Study of the Process $\pi^+ + d \to p + p$ at Low Energies*†

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The process $\pi^+ + d \to p + p$ has been studied in the energy range 2–16 MeV. The data are fitted with the phenomenological equation $\sigma = \frac{2}{3}(p^2/\mu^2c^2)(\alpha/\eta + \beta\eta)$, where p is the proton c.m. momentum, μ is the pion mass, and η is the pion c.m. momentum in units of μc. The fitted values are $\alpha = 0.240 \pm 0.02$ mb and $\beta = 0.52 \pm 0.2$ mb. The relationship of this measurement to the results of π^+ photoproduction and pion elastic scattering experiments is reviewed.

I. INTRODUCTION

CHARGE independence of strong interactions provides a means of relating low-energy pion production and scattering cross sections at corresponding pion center-of-mass energies. The necessary steps in establishing these connections between the various processes are summarized in Table I. These connections were known for a long time[1]; as more precise experimental data and better theoretical extrapolation procedures became available, inconsistencies among the reactions became apparent.

From Table I we see that comparisons of the cross sections for pion photoproduction, for elastic pion-nucleon scattering, and for the reaction $p + p \to \pi^+ + d$ provide a check on the whole series of relationships. Applying this over-all test, we note that:

(1) The cross section for π^+ photoproduction can be determined by direct measurement and can be computed from pion-nucleon scattering and from the reaction $p + p \to \pi^+ + d$ via the relationships of Table I.

(2) Pion photoproduction has been intensively studied both experimentally[2,3] and theoretically[4] with consistent results.

(3) The pion-nucleon scattering lengths can be obtained either through extrapolation to threshold of the measured phase shifts or through the use of forward dispersion relations and total cross sections for pion-nucleon scattering.

(4) Experimental results on elastic scattering of π^+ and π^- on protons are consistent with the results of experiments on the related charge-exchange scattering.[5,6]

(5) At present, discrepancies exist between the values of the cross section for π^+ photoproduction determined experimentally and the values determined via the relationships of Table I.

(6) The s-wave contribution to $p + p \to \pi^+ + d$ is dominant only in the low-energy region where experiments are difficult. Only one determination of the s-wave term has been made, and this one in an energy range where the p-wave term is dominant.[7]

TABLE I. Connections between low-energy pion processes:

$R = w(\gamma + n \to \pi^- + p)/w(\gamma + p \to \pi^+ + n)$;
$P = w(\pi^- + p \to n + \pi^0)/w(\pi^- + p \to n + \gamma)$;
$S = w(\pi^- + d \to n + n)/w(\pi^- + d \to n + n + \gamma)$;
$T = w(\pi^- + d \to n + n + \gamma)/w(\pi^- + p \to n + \gamma)$.
DB = detailed balance;
CI = charge independence;
EZE = extrapolation to zero energy.

$$\begin{tikzcd}[column sep=small]
 & & \sigma(p+p \to \pi^+ + d) \arrow[d, shift left, "\mathrm{DB}"] & \\
 & & \sigma(\pi^+ + d \to p+p) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{CI}"] & \\
 & & \sigma(\pi^- + d \to n+n) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{EZE}"] & \\
\sigma(\gamma + p \to \pi^+ + n) \arrow[d, shift left, "\mathrm{R}"] & & w(\pi^- + d \to n+n) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{S}"] & \\
\sigma(\gamma + n \to \pi^- + p) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{DB}"] & & w(\pi^- + d \to n+n+\gamma) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{T}"] & \\
\sigma(\pi^- + p \to n+\gamma) \arrow[u, shift left] \arrow[rr, leftrightarrow, "\mathrm{EZE}"'] & & w(\pi^- + p \to n+\gamma) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{P}"] & \\
 & & w(\pi^- + p \to n+\pi^0) \arrow[u, shift left] \arrow[d, shift left, "\mathrm{EZE}"] & \\
 & & \sigma(\pi^- + p \to n+\pi^0) \arrow[u, shift left] & \\
 & \sigma(\pi^- + p \to \pi^- + p) \arrow[ur, "\mathrm{CI}"] & & \sigma(\pi^+ + p \to \pi^+ + p) \arrow[ul]
\end{tikzcd}$$

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[2] J. D. Simpson, Ph.D. thesis, University of Illinois, 1964 (unpublished).

[3] M. I. Adamovich, E. G. Gorzhevskaya, V. G. Larionova, V. M. Popova, S. P. Kharlamov, and F. R. Yagudina, Zh. Eksperim. i Teor. Fiz. **38**, 1078 (1960) [English transl.: Soviet Phys.—JETP **11**, 779 (1960)]; A. Barbaro, E. L. Goldwasser, and D. Carlson-Lee, Bull. Am. Phys. Soc. **4**, 23 (1959); G. M. Lewis, R. E. Azuma, E. Gabathuler, D. W. G. S. Leith, and W. R. Hogg, Phys. Rev. **125**, 378 (1962); D. A. McPherson, D. C. Gates, R. W. Kenney, and W. P. Swanson, *ibid.* **136**, B1465 (1964).

[4] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337, 1345 (1957).

[5] R. A. Donald, W. H. Evans, W. Hart, P. Mason, D. E. Plane, and E. J. C. Read, Proc. Phys. Soc. (London) **87**, 445 (1966).

[6] V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965).

We therefore have re-examined the s-wave component of the reaction $p+p \rightarrow \pi^{+}+d$ by studying, with good energy resolution, the inverse reaction $\pi^{+}+d \rightarrow p+p$ at low energies. We find the s-wave component to be considerably higher than previously reported.[7]

In the following sections we shall first examine all the elements of Table I in order to determine possible weaknesses in the relationships, and then look at the status of the relationships. Next we shall describe our experiment and its results, and finally we shall review the consistency of the low-energy pion relationships in view of our results.

II. DERIVATION OF THE RELATIONSHIPS BETWEEN LOW-ENERGY PION PHENOMENA

In discussing the relationships between reactions, we shall make use of the following:

A. Relation Between Absorption Rates and Corresponding Low-Energy Cross Sections

In order to compare low-energy scattering experiments with absorption experiments, we will have to make use of the relation between absorption rate and cross section, viz., $\lambda=\sigma v|\phi(0)|^{2}$, where λ is the absorption rate, σ the cross section, v the relative velocity, and $\phi(0)$ the bound-state wave function at the nucleus. Processes measured above threshold must be extrapolated to threshold in order to compare them with absorption experiments. It is usually more fruitful to use an extrapolation suggested by theory, than to use purely mathematical methods.[4,8]

B. Principle of Detailed Balance

The ratio of the cross section σ_{AB} for a process $A+a \rightarrow B+b$ to the cross section σ_{BA} for the inverse process at corresponding energy is, by detailed balancing,

$$\frac{\sigma_{AB}}{\sigma_{BA}}=\frac{(2S_B+1)(2S_b+1)}{(2S_A+1)(2S_a+1)}\frac{p_B{}^2}{p_A{}^2},$$

where S_i is the spin of the ith particle and $p_{A(B)}$ is the momentum in state $A(B)$. For states with identical fermions (e.g., two protons or two neutrons) the statistical weight factor must be multiplied by $\frac{1}{2}$.

C. Charge Independence

The principle of charge independence has been accurately verified for some low-energy pion processes not contained in Table I. For example, the reactions $n+p \rightarrow \pi^{0}+d$ and $p+p \rightarrow \pi^{+}+d$ are found to have identical angular distributions,[9] and cross sections[10] in the ratio 1:2, i.e., the ratio of the $I=1$ components of the initial states. It is therefore important to examine any apparent departures from the principle of charge independence for the reactions which do appear in Table I.

D. Establishing the Relationships of Table I

We begin with the cross section for positive-pion photoproduction $\sigma(\gamma+p \rightarrow \pi^{+}+n)$. This has been measured very close to threshold by Simpson[2] and by others[3] at values slightly above threshold. Chew, Goldberger, Low, and Nambu[4] (CGLN), using fixed momentum-transfer dispersion relations, provided a theoretical expression for the above cross section. Simpson found his data to be in good agreement with the prediction of CGLN. He therefore was able to use the CGLN formula to extrapolate his results to threshold. His result, $\sigma(\gamma+p \rightarrow \pi^{+}+n)=(0.201\pm0.005)\eta$ mb, where η is the momentum of the pion in units of the pion mass times the velocity of light, is in agreement with other experiments.[3]

Next we use

$$R\equiv\frac{\sigma(\gamma+n \rightarrow \pi^{-}+p)}{\sigma(\gamma+p \rightarrow \pi^{+}+n)} \tag{1}$$

to relate positive-pion photoproduction to negative-pion photoproduction. The factor[11] $1/R$ is usually taken equal to the experimentally measured ratio

$$\frac{\sigma(\gamma+d \rightarrow \pi^{+}+n+n)}{\sigma(\gamma+d \rightarrow \pi^{-}+p+p)}=1/R_{\mathrm{D}}.$$

The justification for doing so is based on an application of the impulse approximation.[12] The photon is assumed to interact with one nucleon only, the other being a "spectator." Any final-state interactions are assumed to be the same for either final state and thus to cancel out in the ratio. Baldin[13] has discussed the necessary Coulomb corrections; Swanson *et al.*,[14] applying these, give $R_{\mathrm{D}}=1.28\pm0.13$.

In a different approach, Yoon[15] has applied the Chew-Low[16] extrapolation method to obtain the cross section for negative-pion photoproduction on a neutron from the negative-pion photoproduction cross section in deuterium. This method is based on the occurrence of a pole, corresponding to the exchange of a nucleon, in

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[8] J. Hamilton and W. S. Woolcock, Phys. Rev. **118**, 291 (1960).

[9] R. H. Hildebrand, Phys. Rev. **89**, 1090 (1953).

[10] R. A. Schluter, Phys. Rev. **95**, 639(A) (1954).

[11] See, for example, W. R. Hogg, Proc. Phys. Soc. (London) **80**, 729 (1962), for a review of data on R.

[12] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952).

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[14] W. P. Swanson, D. C. Gates, T. L. Jenkins, and R. W. Kenney, Phys. Rev. **137**, B1188 (1965).

[15] T. S. Yoon, Ph.D. thesis, University of Illinois, 1964 (unpublished).

[16] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

the transition matrix. The residue at the pole is then a product of the $n+p\to d$ vertex which is assumed known and of the vertex for photoproduction on a free nucleon. Yoon obtained R directly by using the cross section for π^- photoproduction found in this manner and the cross section for π^+ photoproduction in hydrogen found by Simpson.[2] The value Yoon obtained is $R=1.23\pm0.06$. Both Yoon's and Simpson's experiments were performed using the same techniques and equipment.

A third approach is to apply the theoretical arguments of CGLN[4] to obtain an expression for R.

$$R=\left(\frac{1+(g_p+g_n)(\mu/2M)+1.15N^{(-)}}{1-(g_p+g_n)(\mu/2M)+1.15N^{(-)}}\right)^2,$$

where $g_p=2.79$, and $g_n=-1.91$, are the proton and neutron magnetic moments, μ and M are the pion and nucleon mass, and $N^{(-)}$ is a small correction arising from the electric dipole interaction. $N^{(-)}$ is usually set equal to zero. This purely theoretical value of R is 1.30 ± 0.14. Though all three values for R are in agreement, we shall use Yoon's value, $R=1.23\pm0.06$, which is the nearest of the three to being a direct determination of the ratio.

The cross section for the reaction $(\gamma+n\to\pi^-+p)$ is obtained from the inverse process, viz.,

$$\sigma(\gamma+n\to\pi^-+p)=\frac{p_\pi^2}{2p_\gamma^2}\sigma(\pi^-+p\to n+\gamma),\tag{2}$$

by detailed balance. At threshold, Eq. (2) becomes

$$\sigma(\gamma+n\to\pi^-+p)=\frac{p_\pi^2}{2\mu^2c^2}\frac{(1+\mu/M)^2}{(1+\mu/2M)^2}\sigma(\pi^-+p\to n+\gamma).\tag{3}$$

Proceeding along the branch of Table I which leads to elastic scattering, we next invoke the Panofsky ratio[17]

$$\frac{1}{P}=\frac{\lambda(\pi^-+p\to n+\gamma)}{\lambda(\pi^-+p\to n+\pi^0)}$$

to express $\sigma(\pi^-+p\to n+\gamma)$ as

$$\sigma(\pi^-+p\to n+\gamma)=(1/P)\sigma(\pi^-+p\to n+\pi^0).\tag{4}$$

The Panofsky ratio has been measured by several different methods[18] which have yielded the consistent result[19] $P=1.53\pm0.02$.

Then, by charge independence, we relate

$$\sigma(\pi^-+p\to n+\pi^0)\quad\text{to}\quad\sigma(\pi^-+p\to\pi^-+p)$$

and

$$\sigma(\pi^++p\to\pi^++p).$$

This we do by noting that

$$\sigma(\pi^-+p\to n+\pi^0)=\frac{2}{9}\frac{\rho_0}{f_0}|R_1-R_3|^2,$$

$$\sigma(\pi^-+p\to\pi^-+p)=\frac{1}{9}\frac{\rho_-}{f_-}|R_3+2R_1|^2\underset{\eta\to0}{=}\frac{4\pi}{9}(a_3+2a_1)^2,$$

and

$$\sigma(\pi^++p\to\pi^++p)=\frac{\rho_+}{f_+}R_3^2\underset{\eta\to0}{=}4\pi a_3^2,$$

where ρ is the phase space for the process, f is the incident flux, R_1 and R_3 are the pure isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes, and a_1 and a_3 are the pure isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ scattering lengths. Thus, we obtain

$$\sigma(\pi^-+p\to n+\pi^0)\underset{\eta\to0}{=}\frac{8}{9}\pi\frac{\rho_0}{\rho_-}(a_3-a_1)^2,\tag{5}$$

since $f_0=f_+=f_-$ and $\rho_-=\rho_+$ for the same incident pion momentum. At threshold we find

$$\frac{\rho_0}{\rho_-}=\frac{\mu_0^2}{\mu^2}\gamma_0\frac{\eta_0}{\eta}\frac{(1+\mu/M)}{(1+\mu_0/M_n)},\tag{6}$$

where μ_0 is the mass of π^0, M_n is the neutron mass, η_0 is the π^0 momentum in units of μ_0c, and γ_0 is the total π^0 energy in units of μ_0c^2; a_1 and a_3 are found by extrapolating the experimentally measured phase shifts to zero energy. Hamilton and Woolcock[8] have used the CGLN[4] formula for the s-wave phase shifts derived from fixed momentum-transfer dispersion relations as a basis for extrapolating the experimental data. They also calculated the additional inner Coulomb correction which arises from the Coulomb field within the range of nuclear forces. Previous corrections had assumed no Coulomb force within a sphere of radius r_0, usually taken to be equal to the pion Compton wavelength. These outer Coulomb corrections depended on the value chosen for r_0. The inner Coulomb corrections were such as to remove this dependence on r_0 in the total Coulomb correction. Hamilton and Woolcock obtained for the scattering-length combination $(a_1-a_3)=0.245\pm0.007$ in units of the pion Compton wavelength. Collecting Eqs. (1)–(6), we have

$$\sigma(\gamma+p\to\pi^++n)=\frac{1}{P}\frac{1}{R}\frac{4\pi}{9}\frac{(1+\mu/M)^3}{(1+\mu/2M)^2}\frac{1}{(1+\mu_0/M_n)}\frac{\mu_0^2}{\mu^2}\times\gamma_0\eta_0(a_3-a_1)^2\eta.\tag{7}$$

Hamilton and Woolcock's[8] treatment of pion-nucleon scattering data yielded a value of (a_1-a_3) consistent with Eq. (7), the relation between pion photoproduction and pion-nucleon scattering. The value of (a_1-a_3) found

[17] W. K. H. Panofsky, R. L. Aamodt, and J. Hadley, Phys. Rev. **81**, 565 (1951).
[18] See, for example, J. W. Ryan, Phys. Rev. **130**, 1554 (1963), for references on the measurement of P.
[19] This is the weighted average for P as given in Ref. 18.

TABLE II. Allowed angular-momentum states leading to s- and p-wave pions in the reaction $p+p\rightarrow\pi^{+}+d$.

Initial pp	Final $np(d)$	π^{+}
$^3P_1$	$^3S_1$	S
$^1D_2$	$^3S_1$	P
$^1S_0$	$^3S_1$	P

through Eq. (7) was 0.245 ± 0.01, compared to the value $(a_1-a_3)=0.245\pm0.007$ determined by Hamilton and Woolcock's extrapolation of experimentally measured phase shifts.

Recently, however, Donald *et al.*,[5] in a bubble-chamber experiment, measured the charge-exchange cross section. They obtained a value of $(a_1-a_3)=0.291\pm0.015$. Shortly after the experiment of Donald *et al.*, Samaranayake and Woolcock[6] made a new determination of (a_1-a_3) using forward dispersion relations and the most recent total pion-proton cross-section data; they obtained $(a_1-a_3)=0.292\pm0.020$, in good agreement with Donald *et al.* and also with a determination by Höhler and Baacke.[20] This obviously led to an inconsistency in Eq. (7).

It has been suggested[6] that in view of the possibility of a violation of time-reversal invariance in electromagnetic reactions, the cause of the inconsistency might lie in the detailed balance argument used to go from $\sigma(\gamma+p\rightarrow\pi^{+}+n)$ to $\sigma(\pi^{+}+n\rightarrow\gamma+p)$. However, Christ and Lee[21] have shown that at threshold this reciprocity relation can be derived by using the Hermiticity property of the interaction Hamiltonian H_γ, and is independent of the transformation property of H_γ under time reversal, provided all higher order terms in the fine-structure constant are neglected.

If we proceed along the branch of Table I leading to $p+p\rightarrow\pi^{+}+d$, we eliminate $\sigma(\pi^{-}+p\rightarrow n+\gamma)$ by using the expression

$$\sigma(\pi^{-}+p\rightarrow n+\gamma)=\frac{\sigma(\pi^{-}+p\rightarrow n+\gamma)}{\sigma(\pi^{-}+d\rightarrow 2n+\gamma)}\sigma(\pi^{-}+d\rightarrow 2n+\gamma).\quad(8)$$

The first ratio can be expressed in terms of the corresponding rates

$$\frac{\sigma(\pi^{-}+p\rightarrow n+\gamma)}{\sigma(\pi^{-}+d\rightarrow 2n+\gamma)}=\frac{w(\pi^{-}+p\rightarrow n+\gamma)}{w(\pi^{-}+d\rightarrow 2n+\gamma)}\frac{v_{\pi d}}{v_{\pi p}}=\frac{1}{T}\frac{v_{\pi d}}{v_{\pi p}},\quad(9)$$

where $v_{\pi p}$ and $v_{\pi d}$ are the relative velocities of the pion and the proton or deuteron.

[20] G. Höhler and J. Baacke (to be published).
[21] N. Christ and T. D. Lee, Phys. Rev. **148**, 1520 (1966).

Traxler[22] has calculated the ratio of the two rates. He used the impulse approximation for the deuteron process and included the effects of the final-state interaction between the two nucleons and the effect of the Pauli exclusion principle. His value for the ratio is $T=0.83\pm0.08$.

It is important to note that at this point we begin to consider processes which involve one meson and two nucleons rather than processes with one meson and one nucleon such as we have been considering.

In the center-of-mass frame we have

$$\frac{v_{\pi d}}{v_{\pi p}}=\frac{(1+\mu/2M)}{(1+\mu/M)}\quad(10)$$

for the same pion momentum.

At this point, we use

$$S\equiv\frac{\sigma(\pi^{-}+d\rightarrow 2n)}{\sigma(\pi^{-}+d\rightarrow 2n+\gamma)}\quad(11)$$

to go to the reaction $\pi^{-}+d\rightarrow 2n$. S has been measured by Ryan[18] and by Kloeppel[23] with consistent results; they find $S=3.02\pm0.1$.[24] Now, with the use of charge independence and detailed balancing, we obtain

$$\sigma(\pi^{-}+d\rightarrow 2n)=\sigma(\pi^{+}+d\rightarrow 2p)=\frac{2}{3}\frac{p_p^2}{p_\pi^2}\sigma(p+p\rightarrow\pi^{+}+d).\quad(12)$$

The extrapolation to threshold for this reaction depends on the phenomenological theory first presented by Watson and Brueckner,[25] and later reviewed by Rosenfeld[26] and by Gell-Mann and Watson.[27] The theory assumes only that the pion-nucleon interaction has a finite range of the order of the pion Compton wavelength and that the interaction conserves parity, isotopic spin, and angular momentum.

They find that the matrix element A for the lth wave contribution varies as p^l, where p is the momentum of the emitted pion. Squaring the matrix element A and multiplying by the phase-space factor, we have for the s- and p-wave cross sections $\sigma_s\sim p$ and $\sigma_p\sim p^3$.

Conservation of parity, isotopic spin, and angular momentum requires the initial states for s- and p-wave pions to be as shown in Table II.

Interference between the two initial states leading to the P state produces an angular distribution of the form $X+\cos^2\theta$, where X is a function of δ_0, the ratio of the

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[23] P. K. Kloeppel, Nuovo Cimento **34**, 11 (1964).
[24] This is the average of the values of S reported in Refs. 18 and 23.
[25] K. M. Watson and K. A. Brueckner, Phys. Rev. **83**, 1 (1951).
[26] A. H. Rosenfeld, Phys. Rev. **96**, 139 (1954).
[27] M. Gell-Mann and K. M. Watson, Ann. Rev. Nucl. Sci. **4**, 219 (1954).

complex amplitude for ${}^1S_0\rightarrow P$ to ${}^1D_2\rightarrow P$, and is

$$X=\left[\left|\frac{2-\sqrt{2}\delta_0}{1+\sqrt{2}\delta_0}\right|^2-1\right]^{-1}.$$

Combining the momentum dependence and the angular distribution of the s- and p-wave contributions gives the phenomenological expression

$$4\pi\frac{d\sigma}{d\Omega}=\alpha\eta+\beta\eta^3\frac{X+\cos^2\theta}{X+\frac{1}{3}}. \quad (13)$$

Lichtenberg,[28] using the impulse approximation to calculate the p-wave part of the reaction, has suggested that a better expression would be

$$4\pi\frac{d\sigma}{d\Omega}=\left(\frac{p_0}{p_p}\right)^3\left(\alpha'\eta+\beta'\eta^3\frac{X+\cos^2\theta}{X+\frac{1}{3}}\right), \quad (14)$$

where p_0 is the threshold proton momentum. His calculations indicate that β' at first slightly decreases, then increases with energy. He does not attempt to calculate the s-wave term.

Crawford and Stevenson[7] have measured the differential cross section for $p+p\rightarrow\pi^++d$ at values of η from 0.377 to 0.577, and produced least-squares-fitted values of

$$\alpha=0.138\pm0.015 \text{ mb},$$
$$\beta=1.01\pm0.08 \text{ mb},$$
$$X=0.082\pm0.34.$$

Other experiments[29–39] agree with the cross sections obtained using these values in Eq. (13). But in this momentum range, the term in β is dominant, and α and β, though assumed to be constants, may not be and may no longer have their threshold values.

At threshold in the center-of-mass frame, we find for the momentum factor in Eq. (12)

$$\frac{p_p^2}{p_\pi^2}=\frac{\mu Mc^2}{p_\pi^2}\left(1+\frac{\mu}{4M}\right). \quad (15)$$

[28] D. B. Lichtenberg, Phys. Rev. **105**, 1084 (1957).
[29] M. G. Meshcheryakov *et al.*, Dokl. Akad. Nauk SSSR **100**, 677 (1955).
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[33] H. Stadler, Phys. Rev. **96**, 496 (1954).
[34] T. H. Fields, J. G. Fox, J. A. Kane, R. A. Stallwood, and R. B. Sutton, Phys. Rev. **95**, 638 (1954).
[35] R. A. Schluter, Phys. Rev. **96**, 734 (1954).
[36] R. Durbin, H. Loar, and J. Steinberger, Phys. Rev. **84**, 581 (1951).
[37] D. L. Clark, A. Roberts, and R. Wilson, Phys. Rev. **83**, 649 (1951).
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[39] A. H. Schulz, Lawrence Radiation Laboratory Report No. UCRL-1756, 1952 (unpublished).

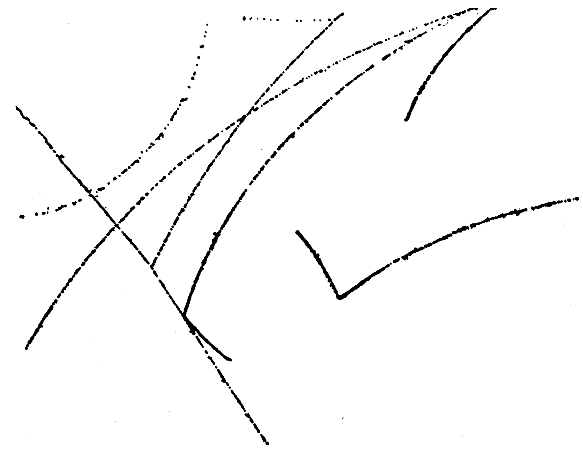

FIG. 1. A typical $\pi^++d\rightarrow p+p$ event as seen in the bubble chamber. The pion enters from upper right. Pions which did not interact are also seen. They stop in the chamber and decay $\pi^+\rightarrow\mu^+\rightarrow e^+$.

Combining Eqs. (8)–(12) and (15), we have

$$\sigma(\gamma+p\rightarrow\pi^++n)=\frac{1}{T}\frac{1}{S}\frac{1}{R}\frac{1}{3}\frac{M}{\mu}\left(1+\frac{\mu}{4M}\right)\frac{(1+\mu/M)}{(1+\mu/2M)}\times\sigma(p+p\rightarrow\pi^++d). \quad (16)$$

If we put in the values for the terms in this equation, we again find an inconsistency.

In order to see the magnitude of the discrepancy and to compare the two relationships, we look at the value obtained experimentally for the pion-photoproduction cross section and at the values determined through the relationships, which are

$$\sigma(\gamma+p\rightarrow\pi^++n)=(0.201\pm0.005)\eta \text{ mb}$$

experimentally,[2]

$$\sigma(\gamma+p\rightarrow\pi^++n)=(0.111\pm0.02)\eta \text{ mb}$$

via $p+p\rightarrow\pi^++d$,

$$\sigma(\gamma+p\rightarrow\pi^++n)=(0.285\pm0.02)\eta \text{ mb}$$

via pion scattering using values of (a_1-a_3) from forward dispersion relations,

$$\sigma(\gamma+p\rightarrow\pi^++n)=(0.202\pm0.01)\eta \text{ mb}$$

via pion scattering using values of (a_1-a_3) from extrapolated phase shifts, where the errors are obtained by combining the experimental errors quadratically.

In view of the discrepancies in the values above, it is appropriate to re-examine the experimental situation; in particular, we have tested the accepted threshold value of $\sigma(p+p\rightarrow\pi^++d)$ by studying the inverse process. We next describe the experiment.

III. EXPERIMENTAL METHOD

Positive pions from the University of Chicago synchrocyclotron were stopped in the Chicago 10-in.

deuterium bubble chamber operating in a magnetic field of 20.7 kG. A total of 24 000 pictures were taken with 8 stopping pions per picture.

A. Scanning

The film was scanned for events ($\pi^+ + d \to p + p$) and stops ($\pi^+ \to \mu^+ \to e^+$). Stops were recorded by the scanners in every hundredth frame. All pictures were scanned twice, and rolls that had a combined scanning efficiency less than 95% were scanned a third time. A histogram of the distribution in the azimuthal angle of the outgoing protons taken around the direction of the pion was made in order to check efficiency for protons that pointed toward the camera. This distribution was found to be flat, indicating that there was no appreciable scanning loss from this effect. A typical event along with several stops is shown in Fig. 1.

B. Reconstruction

A total of 3755 events were found. The total number of stops was calculated to be 1 950 000 from the total number of stops in every hundredth frame (19 400). Stops and events were digitized on the image-plane measuring machines developed at the University of Chicago. These measurements were processed by the Argonne National Laboratory versions of the Harwell geometry reconstruction program and the CERN GRIND kinematic fitting program. The geometry program fits a helix with energy-loss corrections to the tracks. The events were tested for the four-constraint (4C) fit to $\pi^+ + d \to p + p$ and the one-constraint (1C) fit to $\pi^+ + d \to p + p + \gamma$. The result of greatest interest from the fitting program was the momentum vector of the incident particle at the interaction vertex. In the case of the stops, there was no fitting necessary at the vertex; that is, all the π's stopped before decaying. This was verified by observing the range distribution of the decay muons.

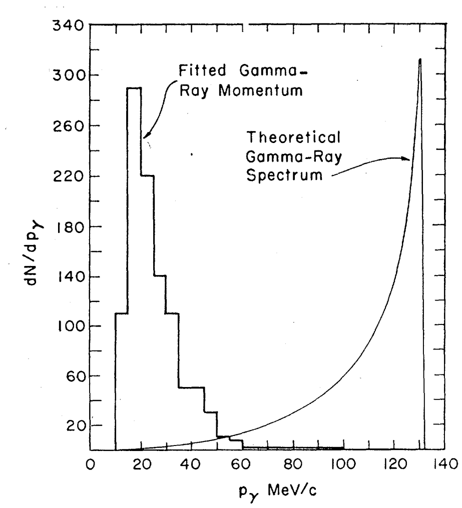

FIG. 2. The theoretical gamma-ray momentum spectrum for the process $\pi^+ + d \to p + p + \gamma$ for zero-energy pions. The fitted gamma-ray spectrum is superimposed.

The average length of the muon track (1.038 ± 0.04 cm) in the $\pi^+ \to \mu^+ \to e^+$ decay was used to determine the density of liquid deuterium. Using the range-momentum tables of Clark and Diehl,[40] we have

$$(\text{density of D}) = \frac{M_d}{M_p} \frac{(\text{range in H})}{(\text{range in D})} \times \text{density of H}$$
$$= (1.245 \pm 0.05) \times 10^{-1}\ \text{g/cm}^3.$$

C. Beam Momentum and Fiducial Volume

A fiducial volume smaller than the one used in scanning was established, and events and stops which fell outside this smaller fiducial volume were eliminated from the sample.

The momentum spectrum of the pion beam at the entrance to the fiducial volume was determined from the pion residual-range distribution at the fiducial boundary. The residual range was found by following the path of the stopping pion back to the point of crossing of the fiducial boundary. The retracing of the path was done in small intervals of path length Δs. The momentum loss per unit path length, dp/ds, was taken to be constant within the interval and equal to $\Delta p/\Delta s$, where Δp is the total momentum loss within the interval and is found from the range-momentum tables. The equations of motion were then solved using this approximation. A computer program was written to perform the retracing of the path. The retracing program used for initial values the position of the stopping vertex and the particle's momentum vector just before stopping, both given by the geometry program. Tracks

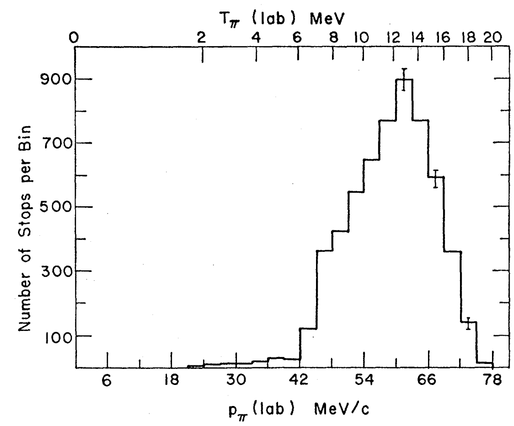

FIG. 3. The momentum distribution of the pion beam at the entrance to the fiducial volume. The momentum is determined from the pion's residual range.

[40] G. Clark and W. Diehl, Lawrence Radiation Report No. UCRL-2426 (rev) II, 1957 (unpublished).

[41] K. M. Watson and R. N. Stuart, Phys. Rev. 82, 738 (1951).

were generated with the program and compared against measured tracks with the same momentum. There was no apparent difference in the paths. The error in track length for a retraced path was 0.1 cm, corresponding to the length of the interval in which the track crossed the fiducial boundary. The error in range of 0.1 cm corresponds to an error in momentum of $\sim$1 MeV/c for the average momentum of this experiment (60 MeV/c).

It was necessary to exclude those events in which the pion would have stopped outside the fiducial volume if it had not interacted. This was done by extrapolating the tracks in the forward direction a length equal to the residual range at the point of interaction. The same program used for retracing the path was used for this

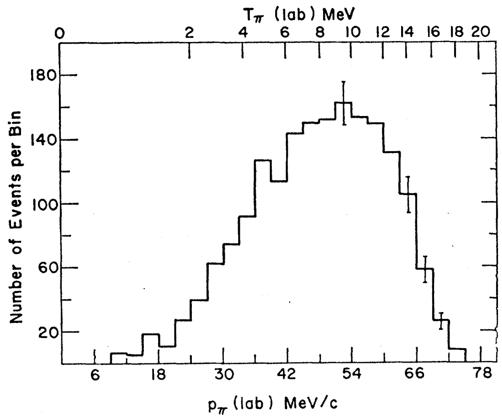

FIG. 4. The distribution of pion momentum at the interaction vertex for $\pi^+ + d \to p + p$ events. The momentum shown is the result of the kinematic fitting program.

purpose. If the path crossed the fiducial boundary before the end of the range, the event was excluded. Approximately 30% of the events were excluded for this reason.

D. Background Reactions

We have considered the following background reactions:

$$\pi^+ + d \to \pi^+ + d\,,$$
$$\pi^+ + d \to p + p + \pi^0\,,$$
$$\pi^+ + d \to \pi^+ + p + n\,,$$
$$\pi^+ + d \to p + p + \gamma\,.$$

The first three of these reactions were eliminated during scanning by requiring that each of the outgoing tracks be >3.0 cm, which is greater than the maximum range for the heavy particle of each reaction at our pion energies. The theoretical gamma-ray spectrum[41] resulting from reaction four is shown in Fig. 2 for a p-p scattering length of $a_p = -7.7 \pm 0.04$ F.[42] Most of the protons from this reaction are left with a range of less than

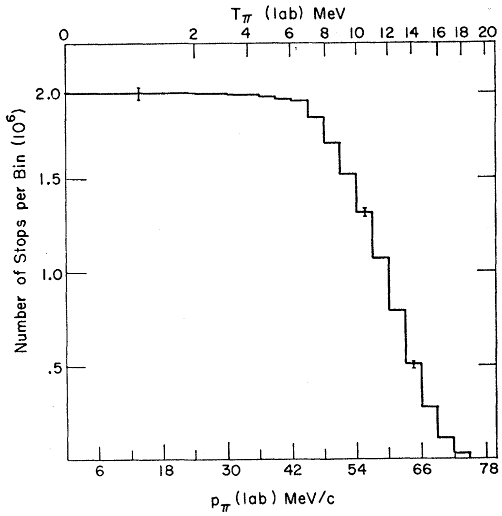

FIG. 5. The total number of pions passing through each momentum bin. This spectrum is computed from the spectrum of Fig. 4 normalized to the total number of stops.

3.0 cm and the reaction was thereby eliminated during scanning.

To be sure that all the $\pi^+ + d \to p + p + \gamma$ events had been eliminated, a fit to this 1C hypothesis was attempted for all events. We would expect that a large percentage of the true 4C events would also fit with a low-energy gamma ray to the 1C hypothesis. The fitting procedure of the kinematics program is such that very few events would accidentally fit to a very low- or zero-energy gamma ray. About 50% of the events did fit to the 1C hypothesis; however, all fit with gamma-ray momenta of <50 MeV/c. Events which fit the 1C hypothesis fit the 4C hypothesis equally well. That there are no events in the region where true radiative events would most likely occur is evident from a comparison of the theoretical spectrum with the histogram of fitted gamma-ray momenta, shown in Fig. 2. Also evident are the expected features of the accidental fitting of low-energy gamma rays and the suppression of fitting to zero-energy gamma rays. Since the 4C hypothesis is more stringent, requiring coplanarity, for example, and because of the discussion above, the 4C hypothesis was chosen in all cases where there was a 4C and a 1C fit. The possibility of ambiguous events in the tail of the distribution introduces an error of $\sim$1%.

E. Treatment of Data

Both the beam-entrance momentum and the interaction momentum were divided into momentum bins with widths in pion lab momentum of 3 MeV/c (Figs. 3 and 4). To find the total path length in each bin, we constructed a cumulative spectrum of the beam. That is, for each bin we took for the number of pions going through that interval the total number of pions with greater momentum plus one-half of the number with entrance momentum within that interval. The resulting

[42] H. P. Noyes, Phys. Rev. Letters 12, 171 (1964).

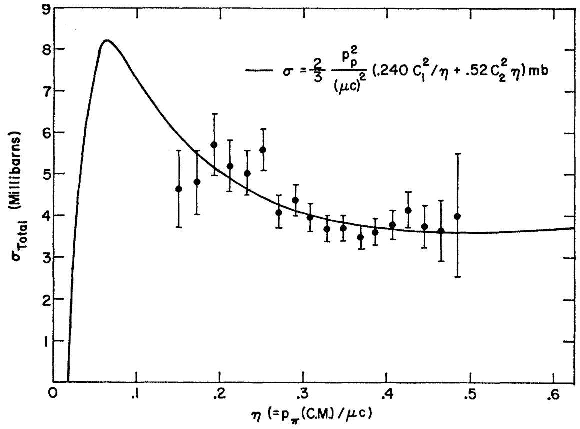

FIG. 6. Experimental points and the fitted curve for this experiment alone for the process $\pi^+ + d \rightarrow p+p$.

cumulative spectrum is shown in Fig. 5. The path length of a pion going through the momentum interval was found by integrating the range-momentum relation

$$\frac{dp}{ds}=\rho p^{\gamma} \text{ and obtaining } \Delta s=\int_{p_1}^{p_2}\frac{dp}{\rho p^{\gamma}},$$

where Δs is the path length to go from momentum p_1 to p_2, and $\rho = 1.415\times 10^5$ and $\gamma = -2.68$ are constants appropriate for pions stopping in deuterium of the density of this experiment. For each interval, we computed a cross section using the corresponding path length and number of events. For η we took the value corresponding to the space-average of p in the interval, i.e.,

$$\langle p\rangle_{\text{av}}=\frac{1}{\Delta s}\int_{s_1}^{s_2} p(s)ds.$$

We then transformed this value to the center-of-mass system to give cross sections in terms of η. The cross section for each interval is given in Table III.

TABLE III. Cross sections and Coulomb corrections for $\pi^+ + d \rightarrow p+p$.

η	σ(exp) (mb)	σ(calc)[a] (mb)	$C_1(\eta)$	$C_2(\eta)$
0.150	4.64±0.90	5.91	0.78	0.75
0.172	4.80±0.77	5.51	0.82	0.78
0.192	5.68±0.73	5.17	0.84	0.80
0.211	5.19±0.61	4.88	0.86	0.82
0.231	5.02±0.53	4.65	0.87	0.83
0.251	5.58±0.50	4.44	0.88	0.85
0.270	4.09±0.39	4.28	0.90	0.86
0.290	4.33±0.37	4.13	0.91	0.87
0.308	3.95±0.33	4.02	0.92	0.88
0.328	3.68±0.30	3.92	0.92	0.89
0.348	3.71±0.30	3.84	0.93	0.90
0.367	3.48±0.29	3.77	0.94	0.90
0.385	3.61±0.30	3.72	0.94	0.90
0.406	3.77±0.34	3.67	0.95	0.91
0.425	4.15±0.42	3.64	0.95	0.92
0.445	3.74±0.51	3.62	0.96	0.92
0.463	3.63±0.74	3.61	0.96	0.93
0.484	4.01±1.48	3.60	0.96	0.93

[a] Calculated from Eq. (18) with $\alpha = 0.240$ mb, $\beta = 0.52$ mb.

E. Coulomb Concentrations

We used the method of Schnitzer[43] in applying the necessary Coulomb corrections. Essentially the method is to join the logarithmic derivatives of the solution to the Schrödinger equation outside the region of strong interactions to the logarithmic derivative of the solution within the region. The solution outside is given in terms of the phase shifts referred to an asymptotic Coulomb wave, i.e., the measured phase shifts. The solution within the range of nuclear forces, where the Coulomb field can be treated as a perturbation,[44] is given as a sum of the unperturbed regular solution and a term which is a function of the regular and irregular unperturbed solution. This term is of order $e^2/\hbar\beta$, where β is the relative motion of the colliding particles in units of c. The unperturbed solutions are given in terms of the pure nuclear phase shifts. This procedure results in an expression for the resultant phase shifts δ_l^r in terms of the nuclear phase shifts δ_l. The treatment is general so that δ_l may be complex as it is for inelastic reactions. We expanded the expression to first order in the nuclear phase shifts to obtain $\delta_l^r = C_l(\eta)\delta_l$, where $C_l(\eta)$ is the Coulomb correction factor. The phenomenological expression, with Coulomb corrections, for the total cross section for $p+p\rightarrow \pi^+ + d$ then becomes

$$\sigma = C_1^2(\eta)\alpha\eta + C_2^2(\eta)\beta\eta^3. \tag{17}$$

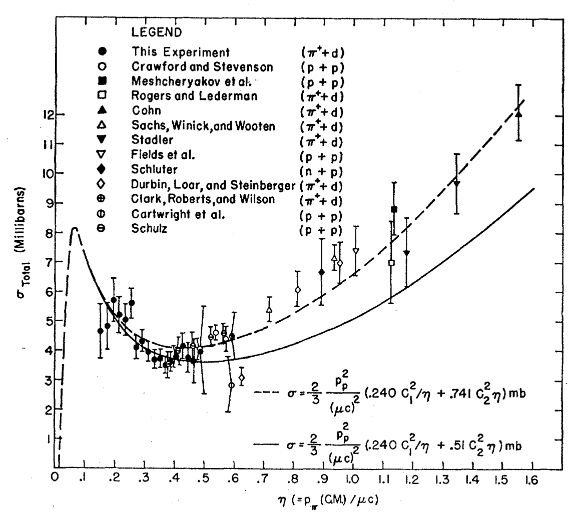

FIG. 7. Experimental points for all experiments on $\pi^+ + d \rightarrow p+p$ and the inverse process with $\eta < 1.5$. Also shown is the curve fitted to all the data shown. Data are taken in the order listed, from this paper; Refs. 7, 30–40.

[43] H. J. Schnitzer, Nuovo Cimento **28**, 752 (1963).

[44] G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949).

Using the least-squares method to fit the cross sections to the expression

$$\sigma(\pi^+ + d \to p+p) = \frac{2}{3}\frac{p_p^2}{\mu^2 c^2}\left[C_1^2(\eta)\frac{\alpha}{\eta} + C_2^2(\eta)\beta\eta\right], \quad (18)$$

we obtain

$$\alpha = 0.240 \pm 0.02 \text{ mb},$$
$$\beta = 0.52 \pm 0.2 \text{ mb},$$

with a χ^2 of 13.4 for 16 degrees of freedom. Using this value of α in Eq. (16), we obtain

$$\sigma(\gamma + p \to n + \pi^+) = (0.193 \pm 0.03)\eta \text{ mb},$$

compared to the experimentally measured value of

$$\sigma(\gamma + p \to n + \pi^+) = (0.201 \pm 0.005)\eta \text{ mb},$$

and we see that the former discrepancy in the figures discussed in Sec. IIF no longer appears. The points from this experiment are shown with the fitted curve in Fig. 6. In the region of this experiment ($0.2 \le \eta \le 0.4$) the fit to the data is relatively insensitive to β. Therefore, we have also fitted Eq. (18) keeping α fixed and equal to 0.240 mb, to the data from this experiment and the data from experiments at higher energies. Differential cross sections from any other experiment to be fitted were converted to total cross section by the use of the values of A and $d\sigma/d\Omega$ reported for that experiment, and through the use of

$$\frac{d\sigma}{d\Omega}\Big/\sigma = \frac{A + \cos^2\theta}{A + \frac{1}{3}}.$$

Fitting again to Eq. (18) we obtain

$$\alpha = 0.240 \pm 0.02 \text{ mb}, \quad \beta = 0.741 \pm 0.05 \text{ mb},$$

with a χ^2 of 69 for 39 degrees of freedom. All earlier experimental values including those from this experiment are shown along with the fitted curve in Fig. 7. We also attempted fitting to Eq. (14) with Coulomb corrections and found a reasonable solution ($\chi^2 = 15$) for the points of this experiment, since $p_0/p \sim 1$ in this region, but we found no good fit ($\chi^2 = 215$) when all experimental data were included. Hence we conclude that Eq. (18) gives a better fit to the data than Eq. (14).

IV. CONCLUSION

Using the values for $\sigma(\pi^+ + d \to p + p)$ obtained in this experiment, we find that the relationship leading from positive-pion photoproduction to the reaction $\sigma(p + p \to \pi^+ + d)$ is now consistent. This indicates that to the precision of this study, the extrapolation procedures, the calculation of T, and the experimental data which are used in Eq. (16) are satisfactory. Table I shows that Eqs. (7) and (16) have several elements in common. In particular, both use detailed balance to go from

$$\sigma(\gamma + n \to \pi^- + p) \quad \text{to} \quad \sigma(\pi^- + p \to \gamma + n).$$

Since Eq. (16) is consistent, all of these common elements are also presumably consistent. Therefore, there is no evidence of a failure in detailed balancing,[6] in charge independence, or in the extrapolation procedures. The only elements left in Eq. (7) which are not common to Eq. (16) are the Panofsky ratio, which has been measured many times with consistent results,[18] and the π-p scattering lengths. The π-p scattering lengths also appear to be consistent since the same values were found by Donald *et al.*[5] in their charge-exchange experiment, and by Samaranayake and Woolcock[6] in their dispersion-relation analysis of total cross section data. However, in view of the possibility suggested by Donald *et al.* that a different set of p-wave phase shifts might alter their values, and presumably could affect the results of Samaranayake and Woolcock, it may be that more work on the determination of the parameters of low-energy π-p scattering is needed.

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