1}{2\pi i}\frac{\alpha_0(\infty)X(+\infty)}{S_0^-(\infty)X^+(-\infty)}. \quad (4.22)$$

From the definition of the function $\Gamma(x)$ [Eq. (3.21)], it follows that [see also Appendix B, Eq. (B16c)]:

$$\lim_{z\to\infty}\Gamma(z)=(m+\tfrac{1}{2})\ln(1+z)-[2\delta(\infty)/\pi]\ln z -[2\delta(\infty)/\pi]\ln 2+(2/\pi)\delta(\infty) \quad (4.23)$$

and

$$\lim_{z\to-\infty+i\epsilon}\Gamma(z)=(m+\tfrac{1}{2})\ln[|z|-1]+i\pi(m+\tfrac{1}{2}) -\frac{2\delta(\infty)}{\pi}\ln 2-\frac{2\delta(\infty)}{\pi}\ln|z|-2i\delta(\infty)+\frac{2}{\pi}\delta(\infty). \quad (4.24)$$

The values of $\alpha(z)$ and $S_0(z)$ at infinity are

$$\lim_{z\to\infty}\alpha_0(z)=2\cos[2\delta(\infty)], \quad (4.25)$$

$$\lim_{z\to\infty}S_0^-(z)=\exp[+2i\delta(\infty)]. \quad (4.26)$$

Collecting the last results (4.23)–(4.26), we obtain for λ

$$\lambda=(-1)^n[\cos 2\delta(\infty)/\pi] \quad (n=m-\kappa). \quad (4.27)$$

For $\delta(\infty)\neq k\pi$, $|\lambda|<1/\pi$ as is required by the condition (4.18).

The last parameter to be determined is n.

As is shown in Appendix B for the Riemann problem with open contours, singularities may occur at the endpoints of the contour. On the other hand, Eq. (4.21a) is of the Hilbert–Schmidt type when $\bar{h}(x)$ belongs to the class $L^2[0,1]$. ($\mathbf{R}$ being bounded, if $\bar{h}(x)\in L^2[0,1]$, then also $\bar{\mathfrak{f}}(x)\in L^2[0,1]$.) This condition can be satisfied by a convenient choice of the integer κ, i.e., of the integer $n=m-\kappa$.

Assuming that

$$\alpha(z)-\alpha_0(z)\xrightarrow[z\to\infty]{} Cz^{-\nu} \quad (\nu>0), \quad (4.28)$$

where C is a constant, the integral from the definition (4.4) of $\bar{h}(x)$ is convergent if

$$n>[2\delta(\infty)/\pi]-\tfrac{1}{2}-\nu. \quad (4.29)$$

But the behavior of the integral in Eq. (4.4) at the end of the integration path $x=0$ is of the type $1/x^{2\delta(\infty)/\pi-n+\frac{1}{2}-\nu}$. The square-integrability of $\bar{h}(x)$ implies

$$n>[2\delta(\infty)/\pi]-\nu \quad (4.30)$$

which is a more stringent condition than (4.29).

As we are looking for a square-integrable solution of the Eq. (4.21), we must have the following behavior at $x=0$ for $\bar{\phi}(x)$:

$$\bar{\phi}(x)\sim(x^{\frac{1}{2}-\epsilon})^{-1} \quad \text{with} \quad \epsilon>0. \quad (4.31)$$

Then, from (3.31) and (4.1), we obtain for the function $\phi(z)$ the following behavior:

$$\phi(z)\sim z^{n-(2\delta(\infty)/\pi)-\epsilon}. \quad (4.32)$$

But we have assumed that $\phi(z)\to 0$ when $z\to\infty$, so that

$$n\leqslant 2\delta(\infty)/\pi. \quad (4.33)$$

Hence, from (4.30) and (4.33),

$$[2\delta(\infty)/\pi]-\nu<n\leqslant 2\delta(\infty)/\pi. \quad (4.34a)$$

If, as is usually assumed, $\alpha(z)$ can be expanded into a $1/z$ series at infinity, then ν is equal to 1 and the relation (4.34a) takes the form

$$n=[2\delta(\infty)/\pi], \quad (4.34b)$$

where the symbol [ ] denotes the greatest integer less than or equal to $2\delta(\infty)/\pi$. In exceptional cases ν maybe 2, 3 ⋯; then n will no longer be unique, and we are led to a broader choice of characteristic X functions. In order to avoid unpermitted singularities in the equations for $\bar{\phi}(z)$, produced by the behavior of $X(z)$ at $z=-1$ [see Appendix B, Eq. (B26)] we have to choose n less than $\frac{1}{2}$, i.e., $n\leqslant 0$. Hence, we have to add to

Eq. (4.34b) the condition

$$\delta(\infty)<\pi/2\,. \tag{4.34c}$$

This is possible remembering that $\delta(\infty)$ is determined by the phase at infinity only modulo π.

We return to the Hilbert–Schmidt equation (4.21)

$$\bar{\phi}(x)=\bar{\mathfrak{f}}(x)+\int_0^1 \mathfrak{K}(x,y)\bar{\phi}(y)dy \tag{4.35}$$

with $\bar{\mathfrak{f}}(x)=\mathbf{R}\bar{h}(x)$.

The explicit form of $\mathbf{R}$ can be found by solving the Dixon equation

$$(1-\lambda\mathbf{A})\bar{\mathfrak{f}}(x)=\bar{h}(x) \tag{4.36}$$

which is equivalent to the expression (4.21b) for the inhomogeneous term of the Hilbert-Schmidt Eq. (4.21a).

Changing to the variable $z=1/x$ $[\bar{h}'(z)=\bar{h}(1/z)/z]$, one gets

$$\bar{\mathfrak{f}}'(z)=\bar{h}'(z)+\lambda\int_1^\infty \frac{\bar{\mathfrak{f}}'(\tau)}{z+\tau}d\tau\,. \tag{4.37}$$

This equation can be solved by the Mehler transformation[12,13]

$$\bar{H}'(\mu)=\mu\,\tanh\pi\mu\int_1^\infty P_{-\frac{1}{2}+i\mu}(z)\bar{h}'(z)dz \tag{4.38a}$$

and its inverse

$$h'(z)=\int_0^\infty P_{-\frac{1}{2}+i\mu}(z)\bar{H}'(\mu)d\mu\,,\quad (1\leqslant \mathrm{Re}z<+\infty) \tag{4.38b}$$

together with the Cauchy integral for Legendre functions

$$P_{-\frac{1}{2}+i\mu}(z)=\frac{\cosh\pi\mu}{\pi}\int_1^\infty \frac{P_{-\frac{1}{2}+i\mu}(z')}{z'+z}dz',\quad (-\tfrac{1}{2}<\mathrm{Im}\mu<\tfrac{1}{2}). \tag{4.38c}$$

The Mehler transform of the Eq. (4.37) has the simple algebraic form

$$\bar{\mathcal{K}}'(\mu)=\bar{H}'(\mu)+(\lambda\pi/\cosh\pi\mu)\bar{\mathcal{K}}'(\mu)\,, \tag{4.39}$$

i.e., $\bar{\mathcal{K}}'(\mu)=\bar{H}'(\mu)[1-\lambda\pi/\cosh\pi\mu]^{-1}$ which can be inverted by (4.38). Returning to the x variable we obtain

$$\bar{\mathfrak{f}}(x)=\mathbf{R}\bar{h}(x)=\int_0^\infty\int_0^1 \frac{\tau\,\tanh\pi\tau}{1-(\lambda\pi/\cosh\pi\tau)} \times\frac{P_{-\frac{1}{2}+i\tau}(1/x)P_{-\frac{1}{2}+i\tau}(1/x')}{xx'}\bar{h}(x')d\tau dx'\,. \tag{4.40a}$$

Similarly

$$\mathfrak{K}(x,y)=\mathbf{R}K(x,y)=\int_0^\infty\int_0^1 \frac{\tau\,\tanh\pi\tau}{1-(\lambda\pi/\cosh\pi\tau)} \times\frac{P_{-\frac{1}{2}+i\tau}(1/x)P_{-\frac{1}{2}+i\tau}(1/x')}{xx'}\frac{M(x',y)-\lambda}{x'+y}d\tau dx'\,. \tag{4.41}$$

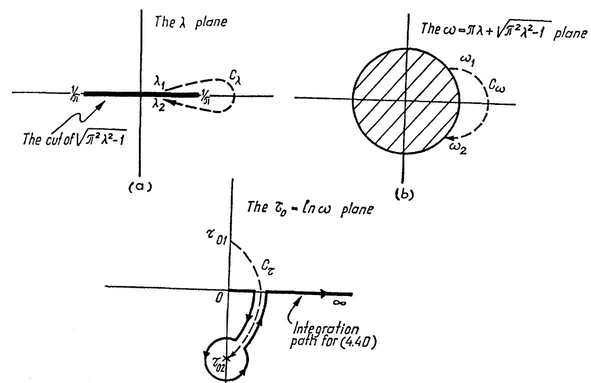

FIG. 4. The analytic continuation in the λ plane of the solution (4.40a) of Eq. (4.36).

According to the argument developed at the beginning of this section, the solution (4.40a) is the unique solution in $L^2[0,1]$ of Eq. (4.36). However, by analytic continuation over the parameter λ, we find a whole set of solutions of Eq. (4.36) which, of course, no longer belong to the $L^2[0,1]$ class but which may present physical interest. (There is no apparent reason to avoid such solutions.)

As one sees from the denominator $(\cosh\pi\tau-\lambda\pi)$ of the integrand in Eq. (4.40a), the point $\lambda=1/\pi$ is an "end-point" singularity for $\bar{\mathfrak{f}}$ as a function of λ. Indeed, solving the equation

$$\cosh\pi\tau_0-\lambda\pi=0\,, \tag{4.40b}$$

we find for the position of the pole of the integrand

$$\tau_n=(1/\pi)\ln[\pi\lambda+(\pi^2\lambda^2-1)^{1/2}]+2ni\,,$$

where both determinations of the square root are to be taken into account.

If now, for instance, λ varies along the path C_λ [Fig. 4(a)] from a physical value λ_1 to the physical value λ_2 $(-1/\pi<\lambda_{1,2}<1/\pi)$, the pole

$$\tau_0=(1/\pi)\ln[\pi\lambda+(\pi^2\lambda^2-1)^{1/2}]$$

varies along the curve C_τ [Fig. 4(c)] from τ_{01} to τ_{02}, both situated on the imaginary axis. Deforming the integration path of (4.40a) correspondingly, we get, besides the integral along the positive real axis, the residue of the pole τ_0, i.e.,

$$A(1/x)P_{-\frac{1}{2}+i\tau_0}(1/x)\,, \tag{4.40c}$$

with

$$A=\int_0^1 \frac{(\tau_0\sinh\tau_0\pi)P_{-\frac{1}{2}+i\tau_0}(1/x)}{x}\bar{h}(x)dx\,.$$

But from (4.38c) we see that

$$\frac{P_{-\frac{1}{2}+i\tau_0}(1/x)}{x}=\lambda\int_0^1\frac{P_{-\frac{1}{2}+i\tau_0}(1/x')}{x'(x'+x)}dx'\,,$$

so that (4.40c) is a solution of the homogeneous part of Eq. (4.36) if $|\mathrm{Im}\tau_0|<\frac{1}{2}$. Of course, as

$$P_{-\frac{1}{2}+i\tau_0}(z)\sim z^{-\frac{1}{2}+|\mathrm{Im}\tau_0|}, \quad (z\to\infty)$$

this solution behaves like $x^{-\frac{1}{2}-|\mathrm{Im}\tau_0|}$ for $x\to 0$ and therefore is no longer of the class $L^2[0,1]$.

The general solution of our integral equation is then

$$\bar{\phi}(x)+\Delta_1\bar{\phi}(x)$$

where $\Delta_1\bar{\phi}(x)$ satisfies the equation

$$(\mathbf{I}-\mathfrak{K})\Delta_1\bar{\phi}(x)=\mathrm{const}(1/x)P_{-\frac{1}{2}+i\tau}(1/x). \quad (4.21b)$$

The integral Eq. (4.21) for $\bar{\phi}(x)$ will be solved by means of the Hilbert–Schmidt method.

$\mathfrak{K}$ being a Hilbert–Schmidt operator, it has eigenvalues λ_i^{-1} and eigenfunctions φ_i which can be computed by a computer for each particular $\mathcal{A}(t)$ from the equations:

$$\lambda_i\mathfrak{K}\varphi_i(x)=\varphi_i(x). \quad (4.42)$$

As is known, the kernel $\mathfrak{K}$ of the integral equation (4.35) can be expanded into the form:

$$\mathfrak{K}(x,y)=\sum_i \varphi_i(x)\varphi_i(y)/\lambda_i \quad (4.43)$$

so that the solution of the equation (4.35) takes the final form:

$$\bar{\phi}(x)=\sum_i [\lambda_i\bar{\mathfrak{f}}_i/(\lambda_i-1)]\varphi_i(x) \quad (4.44)$$

where

$$\bar{\mathfrak{f}}_i=\int_0^1 \bar{\mathfrak{f}}(x)\varphi_i(x)dx. \quad (4.45)$$

The inhomogeneous term of Eq. (4.21b) for $\Delta_1\bar{\phi}(x)$ is no longer of the class $L^2[0,1]$, so that the usual Hilbert–Schmidt procedure cannot be applied. However, our belief is that for a wide class of potentials, the Legendre polynomials $P_{-\frac{1}{2}+i\tau}(1/x)$ can be expanded in terms of the eigenfunctions of the kernel $\mathfrak{K}$.

The advantage of the Hilbert–Schmidt method consists in the considerably smaller computer time required for the calculations. Indeed, a new CDD pole does not change the eigenfunctions $\varphi_i(x)$, for they depend only on the left-hand discontinuity of the amplitude, via Eq. (4.42). Once the $\varphi_i(x)$ are known, for each new input particle or CDD pole

$$\Delta\Pi(k)=\sum_j \frac{ir_j}{k-im_j}+\sum \frac{a_j+ikb_j}{(k-c_j)(k+c_j^*)} \quad (4.46)$$

introduced in the equations, one has to add to the old solution a term:

$$\Delta_2\bar{\phi}(x)=\sum_j \frac{\lambda_i\Delta\bar{\mathfrak{f}}_i}{\lambda_i-1}\varphi_i(x);$$

$$\Delta\bar{\mathfrak{f}}_i=\int_0^1 \varphi_i(x)\mathbf{R}\left(\frac{\mathcal{A}(1/x)X(1/x)}{S_0^-(1/x)X^+(-1/x)}\right)^{1/2} \times\frac{\Delta\Pi(1/x)}{x}dx. \quad (4.47)$$

Hence [see also (4.21a) and (4.1)]:

$$f(k)\equiv 1+\phi(k)+\Delta_1\phi(k)+\Delta_2\phi(k).$$

V. SUMMARY AND CONCLUSIONS

In this paper we have first written dispersion relations and integral equations for the Jost function in the complex plane of the variable k, which has the property of uniformizing the two Riemann sheets of the elastic scattering amplitude.

Finally, the Jost function has been written in terms of the eigensolutions of a Fredholm equation with a Hilbert–Schmidt kernel, depending on the actual form of $\omega(s)$, the left-hand discontinuity of the amplitude.

For practical purposes, we summarize here the main steps of this work:

First, the S matrix was written for the general case when the phase shift at infinity[10]

$$\delta(\infty)=\arcsin[\omega(-\infty)]^{1/2} \quad (\text{modulo } \pi)$$

is not a multiple of π:

$$S(k)=\exp[2id_0(k)][f(-k)/f(k)],$$

where the auxiliary phase $d_0(k)$ is defined in (3.7).

Then, $f(k)$ was written—see (3.11) and (4.1)—in the form

$$f(x)=1+x\left[\frac{S_0^-(1/x)X(1/x)X^+(-1/x)}{\mathcal{A}(1/x)}\right]^{1/2} \times\bar{\phi}(x), \quad x=i/k,$$

where $\bar{\phi}$ is the solution (4.44) of the Hilbert–Schmidt equation (4.35)—see also Eqs. (4.40a) and (4.41).

The quantities $\bar{h}$, M, and λ which appear in these formulas are defined in (4.4), (4.3), and (4.27), respectively, by means of the auxiliary functions S_0 [Eq. (3.18a)] and X [Eq. (3.26)]. The index n is evaluated in Eq. (4.34a)–(4.34c).

For the equal-mass case we have just studied, where lighter particle exchange does not exist, the whole physical input in our equations is given by the known function $\mathcal{A}(k)=S^+(k)+S^-(k)$, defined on the left-hand cut of the s plane [for the connection between $\mathcal{A}(k)$ and $w(k)$, see Eq. (3.8a)], together with the meromorphic function $\Pi(k)$ [Eqs. (4.4) and (2.3)]. The function Π represents the contribution to the scattering process from the actual elementary particles (not from composite particles) and from the CDD poles.

As was discussed in Sec. II, if a sufficient number of composite particles exists, the CDD poles can be replaced by some new elementary-particle poles. If this condition is not fulfilled, some of the CDD poles will remain and must be taken into account in the input Π function.

The method presented here can be extended easily to the many-channel case, where a generalized Jost function exists[3] and the elements of the S matrix are given by the ratios of the values of the Jost function on its various Riemann sheets.

Now, in the complex plane of a suitably chosen variable all the right-hand cuts of the scattering amplitude are uniformized, and therefore only the left-hand cuts will remain. A Riemann problem analogous to (3.14) can be stated. The corresponding integral equations for the generalized many-channel Jost function will be regularized in a subsequent paper.

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APPENDIX A. PHRAGMÉN–LINDELÖF THEOREM

In the following we prove the relations (3.1) and (3.2) using the Phragmén–Lindelöf theorem. This theorem may be stated as follows.

Let the function $f(z)$ be analytic and regular in the domain D confined between two rays Γ_1 and Γ_2 which form an angle with the vertex at the origin. Further, let $f(z)$ be bounded in the sector D:

$$|f(z)|\leqslant C.$$

We denote by E_i, $i=1$, 2 the set of limit points of $f(z)$ when $z\to\infty$ along the ray Γ_i. The sets E_i contain either only one point, or a continuum. Then, either: (1) the sets E_1 and E_2 have a common point, or (2) one of them encircles the second set.

In particular, if there exist finite limits a_1 and a_2 when $z\to\infty$ along Γ_1 and Γ_2, then $a_1=a_2=a$, and $f(z)\to a$ uniformly in D when $z\to\infty$. Consequently, if $f(z)$ has only one cut and remains bounded at infinity, then if we take Γ_1 and Γ_2 on opposite sides of the cut, a_1 being equal to a_2, the discontinuity tends to zero uniformly when $z\to\infty$.

We apply the theorem for the S matrix in the sector $D\equiv(-\pi/2+\epsilon,\pi/2-\epsilon)$. In this domain, the S matrix has no cuts and has only a finite number of poles. From (1.2) we obtain that the S matrix has the same limit $\exp[2i\delta(\infty)]$ along the rays Γ_1 and Γ_2 which define the sector D.

In order to apply the theorem, it is necessary to subtract the poles. But the poles' contribution tends to zero when $|z|\to\infty$, so that

$$\lim_{k\to\infty,\,k\in D}[S(k)-\text{poles}]=\lim_{k\to\infty,\,k\in D}S(k)=\exp[2i\delta(\infty)]. \quad \text{(A1)}$$

APPENDIX B: CAUCHY SINGULAR INTEGRAL EQUATIONS AND THE RIEMANN PROBLEM

We summarize here the main facts from the theory of singular linear integral equations. Now, a Cauchy singular integral equation being given, one can always separate its kernel $(2\pi i)^{-1}H(t,t')/(t'-t)$ into a pure Cauchy kernel $(2\pi i)^{-1}H(t,t)/(t'-t)$ and a regular one, $K(t,t')=(2\pi i)^{-1}(H(t,t')-H(t,t))/(t'-t)$. Therefore, as is known, the initial Cauchy singular linear integral equation is equivalent to a Riemann problem whose coefficient $G(t)$ [see Eq. (B1)] is equal[21] to $[1-H(t,t)]^{-1}$, the inhomogeneous term $g(t)$ being built from the regular part of the integral equation together with its inhomogeneous term.

The Cauchy and regularized equations having been thoroughly studied in Secs. III and IV, we deal here only with the Riemann problem: first in its classical form, for closed contours, then, in a slightly generalized version (allowing poles) for open contours.

Let us begin with the Riemann problem stated for a closed contour C separating the domains D^+ and D^- (D^+ is defined on the left when going along C counterclockwise; further, the origin is supposed to be in D^+). The Riemann problem is the following[22]:

Given two functions $G(t)$ and $g(t)$ defined on C, satisfying Hölder's condition, to find two functions $\phi^+(z)$ and $\phi^-(z)$ which are holomorphic in D^+ and D^- respectively, such that on C we have:

$$\phi^+(t)=G(t)\phi^-(t)+g(t). \quad \text{(B1)}$$

To solve this problem, we first define κ, the index of

[21] It is supposed that t contains an infinitesimal part which places $\phi(t)$ on the left boundary of the contour. If not, the principal value of the integral is understood, and one goes to the auxiliary function $\bar\phi$ defined by $\bar\phi^+(t)-\bar\phi^-(t)=\varphi(t)$, $[\bar\phi^+(t)+\bar\phi^-(t)]/2=(2\pi i)^{-1}\mathcal{P}\int[\varphi(t')/(t'-t)]dt'$. The Riemann problem for $\bar\phi(t)$ has a coefficient $\bar G$ equal to $[H(t,t)+2]/[H(t,t)-2]$.

[22] F. D. Gakhov, *Kraevye Zadatchi* (G.I.F.M.L., Moscow, 1963).

the function $G(t)$, with the help of the variation of $\arg G(t)$ on C:

$$\kappa=\Delta/2\pi i\,,\quad \Delta=\int_C d\ln G(\tau)\,. \qquad \text{(B2)}$$

The contour C being closed, κ is an integer.

In contrast with $\ln G(t)$, $\ln[t^{-\kappa}G(t)]$ is a one-valued function on C, so that we can define the functions

$$\Gamma(z)=\frac{1}{2\pi i}\int_C \frac{\ln[\tau^{-\kappa}G(\tau)]}{\tau-z}d\tau \qquad \text{(B3)}$$

and

$$X^+(z)=\exp\Gamma^+(z)\,,\quad X^-(z)=z^{-\kappa}\exp\Gamma^-(z) \qquad \text{(B4)}$$

which are, respectively, holomorphic in D^+ and D^-. Here $\Gamma^\pm(z)$ are the values of $\Gamma(z)$ for $z\in D^\pm$.

Noting that

$$\begin{aligned} X^+(t)/X^-(t)&=t^\kappa \exp[\Gamma^+(t)-\Gamma^-(t)] \\ &=t^\kappa \exp\ln[t^{-\kappa}G(t)]\equiv G(t) \end{aligned} \qquad \text{(B5)}$$

and writing $g(t)/X^+(t)=\Psi^+(t)-\Psi^-(t)$, where

$$\Psi(z)=\frac{1}{2\pi i}\int_C \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-z}\,, \qquad \text{(B6)}$$

Eq. (B1) may be transformed into the equality

$$[\phi^+(t)/X^+(t)]-\Psi^+(t)=[\phi^-(t)/X^-(t)]-\Psi^-(t)\,. \qquad \text{(B7)}$$

The left-hand side of (B7) represents the limit value on C of a holomorphic function in D^+, while the right-hand side is the boundary value of a function having in D^- at most a κ-order pole at infinity [owing to the κ-order zero of $X^-(z)$ at $z=\infty$].

Equation (B7) tells us that there is no discontinuity along C, and therefore both its sides represent the same function $\mathcal{P}_\kappa(z)$ analytical in $D^+\cup D^-$, having at most a κ-order pole at infinity (i.e., $\mathcal{P}_\kappa$ is a polynomial of degree κ).

Hence, the general solution of (B1) is

$$\phi^\pm(z)=X^\pm(z)[\Psi^\pm(z)+\mathcal{P}_\kappa(z)]\,. \qquad \text{(B8)}$$

We deal with the case of open contours, also allowing the function $\phi(z)$ to have poles of given positions and residues. The contour C is open and is supposed to consist of the uncrossed curves C_k ($C=\bigcup_k C_k$). The argument runs along the same lines as for the closed-contour case, with the only difference that we have to be careful about the possible singularities at the end points of the segments C_k. Of course, the two domains D^+ and D^- lose their meaning, but the Riemann problem can still be stated in the form (B1), where $\phi^+(z)$ and $\phi^-(z)$ (as well as X^+ and X^-, and Ψ^+ and Ψ^-, etc.) are now the boundary values on C of the same analytic function $\phi(z)$.

Since the contour C is not closed, we no longer have troubles with the multivalued property of the phase of $G(t)$ and we can simply write

$$\Gamma(z)=\frac{1}{2\pi i}\int_C \frac{\ln G(\tau)}{\tau-z}d\tau\,. \qquad \text{(B9)}$$

The characteristic function $X(z)$ is defined by

$$X(z)=\prod_k (z-b_k)^{-\kappa_k}\exp[\Gamma(z)]\,. \qquad \text{(B10)}$$

Writing

$$\Psi(z)=\frac{1}{2\pi i}\int_C \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-z} \qquad \text{(B11)}$$

we observe, as for the closed-contour case, that the function $\phi/X-\Psi$ no longer has discontinuities along the contours C_k. Thus

$$\phi(z)=X(z)[\Psi(z)+\Pi(z)+\mathcal{P}_n(z)]\,, \qquad \text{(B12)}$$

where $\Pi(z)$ contains the poles of $\Phi(z)$, and $\mathcal{P}_n(z)$ is a polynomial. If none of the contours C_k goes to infinity, then $\Gamma(\infty)=0$ and

$$X(z)\xrightarrow[(z\to\infty)]{}(1/z)^{\Sigma\kappa_k}\,. \qquad \text{(B13)}$$

The resulting degree of the function $\phi(z)$ at infinity is therefore $\kappa-\sum_k \kappa_k$.

In order to establish the main difference between the open-contour case and the closed-contour one, we have to examine the possible singularities of the function $\phi(z)$ at the end points a_k, b_k of the segments C_k.

Indeed, a function $f(z)$ defined by the integral

$$f(z)=\frac{1}{2\pi i}\int_a^b \frac{\sigma(z')}{z'-z}dz' \qquad \text{(B14)}$$

has a behavior at one of the end points which depends strongly on that of $\sigma(z')$.

Denoting by z_0 either a or b, then, if

$$\sigma(z)\underset{(z\to z_0)}{\sim}\text{const}\times(z-z_0)^{-\gamma} \qquad \text{(B15a)}$$

it follows that

$$f(z)\underset{(z\to z_0)}{\sim}\pm\text{const}\frac{e^{\pm i\gamma\pi}}{2i\sin\gamma\pi}(z-z_0)^{-\gamma}\,,\quad \text{for}\quad 0<\operatorname{Re}\gamma<1\,, \qquad \text{(B15b)}$$

and

$$f(z)\underset{(z\to z_0)}{\sim}\mp\frac{\text{const}}{2\pi i}\ln(z-z_0)\,,\quad \text{for}\quad \gamma=0\,. \qquad \text{(B15c)}$$

When one of the limits of the integral (B14) tends to infinity, then, if

$$\sigma(z)\underset{(z\to\infty)}{\sim}\text{const}z^{\gamma-1} \qquad \text{(B16a)}$$

it follows that (the upper sign corresponds to $a\to\infty$

while the lower one to $b\to\infty$)

$$f(z) \underset{(z\to\infty)}{\sim} \mp\text{const}\frac{e^{\mp i\pi\gamma}}{2i\sin\pi\gamma}z^{\gamma-1}, \quad \text{for} \quad 0<\text{Re}\gamma<1, \tag{B16b}$$

and

$$f(z) \underset{(z\to\infty)}{\sim} \pm\frac{\text{const}}{2\pi i}\frac{\ln z}{z}, \quad \text{for} \quad \gamma=0. \tag{B16c}$$

With these results in mind, we can now study the singularities of $\phi(z)$ at the end points a_k, b_k of the contours C_k, when they are finite.

We write

$$\ln G(a_k)=i\theta(a_k)+\ln\rho(a_k), \tag{B17}$$

where $\theta(a_k)$ is almost arbitrary (it is fixed up to a multiple m of 2π).

From (B17) and (B15c) we obtain the behavior of the function $\Gamma(z)$ for $z\to a_k$, and hence, of the function $X(z)$ defined by (B10):

$$X(z)\sim(z-a)^{-\theta(a_k)/2\pi}. \tag{B18}$$

(We have neglected here an oscillating factor.)

As for $\Psi(z)$, if the coefficient $g(z)$ behaves like $(z-a_k)^q$ in the neighborhood of a_k, we have to choose $\theta(a_k)$ so that $g(\tau)/X^+(\tau)$ is integrable, i.e., $\text{Re}(q+\theta(a_k)/2\pi)$ must be greater than -1.

Two cases may occur:

(1) $$-1<\text{Re}\{q+[\theta(a_k)/2\pi]\}<0 \tag{B19a}$$

and then

$$\Psi(z)\sim(z-a_k)^{q+[\theta(a_k)-2\pi]}; \tag{B19b}$$

(2) $$0<\text{Re}\{q+[\theta(a_k)/2\pi]\} \tag{B20a}$$

and then

$$\Psi(z)\sim\text{constant}. \tag{B20b}$$

In the first case, $\phi(z)$ would behave either like $(z-a_k)^q$ (from the term $X\Psi$), or like $(z-a_k)^{-\theta(a_k)/2\pi}$ (from the term $X\Pi$). The strongest singularity will dominate.

In the second case, $\phi(z)$ behaves like $(z-a_k)^{-\theta(a_k)/2\pi}$.

Now, a few words about the singularity at the second end point b_k. Assuming $\theta(a_k)$ fixed, the phase of $G(t)$ at $t=b_k$ is given by $\theta(a_k)+\Delta_k$, where Δ_k is the variation of $\arg G(t)$ along C_k. Defining

$$\Theta(b_k)=\theta(a_k)+\Delta_k-2\pi\kappa_k, \tag{B21}$$

the behavior of $X(z)$ in the neighborhood of b_k will be

$$X(z) \underset{(z\to b_k)}{\sim} (z-b_k)^{\Theta(b_k)/2\pi}. \tag{B22}$$

The rest of the argument is the same as for the end point a_k, and we have only to replace $\theta(a_k)$ by $\Theta(b_k)$.

If one of the points a_k goes to infinity, then $\Gamma(z)$ is defined with a subtraction—see (3.19)—and from (B16c) and (B10) we obtain

$$X(z) \underset{(z\to a_k=\infty)}{\sim} z^{+[\theta(a_k)/2\pi]-\Sigma\kappa_k}. \tag{B23}$$

In Sec. III, in the particular case of $\Gamma(z)$ given by Eq. (3.21), it results that

$$\begin{aligned}&\theta(-\infty)=(2m+1)\pi-4\delta(\infty),\\&\Delta_{[-\infty,-1]}=4\delta(\infty),\\&\Theta(-1)=(2m+1)\pi-2\pi\kappa\equiv(2n+1)\pi.\end{aligned} \tag{B24}$$

Therefore, the behavior of the characteristic function $X(z)$ at $z=-\infty$ and $z=-1$ is

$$X(z) \underset{(z\to\infty)}{\sim} z^{\theta/2\pi-\kappa}=z^{(n+\frac{1}{2}-2\delta(\infty)/\pi)} \tag{B25}$$

and correspondingly

$$X(z) \underset{(z\to-1)}{\sim} (z+1)^{\Theta/2\pi}=(z+1)^{n+\frac{1}{2}}. \tag{B26}$$