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Multimass Fields, Spin, and Statistics*

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The most general free field which transforms locally under Lorentz transformations according to a unitary representation of the homogeneous Lorentz group is constructed. Such a field is a linear combination of annihilation and creation operators for particles belonging to an infinite tower of irreducible multiplets of the Poincaré group. Different spin multiplets in the tower may have different masses, and appear with arbitrary spin-dependent and mass-dependent weight factors in the field expansion. It is shown that the requirement of causal (anti-)commutation relations for these fields can be satisfied simply if one assumes Bose statistics for the particles in the towers for both integer and half-odd-integer spin. However, by a judicious use of the Gelfand matrices (generalized Dirac matrices) the causality condition can also be satisfied using Fermi statistics. Such Fermi fields are constructed for two particular unitary representations—one of integer and one of half-odd-integer spin. These generalized Fermi fields still do not enable one to construct an index-invariant theory consistent with the unitarity of the S matrix which incorporates particles satisfying Fermi statistics. Thus the incompatibility for an index-invariant causal theory between unitarity and Fermi statistics, which was established in a previous paper under more restrictive assumptions, remains valid for these more general fields.

I. INTRODUCTION

In two previous papers[1,2] we have constructed local[3] field operators which transform as representations of the homogeneous Lorentz group L. The particle multiplets corresponding to these fields could contain either a finite or infinite number of different spins. In I we discussed the well-known result that if the fields transform as finite-dimensional representations of L it is impossible to construct an index-invariant theory (defined in I) which is consistent with the unitarity of the S matrix. In II we established two theorems. Firstly, we showed that by constructing fields which transform as unitary representations of L, one *can* construct a theory which is index invariant and is consistent with the unitarity of the S matrix. Secondly, such a theory was shown to be causal only if all of the particles (whether of integral of half-integral spin) obeyed Bose statistics. In Sec. II we construct the most general local field which transforms as a unitary representation of L. The field contains particles of different mass as well as spin. The different spin multiplets may occur with arbitrary mass- and spin-dependent weight factors in the field expansion.

In Sec. III we summarize some general properties of the homogeneous Lorentz group. In particular, we introduce the generalized unitary Dirac matrices found by Gelfand.

In Sec. IV we examine the implications of the requirement that the fields satisfy causal (anti-)commutation relations. In general it is a simple matter to satisfy the causality condition if the particles satisfy Bose statistics. This leads to fields satisfying causal commutation relations. By exploiting the extra generality of the fields considered here, it is shown that for two particular unitary towers—one of integer, one of half-odd-integer spin—one can choose weight factors for the different spins in the field expansion in such a way as to introduce the Gelfand matrices. This device leads to particles satisfying Fermi statistics and corresponding fields satisfying causal anticommutation relations. These explicit constructions of Fermi or Bose fields for either integer of half-integer spin are in marked contrast to the conventional Pauli relation between spin and statistics for local causal fields which transform as finite representations of the homogeneous Lorentz group.

To conclude Sec. IV we consider whether these unitary Fermi fields can be used to construct index-invariant theories which are consistent with the unitarity of the S matrix. It is shown that they cannot. There is in fact a very direct conflict between the unitarity condition, which requires the sum over spinors be unity, and causality with Fermi statistics, which forces the same sum to be an odd function of the momentum. Thus, these general fields do not alter the conclusion of the causality theorem, established in II under more restrictive assumptions.

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[1] G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **40**, 19 (1966), hereafter referred to as I.

[2] G. Feldman and P. T. Matthews, Phys. Rev. **151**, 1176 (1966), hereafter referred to as II.

[3] By a local field operator we mean one which has simple Poincaré transformation properties. Thus if $\psi_\alpha(x)$ is a local field, under a pure Lorentz transformation, $\psi_\alpha(x) \to S_{\alpha\beta}\psi_\beta(x')$, where $S_{\alpha\beta}$ is a matrix (independent of x and x') in some "spinor" space (finite or infinite dimensional) and x' is connected to x by the usual equations of a Lorentz transformation. Under space-time translation, $\psi_\alpha(x) \to \psi_\alpha(x+l)$. We distinguish a local field from a causal field which is defined below to satisfy causal (anti-)commutation relations.

II. MULTIMASS FIELDS

In this section we develop local multimass field operators,[3] following closely the procedures of I and II (but changing the notation somewhat).

The infinitesimal generators of the Poincaré group are the usual translation and rotation operators[4]

$$P_\mu, J_{\mu\nu}. \tag{2.1}$$

The single-particle states belong to the irreducible unitary representations labeled by mass m and spin s and the components can be labeled by a four-velocity u_μ and spin component s_3, where

$$u_\mu u^\mu \equiv u^2 = 1. \tag{2.2}$$

Thus we shall write the states as

$$|m,s;u_\mu,s_3\rangle, \tag{2.3}$$

such that

$$P_\mu|m,s;u_\mu,s_3\rangle = mu_\mu|m,s;u_\mu,s_3\rangle, \tag{2.4}$$

and

$$W^2|m,s;u_\mu,s_3\rangle = -m^2s(s+1)|m,s;u_\mu,s_3\rangle, \tag{2.5}$$

where

$$W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}P_\nu J_{\lambda\rho}. \tag{2.6}$$

Since the eigenvalues of P_μ form a continuous set we normalize the states (2.3) to a δ function, and, in fact, we write[5]

$$\langle m',s';u_\mu',s_3'|m,s;u_\mu,s_3\rangle(2\pi)\theta(u_0)\delta(u^2-1) = (2\pi)^4\delta(u-u')\delta_{mm'}\delta_{ss'}\delta_{s_3s_3'}. \tag{2.7}$$

We define

$$K_i \equiv J_{0i}. \tag{2.8}$$

Then the boost operator is the pure Lorentz transformation which transforms rest states to moving states, and we may write

$$|m,s;u_\mu,s_3\rangle = N\exp(-i\boldsymbol{\varepsilon}(u)\cdot\mathbf{K})|m,s,s_3\rangle, \tag{2.9}$$

where

$$\cosh|\boldsymbol{\varepsilon}| = u_0, \quad \sinh|\boldsymbol{\varepsilon}| = |\mathbf{u}|, \tag{2.10}$$

and the unit vectors $\hat{\varepsilon}(u)$ and $\hat{u}$ are equal. For later convenience we have chosen the states $|m,s,s_3\rangle$ to be normalized such that

$$\langle m',s',s_3'|m,s,s_3\rangle = \delta_{mm'}\delta_{ss'}\delta_{s_3s_3'}. \tag{2.11}$$

Accordingly, the factor N in (2.9) must be chosen[6] consistent with (2.7). Using only the commutation relations of the Poincaré algebra, it can easily be shown that the states defined in (2.9) do indeed satisfy Eq. (2.4). Note that the boost operator does not depend upon m but only on u_μ, so that the same boost operator can be used to produce states of a definite velocity, even though their masses are different. Accordingly, as already implied by Eqs. (2.7) and (2.11) we shall assume that our states $|m,s;u_\mu,s_3\rangle$ can be some *reducible* representation of the Poincaré group and thus one containing different masses m and different spins s. All the members of this reducible representation have in common only the velocity u_μ.

As usual, we assume the existence of a lowest energy state, $\rangle_0$, the vacuum, and introduce annihilation and creation operators such that

$$a(m,s;u,s_3)\rangle_0 = 0, \tag{2.12}$$

and

$$a^\dagger(m,s;u,s_3)\rangle_0 = |m,s;u_\mu,s_3\rangle. \tag{2.13}$$

It must then follow[7] from (2.7) that

$$[a(m,s;u,s_3),a^\dagger(m',s';u',s_3')]_\pm(2\pi)\theta(u_0)\delta(u^2-1) = (2\pi)^4\delta^4(u-u')\delta_{mm'}\delta_{ss'}\delta_{s_3s_3'}. \tag{2.14}$$

A pure Lorentz transformation can be specified by a four-velocity v_μ (where $v^2=1$). It was demonstrated[8] in I that under such a transformation,

$$U(v) \equiv \exp(-i\boldsymbol{\eta}\cdot\mathbf{K}), \tag{2.15}$$

$$U(v)a(m,s;u,s_3)U^{-1}(v) = \sum_{s_3'}\langle m,s,s_3|D^{-1}(u,v)|m,s,s_3'\rangle a(m,s;u',s_3'), \tag{2.16}$$

where

$$D(u,v) = \exp(i\boldsymbol{\varepsilon}'\cdot\mathbf{K})\exp(-i\boldsymbol{\eta}\cdot\mathbf{K})\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K}), \tag{2.17}$$

and

$$\cosh|\boldsymbol{\eta}| = v_0, \quad \sinh|\boldsymbol{\eta}| = |\mathbf{v}|,$$

$$\hat{\eta} = \hat{v}, \tag{2.18}$$

$$\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}(u'), \tag{2.19}$$

and u_μ' is the four-velocity obtained from u_μ by the Lorentz transformation specified by v_μ.

Similarly, as in (2.46) of I we can write

$$U(v)a^\dagger(m,s;u,s_3)U^{-1}(v) = \sum_{s_3'}\langle m,s,s_3|B^{-1}D^{-1}B|m,s,s_3'\rangle a^\dagger(m,s;u',s_3'), \tag{2.20}$$

where B is a spin-flip matrix defined by (2.45) of I. Under a translation $U(l)$ specified by a four-vector l_μ,

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[4] We use the notation μ, $\nu=0, 1, 2, 3$ and i, $j=1, 2, 3$, with the metric $g_{\mu\nu}$ such that $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$. The commutation relations satisfied by P_μ and $J_{\mu\nu}$ have been written down many times. See, e.g., I, Eqs. (2.1) to (2.3).

[5] One can normalize different irreducible representations (i.e., those with different m and s) arbitrarily. Thus we could multiply the right side of (2.7) by $f(m,s)$, an arbitrary (positive) function of m and s. However, since these states are to be identified with physical states, the normalization factor must be introduced at some stage in calculating probabilities and is most conveniently done at this stage. However, this freedom of normalization will be exploited when we introduce fields below.

[6] Thus the true "rest states" are $N|m,s,s_3\rangle$ and not $|m,s,s_3\rangle$.

[7] The $\pm$ in Eq. (2.14) will refer to the anticommutator or commutator, respectively, i.e., to fermions or bosons. In this paper we will not consider the possibility of the particles obeying parastatistics.

[8] Equation (2.16) is the same as (2.22) of I, incorporating our new notation. It is important to remember that the operator $D(u,v)$ is an operator in the little group and thus, when operating on a rest frame state $|m,s,s_3\rangle$, cannot change the mass m or spin s.

we have

$$U(l)aU^{-1}(l)=e^{-ip_\mu l^\mu}a\,, \tag{2.21}$$

$$U(l)a^\dagger U^{-1}(l)=e^{ip_\mu l^\mu}a^\dagger\,. \tag{2.22}$$

In terms of these operators, a and $a^\dagger$, one can define an auxiliary operator (see I) with simple transformation properties under all the Poincaré transformations:

$$\begin{aligned}A_\alpha{}^{(\sigma)}(u,x)&=\sum_{m,s,s_3}\sum_\beta\langle\sigma;\alpha|\exp[-i\boldsymbol{\varepsilon}(u)\cdot\mathbf{K}]|\sigma;\beta\rangle\\&\qquad\times\langle\sigma;\beta|m,s,s_3\rangle f(s)e^{-imu\cdot x}a(m,s;u,s_3)\\&\equiv\sum_{m,s,s_3}w_\alpha{}^{(\sigma)}(m,u,s,s_3)f(s)e^{-imu\cdot x}a(m,s;u,s_3)\,,\end{aligned} \tag{2.23}$$

where $|\sigma;\alpha\rangle$ is some representation of the auxiliary group which we shall take to be the homogeneous Lorentz group L. These states will be discussed in more detail in Sec. III. Here, let us note that σ labels some *irreducible*[9] representation of L and α the components. The weight factor[10] $f(s)$ is some arbitrary function of s. The factor $w_\alpha{}^{(\sigma)}$ is a generalized spinor and hence $\langle\sigma;\alpha|m,s,s_3\rangle$ is a constant spinor.

From the transformation properties (2.16) and (2.21) one can immediately deduce that

$$U(v)A_\alpha{}^{(\sigma)}(u,x)U^{-1}(v)=\sum_\beta\langle\sigma;\alpha|\exp(i\boldsymbol{\eta}\cdot\mathbf{K})|\sigma;\beta\rangle A_\beta{}^{(\sigma)}(u',x')\,, \tag{2.24}$$

and

$$U(l)A_\alpha{}^{(\sigma)}(u,x)U^{-1}(l)=A_\alpha{}^{(\sigma)}(u,\,x+l)\,. \tag{2.25}$$

The auxiliary operator $A_\alpha{}^{(\sigma)}(u,x)$ defined here differs from the operator $A_\alpha(p)$ defined in I and II. In I and II we consider only those physical states $|m,s;u,s_3\rangle$ which were *irreducible* representations of the Poincaré group and thus were states all of the same single mass m. Thus, if instead of (2.23) we had defined an operator $A_\alpha{}^{(\sigma)}(u)$ without the factor $e^{-imu\cdot x}$ we would have found that $A_\alpha{}^{(\sigma)}(u)$ did not have simple translation properties. However, for the special case, when all the states in $|m,s,s_3\rangle$ are restricted to have the same mass, say M, we can define an $A_\alpha{}^{(\sigma)}(u)$ and we will have

$$A_\alpha{}^{(\sigma)}(u,x)=A_\alpha{}^{(\sigma)}(u)e^{-iMu\cdot x}\,. \tag{2.26}$$

Just as in the single-mass case, we introduce antiparticle operators $b(m,s;u,s_3)$ and auxiliary operators associated with them. Thus,

$$\begin{aligned}\tilde{B}_\alpha{}^{(\sigma)}(u,x)=\sum_{m,s,s_3}\sum_\beta&\langle\sigma;\alpha|\exp[-i\boldsymbol{\varepsilon}(u)\cdot\mathbf{K}]|\sigma;\beta\rangle\\&\times\langle\sigma;\beta|B|\bar{m},s,s_3\rangle g(s)e^{+imu\cdot x}b^\dagger(\bar{m},s;u,s_3)\,,\end{aligned} \tag{2.27}$$

where $g(s)$ is some arbitrary function of s; and $\bar{m}$ runs over the same values as m but refers to an antiparticle state. $\tilde{B}_\alpha{}^{(\sigma)}(u,x)$ transforms in the same way as $A_\alpha{}^{(\sigma)}(u,x)$ under all Poincaré transformations. It is then possible to define a local field

$$\psi_\alpha{}^{(\sigma)}(x)\equiv\int[A_\alpha(u,x)+\tilde{B}_\alpha(u,x)]\times(2\pi)\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4}\,. \tag{2.28}$$

It follows that

$$U(v)\psi_\alpha{}^{(\sigma)}(x)U^{-1}(v)=\sum_\beta\langle\sigma;\alpha|\exp(i\boldsymbol{\eta}\cdot\mathbf{K})|\sigma;\beta\rangle\psi_\beta{}^{(\sigma)}(x')\,, \tag{2.29}$$

and

$$U(l)\psi_\alpha{}^{(\sigma)}(x)U^{-1}(l)=\psi_\alpha{}^{(\sigma)}(x+l)\,, \tag{2.30}$$

showing that $\psi_\alpha{}^{(\sigma)}(x)$ is a local field.

In contrast to the single-mass case, $\psi_\alpha{}^{(\sigma)}(x)$ will, in general, not satisfy any equation of motion—not even a Klein-Gordon equation.

Before going on to examine whether these fields can be made to satisfy causal (anti-)commutation relations, we shall find it useful to review some important features of the homogeneous Lorentz group L.

III. THE HOMOGENEOUS LORENTZ GROUP L

In this section we review some of the properties of the homogeneous Lorentz algebra and its representations. Most of the material can be found (with some notational changes) in the books by Gelfand, Minlos, and Shapiro[11] and Naimark.[12]

The infinitesimal generators of L are the operators $J_{\mu\nu}$. If we define

$$J_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}\,, \tag{3.1}$$

and

$$K_i=J_{0i}\,, \tag{3.2}$$

the commutation relations are

$$[J_i,J_j]=i\epsilon_{ijk}J_k\,, \tag{3.3}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k\,, \tag{3.4}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k\,. \tag{3.5}$$

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[9] The restriction that $|\sigma;\alpha\rangle$ transform as an irreducible representation of L will imply some restriction on what physical states we are combining in our reducible Poincaré representation, $|m,s,s_3\rangle$. As we shall see in Sec. III, the component label α stands for j and j_3, the spin magnitude and spin component labeling a representation of L. In an irreducible representation σ of L a particular j appears no more than once and the value of σ determines which j's do occur, and only j's differing by integers can occur. These restrictions on j imply a restriction on which s appears in $|m,s,s_3\rangle$ since we must have $\langle\sigma;\beta|m,s,s_3\rangle\neq0$. However, we will find it useful at times to introduce other auxiliary operators, such as $A_\alpha{}^{(\tau)}(u,x)$, where τ labels a different irreducible representation of L but with the same spin content as the σ representation.

[10] More generally, we can have an $f(m,s)$ rather than $f(s)$. However, in all of our subsequent work we shall always choose $m=m(s)$. In previous discussions this factor $f(s)$ has been tacitly assumed to be unity.

[11] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups* (Pergamon Press, Inc., New York, 1963).

[12] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., New York, 1964).

All of the irreducible representations of this group are specified by two parameters

$$(k_0,c)\equiv\sigma, \tag{3.6}$$

where k_0 is either an integer or half an odd integer, and c is any complex number. These parameters are related to the values of the Casimir operators in this representation. The Casimir operators are

$$\mathbf{J}^2-\mathbf{K}^2 \quad\text{and}\quad \mathbf{J}\cdot\mathbf{K}. \tag{3.7}$$

Let a state be labeled

$$|\sigma;\alpha\rangle, \tag{3.8}$$

where σ specifies a representation and α labels the component. Then

$$(\mathbf{J}^2-\mathbf{K}^2)|\sigma;\alpha\rangle=(k_0^2+c^2-1)|\sigma;\alpha\rangle, \tag{3.9}$$

and

$$\mathbf{J}\cdot\mathbf{K}|\sigma;\alpha\rangle=-ik_0c|\sigma;\alpha\rangle. \tag{3.10}$$

The components α of a given representation are specified by two parameters

$$(j,j_3)\equiv\alpha, \tag{3.11}$$

which are both either integers or half-odd integers depending on k_0;

$$-j\leq j_3\leq j, \quad\text{and}\quad j\geq|k_0|. \tag{3.12}$$

These states satisfy the equations

$$\mathbf{J}^2|\sigma;\alpha\rangle=j(j+1)|\sigma;\alpha\rangle, \tag{3.13}$$

$$J_3|\sigma;\alpha\rangle=j_3|\sigma;\alpha\rangle. \tag{3.14}$$

Since the operators J_i commute with $\mathbf{J}^2$ they will only connect states of the same j (but possibly with different j_3). The operators K_i do not commute with $\mathbf{J}^2$ and will therefore connect states with different j as well as j_3.

The representation specified by $(-k_0, -c)$ is equivalent to (k_0,c). (See Ref. 11, p. 194.) The finite dimensional representations are nonunitary and are given by

$$|c|=|k_0|+n, \tag{3.15}$$

where n is a positive integer. In this case, the possible values of j reach a maximum.

$$|k_0|\leq j\leq|c|-1. \tag{3.16}$$

The unitary representations are given by

$$\left.\begin{array}{l}k_0=0,\pm\frac{1}{2},\pm1\cdots,\\ c\text{ pure imaginary,}\end{array}\right. \quad\text{(the principal series)} \tag{3.17}$$

or

$$\left.\begin{array}{l}k_0=0,\\ 0<|c|<1.\end{array}\right. \quad\text{(the supplementary series)} \tag{3.18}$$

In both these cases

$$j=|k_0|,\quad |k_0|+1,\quad |k_0|+2,\quad \cdots,$$

and there is no maximum value.

It is easy to see that if we have a representation of J_i and K_i which we label by σ we can find another representation, which we label $\dot\sigma$ and call the conjugate representation, such that

$$\dot\sigma=(k_0,\,-c)=(-k_0,\,c). \tag{3.19}$$

If we write

$$\langle\sigma;\alpha|K_i|\sigma;\beta\rangle\equiv(\kappa_i)_{\alpha\beta}, \tag{3.20}$$

then we shall have[13]

$$\langle\dot\sigma;\alpha|K_i|\dot\sigma;\beta\rangle=-(-1)^{[j]+[j']}(\kappa_i)_{\alpha\beta}, \tag{3.21}$$

where

$$\alpha=(j,j_3),\quad \beta=(j',j_3'), \tag{3.22}$$

and

$$\begin{aligned}[j]&=j,\text{ for } j \text{ an integer}\\ &=j-\tfrac{1}{2},\text{ for } j \text{ half an odd integer.}\end{aligned} \tag{3.23}$$

From (3.19) we immediately deduce that the representations $(0,c)$ and $(k_0,0)$ are self-conjugate. (See Ref. 11, pp. 193–4.) No half-integer finite-dimensional representation is self-conjugate. Thus, for example, if

$$\sigma=(\tfrac{1}{2},\tfrac{3}{2}),\quad \dot\sigma=(\tfrac{1}{2},\,-\tfrac{3}{2}) \tag{3.24}$$

(the usual undotted and dotted representations for spin $\frac{1}{2}$), then we have

$$\langle\sigma;\alpha|K_i|\sigma;\beta\rangle=-\tfrac{1}{2}i(\sigma_i)_{\alpha\beta}, \tag{3.25}$$

and

$$\langle\dot\sigma;\alpha|K_i|\dot\sigma;\beta\rangle=\tfrac{1}{2}i(\sigma_i)_{\alpha\beta}, \tag{3.26}$$

where the σ_i are the usual Pauli matrices and the α, β run over the two values of j_3 in these representations.[14]

We shall also be interested in including the parity operation R in specifying our physical states and thus also our auxiliary states. This satisfies the following commutation relations:

$$RJ_i-J_iR=0, \tag{3.27}$$

and

$$RK_i+K_iR=0. \tag{3.28}$$

From these relations one can immediately deduce that (see Ref. 11, p. 216)

$$R|\sigma;\alpha\rangle=(-1)^{[j]}|\dot\sigma;\alpha\rangle. \tag{3.29}$$

Thus, an irreducible representation of the group $L(+)R$ will contain the two irreducible representations σ and $\dot\sigma$ of L. For self-conjugate representations

$$R|\sigma;\alpha\rangle=\pm(-1)^{[j]}|\sigma;\alpha\rangle. \tag{3.30}$$

In addition to the representation of the generators K_i given formally by (3.20) it is convenient to introduce a set of vectors $|s\rangle$ which span the space of a given

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[13] By redefining the set of basis vectors in the $\dot\sigma$ representation Gelfand is able to absorb the factor $(-1)^{[j]+[j']}$ into the new set of vectors [see Ref. 11, p. 193, but see also the results of changing (k_0,c) to $(k_0,\,-c)$ in Eqs. (13) to (16), pp. 193–4]. It is convenient to keep this factor explicit, especially when we come to discuss the parity operator.

[14] In these representations only one value of j (namely, $\frac{1}{2}$) can occur. See Eq. (3.16).

representation. Thus we may write (3.20) as

$$\sum_\beta \langle\sigma;\alpha|K_i|\sigma;\beta\rangle\langle\sigma;\beta|s\rangle=\sum_\beta(\kappa_i)_{\alpha\beta}\langle\sigma;\beta|s\rangle. \quad (3.31)$$

The label s is really a shorthand for two indices s and s_3, the spin and spin component. We can choose the vectors such that

$$\langle\sigma;\alpha|s\rangle\equiv\langle\sigma;j,j_3|s,s_3\rangle=\lambda^{(\sigma)}\delta_{js}\delta_{j_3s_3}, \quad (3.32)$$

where $\lambda^{(\sigma)}$ is an arbitrary phase factor. The components of these vectors, $\langle\sigma;\alpha|s\rangle$ may be identified with the "constant spinors" introduced in (2.23). This follows since the states $|m,s,s_3\rangle$ are the rest-frame states of the Poincaré group, the Casimir operator W^2 is proportional to $\mathbf{J}^2$ in the rest frame, and s_3 is the eigenvalue of J_3 in the rest frame.

In our attempts in Sec. IV to construct causal fields we shall find it useful to introduce a set of matrices

$$\langle\sigma;\alpha|\Gamma_\mu|\sigma';\beta\rangle, \quad (3.33)$$

which were first introduced in Ref. 11 and by Naimark.[12] These are a set of matrices which "transform as a four-vector." More specifically, if $A_\alpha^{(\sigma)}(u,x)$ is an operator which transforms as given by (2.24) and if $B^{(\sigma')\alpha}(w,y)$ transforms as

$$U(v)B^{(\sigma')\alpha}(w,y)U^{-1}(v)$$
$$=\sum_\beta B^{(\sigma')\beta}(w',y')\langle\sigma';\beta|\exp(-i\boldsymbol{\eta}\cdot\mathbf{K})|\sigma';\alpha\rangle, \quad (3.34)$$

then

$$\sum_{\alpha,\beta} B^{(\sigma')\alpha}(w,y)\langle\sigma';\alpha|\Gamma_\mu|\sigma;\beta\rangle A_\beta^{(\sigma)}(u,x) \quad (3.35)$$

transforms as a four-vector density under the Lorentz transformation $U(v)$.

These matrices Γ_μ have been shown to have the property that

$$\langle\sigma';j',j_3'|\Gamma_0|\sigma;j,j_3\rangle=C_j^{\sigma\sigma'}\delta_{jj'}\delta_{j_3j_3'}, \quad (3.36)$$

and that $C_j^{\sigma\sigma'}$ differs from zero only if

$$\sigma'\equiv(k_0',c')=(k_0\pm1,\,c), \quad (3.37)$$

or

$$\sigma'\equiv(k_0',c')=(k_0,\,c\pm1). \quad (3.38)$$

In Sec. IV we shall restrict the discussion to unitary representations σ and σ' and also either to the self-conjugate representations

$$\sigma=(0,\tfrac{1}{2})=\dot\sigma=\sigma' \quad (3.39)$$

(the self-conjugate integer spin representation),

$$\sigma=(\tfrac{1}{2},0)=\dot\sigma=\sigma' \quad (3.40)$$

(the self-conjugate half-odd-integer spin representation), or to the half-odd-integer spin representation

$$\sigma=(\tfrac{1}{2},c),\quad c\text{ pure imaginary}$$

and

$$\dot\sigma=\sigma'=(-\tfrac{1}{2},\,c)=(\tfrac{1}{2},\,-c). \quad (3.41)$$

For the cases (3.39) and (3.40) one finds (Ref. 11, pp. 351–2)

$$C_j^{\sigma\sigma'}\equiv C_j=(j+\tfrac{1}{2})C, \quad (3.42)$$

and for case (3.41)

$$C_j^{\sigma\sigma'}=C_j^{\sigma\dot\sigma}=C_j^{\dot\sigma\sigma}=(j+\tfrac{1}{2})C, \quad (3.43)$$

and, of course,

$$C_j^{\sigma\sigma}=C_j^{\dot\sigma\dot\sigma}=0. \quad (3.44)$$

The matrix Γ_0 boosts in the following way:

$$\sum_{\delta,\gamma}\langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\delta\rangle\langle\sigma;\delta|\Gamma_0|\sigma';\gamma\rangle$$
$$\times\langle\sigma';\gamma|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma';\beta\rangle$$
$$=\langle\sigma;\alpha|\Gamma_\mu|\sigma';\beta\rangle u^\mu, \quad (3.45)$$

where the relation between $\boldsymbol{\varepsilon}$ and u_μ is given by (2.10).

Equations (3.42) and (3.43) can be summarized in the following way: For the self-conjugate representations, there is a complete set of vectors

$$|s,s_3\rangle\equiv|s\rangle, \quad (3.46)$$

such that

$$\Gamma_0|s\rangle=(s+\tfrac{1}{2})|s\rangle, \quad (3.47)$$

or more precisely

$$\sum_\beta\langle\sigma;\alpha|\Gamma_0|\sigma;\beta\rangle\langle\sigma;\beta|s\rangle=(s+\tfrac{1}{2})\langle\sigma;\alpha|s\rangle, \quad (3.48)$$

so that

$$\langle\sigma;\alpha|\Gamma_0|\sigma;\beta\rangle=\sum_{s,s_3}\langle\sigma;\alpha|s\rangle(s+\tfrac{1}{2})\langle s|\sigma;\beta\rangle. \quad (3.49)$$

Also,[15]

$$\sum_\beta\langle\sigma;\alpha|\Gamma u|\sigma;\beta\rangle w_\beta^{(\sigma)}(u,s)=(s+\tfrac{1}{2})w_\alpha^{(\sigma)}(u,s), \quad (3.50)$$

where by (2.23)

$$w_\alpha^{(\sigma)}(u,s)=\sum_\beta\langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\beta\rangle\langle\sigma;\beta|s\rangle. \quad (3.51)$$

For the non-self-conjugate case (3.41), there is a complete set of vectors,

$$|s\rangle\quad\text{and}\quad|\bar s\rangle, \quad (3.52)$$

such that

$$\Gamma_0|s\rangle=(s+\tfrac{1}{2})|s\rangle, \quad (3.53)$$

$$\Gamma_0|\bar s\rangle=-(s+\tfrac{1}{2})|\bar s\rangle, \quad (3.54)$$

or explicitly,

$$\sum_\beta\langle\sigma;\alpha|\Gamma_0|\dot\sigma;\beta\rangle\langle\dot\sigma;\beta|s\rangle=(s+\tfrac{1}{2})\langle\sigma;\alpha|s\rangle,$$
$$\sum_\beta\langle\dot\sigma;\alpha|\Gamma_0|\sigma;\beta\rangle\langle\sigma;\beta|s\rangle=(s+\tfrac{1}{2})\langle\dot\sigma;\alpha|s\rangle, \quad (3.55)$$

[15] We write $\Gamma_\mu u^\mu=\Gamma u$. Also, the $w_\alpha^{(\sigma)}$ are just the quantities introduced in (2.23).

and

$$\sum_\beta \langle\sigma;\alpha|\Gamma_0|\dot{\sigma};\beta\rangle\langle\dot{\sigma};\beta|\bar{s}\rangle = -(s+\tfrac{1}{2})\langle\sigma;\alpha|\bar{s}\rangle,$$
$$\sum_\beta \langle\dot{\sigma};\alpha|\Gamma_0|\sigma;\beta\rangle\langle\sigma;\beta|\bar{s}\rangle = -(s+\tfrac{1}{2})\langle\dot{\sigma};\alpha|\bar{s}\rangle. \tag{3.56}$$

Now

$$\langle\sigma;\alpha|\Gamma_0|\dot{\sigma};\beta\rangle = \sum_s \{\langle\sigma;\alpha|s\rangle(s+\tfrac{1}{2})\langle s|\dot{\sigma};\beta\rangle - \langle\sigma;\alpha|\bar{s}\rangle(s+\tfrac{1}{2})\langle\bar{s}|\dot{\sigma};\beta\rangle\}, \tag{3.57}$$

and similarly for

$$\langle\dot{\sigma};\alpha|\Gamma_0|\sigma;\beta\rangle;$$

and

$$0 = \langle\sigma;\alpha|\Gamma_0|\sigma;\beta\rangle = \sum_s \{\langle\sigma;\alpha|s\rangle(s+\tfrac{1}{2})\langle s|\sigma;\beta\rangle - \langle\sigma;\alpha|\bar{s}\rangle(s+\tfrac{1}{2})\langle\bar{s}|\sigma;\beta\rangle\}, \tag{3.58}$$

and similarly for

$$\langle\dot{\sigma};\alpha|\Gamma_0|\dot{\sigma};\beta\rangle.$$

It is worth mentioning that for the Dirac representation,[16] i.e., when

$$\sigma = (\tfrac{1}{2},\tfrac{3}{2}), \tag{3.59}$$

and

$$\dot{\sigma} = \sigma' = (\tfrac{1}{2}, -\tfrac{3}{2}), \tag{3.60}$$

the matrix Γ_0 is just the unit matrix in the mixed σ-$\dot{\sigma}$ representation (γ_0 in the Weyl representation) and Eqs. (3.55) and (3.56) reduce to

$$\langle\dot{\sigma};\alpha|s\rangle = \langle\sigma;\alpha|s\rangle,$$
$$\langle\dot{\sigma};\alpha|\bar{s}\rangle = -\langle\sigma;\alpha|\bar{s}\rangle. \tag{3.61}$$

IV. CAUSAL FIELDS

In this section we examine the possibility of constructing causal fields from the local fields defined by Eq. (2.28).

We say a field $\psi_\alpha^{(\sigma)}(x)$ is causal if, together with its Hermitian conjugate, it satisfies (anti-)commutation relations. That is, we examine whether it is possible that

$$[\psi_\alpha^{(\sigma)}(x), \psi_\beta^{(\sigma)}(y)^\dagger]_\pm = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{4.1}$$

The $\pm$ refers to anticommutation and commutation relations, respectively. We examine an expression of the type (4.1) because we have in mind theories of the very general kind considered by Weinberg.[17] They are those theories whose single-particle states are given in terms of creation and annihilation operators out of which the fields of (2.28) are constructed. A Hermitian interaction Lagrangian is constructed from these free fields and the S-matrix operator is assumed to be of the Dyson form involving a time-ordered product. For this operator to be Lorentz invariant it is necessary that the fields satisfy relations of the type (4.1). If in addition, we insist on reflection symmetry we must also introduce[18] operators $\psi_\alpha^{(\dot{\sigma})}(x)$ and show in addition to (4.1) that

$$[\psi_\alpha^{(\sigma)}(x), \psi_\beta^{(\dot{\sigma})}(y)^\dagger]_\pm = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{4.2}$$

From Eqs. (2.28), (2.27), (2.23), and (2.14) we have

$$[\psi_\alpha^{(\sigma)}(x), \psi_\beta^{(\sigma)}(y)^\dagger]_\pm = \int \sum_{\gamma,\delta,s,s_3} \{\langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|m_s,s,s_3\rangle|f(s)|^2 e^{-im_s u\cdot(x-y)}\langle m_s,s,s_3|\sigma;\delta\rangle$$
$$\times\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K}^\dagger)|\sigma;\beta\rangle \pm \langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|B|\bar{m}_s,s,s_3\rangle|g(s)|^2 e^{im_s u\cdot(x-y)}$$
$$\times\langle\bar{m}_s,s,s_3|B^{-1}|\sigma;\delta\rangle\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K}^\dagger)|\sigma;\beta\rangle\}(2\pi)\theta(u_0)\delta(u^2-1)d^4u/(2\pi)^4, \tag{4.3}$$

where the alternative sign on the right arises from the assumption of anticommutation or commutation relations for the single-particle operators.

In writing (4.3) we have already assumed that m will be some function of s (perhaps a constant). For finite-dimensional representations $K=-K^\dagger$. It has been shown by Weinberg[17] that for the representations $(\pm k_0, k_0+1)$, the expression (4.3) satisfies the condition (4.1) only if[19]

$$|f(s)|^2 = |g(s)|^2, \tag{4.4}$$

and we assume the usual connection between spin and statistics (the integer spin representations obey Bose statistics—commutation relations—and the half-odd-integral spin obey Fermi statistics—anticommutation relations). Here, we would like to examine what restrictions, if any, are forced by causality for unitary representations[20] σ. In these cases,

$$K = K^\dagger. \tag{4.5}$$

The field and the representations defined by Eq. (2.28) are quite general. In order to make our points,

[16] This representation is of course not unitary, but it does have the important property $\sigma \neq \dot{\sigma} = \sigma'$.

[17] S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[18] One must show that the reflection operation does in fact produce $\psi_\alpha^{(\dot{\sigma})}(x)$ from $\psi_\alpha^{(\sigma)}(x)$. The reflection operator R has the property that $Ra(m,u,s,s_3)R^{-1} = \pm(-1)^s a(m, -u, s, s_3)$, where by $-u$ we mean $(u_0, -\mathbf{u})$ and where the $\pm$ takes into account a possible intrinsic parity. Operating on Eq. (2.28) and making use of Eqs. (3.21), (3.29), and (3.32) one can deduce that $R\psi_\alpha^{(\sigma)}(x)R^{-1} = (-1)^j\psi_\alpha^{(\dot{\sigma})}(-x)$, where by $-x$ we mean $(x_0, -\mathbf{x})$. One also deduces that $\langle\sigma;\alpha|m,s,s_3\rangle = \pm\langle\dot{\sigma};\alpha|m,s,s_3\rangle$, where the $\pm$ refer to the intrinsic parity of the particles in the reducible representation $|m,s,s_3\rangle$.

[19] For these representations $(\pm k_0, k_0+1)$, there is one value of j and therefore of s. Thus the s dependence of m_s and $f(s)$ is irrelevant.

[20] It is clear that for nonunitarity infinite-dimensional representations we will have enough freedom to find fields which satisfy any kind of statistics for any spins.

it will only be necessary to examine certain special cases. To show that (4.3) is causal we must cast it in the form[21]

$$[\psi_\alpha{}^{(\sigma)}(x),\psi_\beta{}^{(\sigma)}(y)^\dagger]_\pm = f_{\alpha\beta}(\partial)\int (e^{-imu\cdot(x-y)} - e^{imu\cdot(x-y)})2\pi\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4}, \quad (4.6)$$

where

$$\partial \equiv \partial/\partial x_\mu, \quad (4.7)$$

and the crucial feature is the relative minus sign between the positive- and negative-frequency terms. In (4.3) such a minus sign occurs naturally for the case of Bose statistics directly from the commutation of the single-particle operators. The main problem will be to find some field for which a minus sign will appear for Fermi statistics.

Case (a)

$$|f(s)|^2 = |g(s)|^2 = \text{const}, \quad (4.8)$$

$$m_s = \text{const}. \quad (4.9)$$

This is the case discussed in II; namely, those fields out of which an index-invariant S matrix (consistent with unitarity) can be constructed.

Using (3.32) we may write

$$\sum_{s,s_3} \langle\sigma;\gamma|m,s,s_3\rangle\langle m,s,s_3|\sigma;\delta\rangle = \sum_{s,s_3}\langle\sigma;\gamma|B|m,s,s_3\rangle\langle m,s,s_3|B^{-1}|\sigma;\delta\rangle = \delta_{\gamma\delta}. \quad (4.10)$$

Using (4.5), (4.8), (4.9), and (4.10), we have immediately[22] that

$$[\psi_\alpha{}^{(\sigma)}(x),\psi_\beta{}^{(\sigma)}(y)^\dagger]_\pm = \int (e^{-imu\cdot(x-y)} \pm e^{imu\cdot(x-y)}) \times (2\pi)\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4}\delta_{\alpha\beta}. \quad (4.11)$$

Thus, the fields are causal only if we assume that all of the particles are bosons. This is true for any unitary representation σ (whether one containing integer or half-integer spins).

Case (b)

$$|f(s)|^2 = |g(s)|^2 = 1 \quad (4.12)$$

$$m_s \neq \text{const}. \quad (4.13)$$

In each term in (4.3) in the sum over s and s_3 we make a change of variable to

$$p = m_s u \quad (4.14)$$

and introduce a dummy variable m. Then

$$[\psi_\alpha(x),\psi_\beta(y)^\dagger]_\pm = \int \{\langle\sigma;\alpha|P(p,m)|\sigma;\beta\rangle e^{-ip\cdot(x-y)} \pm \langle\sigma;\alpha|\tilde{P}(p,m)|\sigma;\beta\rangle e^{ip\cdot(x-y)}\} \times(2\pi)\theta(p_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^4}\frac{dm}{m^2}, \quad (4.15)$$

where

$$P(p,m) = \sum_{\gamma,\delta,s,s_3}\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|m_s,s,s_3\rangle \times\delta(m-m_s)\langle m_s,s,s_3|\sigma;\delta\rangle\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K}), \quad (4.16)$$

and

$$\tilde{P}(p,m) = \sum_{\gamma,\delta,s,s_3}\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|B|m_s,s,s_3\rangle \times\delta(m-m_s)\langle m_s,s,s_3|B^{-1}|\sigma;\delta\rangle\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K}). \quad (4.17)$$

These operators $P(p,m)$ and $\tilde{P}(p,m)$ in unitary representations σ are projection operators[23] and therefore it is impossible for the relation

$$P(p_\mu,m) = -\tilde{P}(-p_\mu,\ m) \quad (4.18)$$

to hold. This is the relation that is required in order to cast (4.15) into the form (4.6) for the case of Fermi statistics. Again, as in case (a), causality requires that all of our particles must be bosons, if σ labels a unitary representation.

Case (c). Gelfand Equations

Gelfand[11] has shown that if we introduce, in addition to $\psi_\alpha{}^{(\sigma)}(x)$, the field $\psi_\alpha{}^{(\sigma')}(x)$, where σ' is given by Eqs. (3.37) and (3.38), one can find an equation of motion connecting the two fields. For simplicity let us look at those representations given by Eqs. (3.39), (3.40), or (3.41). In these cases $\sigma' = \dot{\sigma}$. If, for the self-conjugate case (3.39) or (3.40), we restrict the rest states to satisfy (3.47) and thus (3.48), one can check, using the boost property (3.45), that if

$$m_s = \kappa/(s+\tfrac{1}{2}), \quad (4.19)$$

where κ is a constant and

$$g(s) = 0, \quad (4.20)$$

then

$$\langle\sigma;\alpha|i\Gamma\partial|\sigma;\beta\rangle\psi_\beta{}^{(\sigma)}(x) = \kappa\psi_\alpha{}^{(\sigma)}(x), \quad (4.21)$$

where

$$\Gamma\partial = \Gamma_\mu\partial^\mu. \quad (4.22)$$

Since $g(s)=0$, there are no antiparticles.

[21] This is not sufficient, since only if $f_{\alpha\beta}(\partial)$ is a polynomial in ∂ can we be sure that (4.6) vanishes for spacelike distances.

[22] We have put $|f(s)|^2 = |g(s)|^2 = 1$ and $m_s = m$.

[23] These operators must be divided by a normalization factor to make them projection operators. Thus if we define $Q = [1/\delta(0)] \times\sum_{s,s_3}|m,s,s_3\rangle\,\delta(m-m_s)\langle m,s,s_3|$, then indeed $Q^2 = Q$.

For the non-self-conjugate representations one finds

$$i\langle\sigma;\alpha|\Gamma\partial|\dot\sigma;\beta\rangle\psi_\beta^{(\dot\sigma)}(x)=\kappa\psi_\alpha^{(\sigma)}(x)\,,\qquad i\langle\dot\sigma;\alpha|\Gamma\partial|\sigma;\beta\rangle\psi_\beta^{(\sigma)}(x)=\kappa\psi_\alpha^{(\dot\sigma)}(x)\,, \tag{4.23}$$

provided that the states $|m,s,s_3\rangle$ and $|\bar m,s,s_3\rangle$ satisfy Eqs. (3.53) and (3.54) and thus (3.55) and (3.56), and again if m_s is restricted by (4.19). In this case, we do have antiparticles and their parities must be opposite. (See Ref. 18.) We now go on to discuss the causality of these fields. For the self-conjugate case it is clear that since there are no antiparticles we cannot have a causal field no matter what statistics we assume for the particles. For the non-self-conjugate representations, if we take

$$|f(s)|^2=|g(s)|^2=\text{const}\,, \tag{4.24}$$

we have a special example of case (b) which implies only Bose statistics is consistent with causality. For other choices of $f(s)$ and $g(s)$ we cannot discuss directly the causality of the (anti-)commutators. However, in Refs. 11 and 12 there is a discussion of a problem related to the spin-statistics question. Out of those fields which satisfy "Gelfand equations," these authors have constructed "energy" and "charge" functions and have discussed the positive definiteness of these quantities for all representations of the Lorentz group. If we assume[24] that Bose systems must have positive definite energy but nonpositive definite charge and Fermi systems have positive definite charge but nonpositive definite energy, then only Bose systems exist for unitary representations which satisfy Gelfand equations. (See Naimark,[12] p. 404.)

Case (d). An Example of a Fermi Field

Let us assume that σ is one of the self-conjugate representations $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$. These are the cases for which

$$\sigma=\dot\sigma=\sigma'\,. \tag{4.25}$$

In the expansions (2.28), (2.27), and (2.23) for $\psi_\alpha^{(\sigma)}(x)$, let

$$|f(s)|^2=|g(s)|^2=s+\tfrac{1}{2}\,, \tag{4.26}$$

and

$$m_s=m=\text{const}\,. \tag{4.27}$$

This gives

$$\begin{aligned}[\psi_\alpha^{(\sigma)}(x),\psi_\beta^{(\sigma)}(y)^\dagger]_\pm=\sum_{s,s_3}\sum_{\gamma,\delta}\{&\langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|m,s,s_3\rangle(s+\tfrac{1}{2})\langle m,s,s_3|\sigma;\delta\rangle\\ &\times\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\beta\rangle e^{-imu\cdot(x-y)}\pm\langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|B|\bar m,s,s_3\rangle\\ &\times(s+\tfrac{1}{2})\langle\bar m,s,s_3|B^{-1}|\sigma;\delta\rangle\langle\sigma;\delta|\exp(+i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\beta\rangle e^{imu\cdot(x-y)}\}(2\pi)\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4}\,. \end{aligned}\tag{4.28}$$

Now, using (3.49) and (3.45) we have

$$[\psi_\alpha^{(\sigma)}(x),\psi_\beta^{(\sigma)}(y)^\dagger]_\pm=\int\{\langle\sigma;\alpha|\Gamma u|\sigma;\beta\rangle e^{-imu\cdot(x-y)}\pm\langle\sigma;\alpha|\Gamma u|\sigma;\beta\rangle e^{imu\cdot(x-y)}\}(2\pi)\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4} \tag{4.29}$$

$$=\frac{1}{m}\langle\sigma;\alpha|i\Gamma\partial|\sigma;\beta\rangle\int(e^{-imu\cdot(x-y)}\mp e^{imu\cdot(x-y)})(2\pi)\theta(u_0)\delta(u^2-1)\frac{d^4u}{(2\pi)^4}\,. \tag{4.30}$$

Accordingly, we will have a causal field if we assume that the particles obey Fermi statistics. Since the representation σ is self-conjugate, we can choose the intrinsic parity of the antiparticle state to be either the same or opposite that of the particle [see Eq. (3.30)]. We have in this category towers of both integer $[(0,\frac{1}{2})]$ and half-odd-integer spin $[(\frac{1}{2},0)]$. For $|f(s)|^2=|g(s)|^2=1$ these representations are included under case (a). Hence we have explicit examples of fields corresponding to unitary towers which can be constructed to satisfy either Bose or Fermi statistics for both integer and half-integer spin. This new freedom has arisen very directly from the inclusion of nontrivial weight factors in the field expansion, and the properties of the Gelfand matrices.

Notice that the Fermi fields of this type do not satisfy any Gelfand equation, since in the first place all masses are equal. Even if we introduce a Gelfand mass spectrum (4.19), then the positive-frequency part of the field satisfies (4.21), while the negative-frequency part (referring to the particles created by $b^\dagger$) satisfies

$$(i\Gamma\partial+\kappa)\psi^{(-)}(x)=0\,. \tag{4.31}$$

A consequence of this is that for both Fermi and Bose statistics the only conserved current which can be formed is

$$J_\mu=i\psi^\dagger(\partial_\mu\psi)-i(\partial_\mu\psi^\dagger)\psi\,. \tag{4.32}$$

For Bose statistics this leads to a "charge" operator very similar to that for a single component (non-Hermitian) scalar field with particles (created by $a^\dagger$)

[24] These energy and charge functions are c-number functions constructed from functions which satisfy the Gelfand equations. For the finite-dimensional representations they are the usual quantities constructed by means of a Lagrangian. For example, for the Dirac representation the "charge" is $\int\psi^\dagger\psi d^3x$ and the energy is $i\int(\psi^\dagger\partial_0\psi-\partial_0\psi^\dagger\psi)d^3x$.

and antiparticles (created by $b^{\dagger}$) of opposite charge. However, for Fermi statistics the charge is proportional to $(2s+1)$ for particles of spin s, and of the same magnitude and sign for the antiparticles.

Let us finally consider the implications of these unitary Fermi fields for the causality theorem (theorem II of II). This is, in fact, still valid because we now show that a theory constructed from such a field will not simultaneously satisfy index invariance and give rise to a unitary S matrix. Thus the crucial term

$$\sum_{s,s_3} \langle\sigma;\gamma|m,s,s_3\rangle(s+\tfrac{1}{2})\langle m,s,s_3|\sigma;\delta\rangle, \qquad (4.33)$$

which occurs in (4.28) and gives rise to the odd function of u, Γu, in (4.29), will also occur in the unitarity sum in II. Thus, Eq. (3.7) of II will in this case be replaced by

$$\sum_{\gamma,\delta,s,s_3} \langle\sigma;\alpha|\exp(-i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\gamma\rangle\langle\sigma;\gamma|m,s,s_3\rangle(s+\tfrac{1}{2}) \times\langle m,s,s_3|\sigma;\delta\rangle\langle\sigma;\delta|\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{K})|\sigma;\beta\rangle =\langle\sigma;\alpha|\Gamma u|\sigma;\beta\rangle\neq\delta_{\alpha\beta}, \qquad (4.34)$$

and thus the unitarity condition destroys the index invariance.

V. CONCLUSIONS

We have constructed, out of free-particle annihilation and creation operators, local fields which transform as unitary representations of the homogeneous Lorentz group L. The most general field need not satisfy any equation of motion. In fact, the field may contain particle operators associated with different spins and different masses. For fields which are irreducible representations, each spin value associated with a particle can occur no more than once. When we examine the possibility of constructing causal fields [i.e., those that obey causal (anti-)commutation relations] which are unitary representations of L, we find that there is no connection between spin and statistics. If we consider those unitary fields which satisfy Gelfand equations or whose spin-dependent weight factors $f(s)$ and $g(s)$ are constants, then we find that we can construct only Bose fields. However, for particular self-conjugate representations, $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$, and for a suitable choice of $f(s)$ and $g(s)$ one can construct a Fermi field.

Our interest in these generalized fields was aroused originally by the problem of constructing index-invariant theories [such as $\tilde{U}(12)$ and $SL(6,c)$] which are both causal and consistent with unitarity. We have shown that index-invariant theories cannot be constructed using the unitary Fermi fields developed above. Our results thus do not contradict the unitarity and causality theorems stated in II, although these were established for less general fields than those considered above.

It may well be that these very general fields bear little relation to the physical world. They should at least provide axiomatic field theorists with much food for thought. In particular, the results established above, using the heuristic methods of Dirac, Wigner, Fock, Pauli, and Heisenberg may no doubt be rewritten with greatly increased rigor using a more conventional mathematical formalism. In addition, one must discover whether a "*CPT* theorem" can be proved if theories are constructed out of these unitary fields rather than the conventional finite-dimensional fields.

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