Measurement of Fourth-Order Coherence Functions

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In this paper we outline a method which allows one, in principle, to measure $\langle E_i(\mathbf{x}_1,t)E_j(\mathbf{x}_2,t)E_k^*(\mathbf{x}_3,t) \times E_l^*(\mathbf{x}_4,t)\rangle$ (in general, $\mathbf{x}_1\neq\mathbf{x}_2\neq\mathbf{x}_3\neq\mathbf{x}_4$), for stationary quasimonochromatic light propagating principally in the forward direction. The procedure makes use of the properties of nonlinear dielectrics in a sequence of interference experiments. No correlation of photoelectron currents is used, and as a special case the contracted moment $\langle |E_i(\mathbf{x}_1,t)|^2|E_j(\mathbf{x}_2,t)|^2\rangle$ measured by Hanbury Brown and Twiss could be obtained in this manner. It is shown that it should be possible to measure the full fourth-order coherence function in the laboratory for laser light, using presently obtainable intensity levels.

1. INTRODUCTION

THE coherence theory of electromagnetic radiation was originally concerned with the classical second-order coherence function $\mathcal{E}_{ij}(\mathbf{x}_1,\mathbf{x}_2,\tau)=\langle E_i(\mathbf{x}_1, t+\tau)E_j^* \times(\mathbf{x}_2t)\rangle$ (see Wolf).[1] [Here $E_i(\mathbf{x},t)$ is the electric field at point $\mathbf{x}$, the bracket denotes a time average, and E_i is an analytic signal.] This theory has been extended in recent years in two principal ways. The classical theory been formulated to include study of nth-order time-averaged coherence functions and has been generalized to treat nonstationarity effects by replacing the time average by an ensemble average. (See Beran and Parrent[2] and Mandel and Wolf[3] for original references.) It also has been pointed out that conceptually the classical statistical problem may be generally treated by specifying the probability density functional associated with $E_i(\mathbf{x},t)$ (see Beran and Parrent.)[2] The second development of coherence theory has been to make it an explicitly quantum theory by defining all nth-order coherence functions in terms of operators associated with $E_i(\mathbf{x},t)$. This has been done principally by Glauber.[4a,b,c] Other work in this direction is well summarized in Glauber[4d] and Mandel and Wolf.[3]

While the formal theoretical aspects of coherence theory were readily generalized, measurement of theoretically defined coherence functions has lagged. For most coherence functions that have been defined not even conceptual measurements of general applicability have been outlined. In fact, the measurement of the second-order coherence function $\mathcal{E}_{ij}(\mathbf{x}_1,\mathbf{x}_2,\tau)$ has only been shown to be possible under very restricted conditions. The Hanbury Brown and Twiss experiment (see Hanbury Brown and Twiss)[5] which ideally measures $\langle|E_i(\mathbf{x}_1,t+\tau)|^2|E_j(\mathbf{x}_2,t)|^2\rangle$, is also very restricted in the class of fields that may be studied.

From a classical point of view there is, of course, no conceptual difficulty in measuring any order coherence function. One simply assumes that $E_i(\mathbf{x},t)$ may be measured at all positions and times for all members of an ensemble and any averages desired may then be calculated. Quantum mechanically, we know, however, that except for highly degenerate signals, this type of point measurement has no meaning even in an approximate sense. If one seeks methods of measurement appropriate to more general radiation fields, it is not the measurement of $E_i(\mathbf{x},t)$ on which we must focus our attention but either on measurement of local spatial and time averages of $E_i(\mathbf{x},t)$ or on a more direct measurement of the coherence functions themselves. In either case considerable care must be taken in interpreting the measurement and a detailed specification of the measurement procedure must be given for every theoretically defined coherence function one choses to measure. Experience has shown that it is a nontrivial problem to specify a measurement procedure even conceptually and that most procedures are suitable only under very restricted conditions.

In this paper we will focus our attention on a measurement procedure for the stationary fourth-order coherence function[6]

$$L_{ijkl}^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\tau_2,\tau_3,\tau_4) =\langle E_i(\mathbf{x}_1,t)E_j(\mathbf{x}_2, t+\tau_2)E_k^*(\mathbf{x}_3, t+\tau_3)E_l^*(\mathbf{x}_4, t+\tau_4)\rangle$$

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[1] E. Wolf, Proc. Royal Soc. (London), **230**, 246 (1955).

[2] M. Beran and G. B. Parrent, Jr., *Theory of Partial Coherence* (Prenctice-Hall, Inc., Englewood Cliffs, New Jersey, 1964), Chap. 13.

[3] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

[4] R. Glauber, (a) Phys. Rev. Letters **10**, 8 (1963); (b) Phys. Rev. **130**, 2529 (1963); (c) **131**, 2766 (1963); (d) in *Quantum Optics and Electronics: The 1964 Les Houches Lectures*, edited by C. DeWitt *et al.* (Gordon and Breach Science Publishers, Inc., New York, 1965).

[5] R. Hanbury Brown and R. Twiss, Proc. Royal Soc. (London) **242**, 300 (1957); **243**, 291 (1957).

[6] The notation

$$\Gamma_{ijkl}^{(2,2)}(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2;\mathbf{x}_3,t_3;\mathbf{x}_4,t_4) =\langle E_i^*(\mathbf{x}_1,t_1)E_j^*(\mathbf{x}_2,t_2)E_k(\mathbf{x}_3,t_3)E_l(\mathbf{x}_4,t_4)\rangle$$

is also used in the literature.

using nonlinear dielectrics in a sequence of interference experiments. As in the second-order case, only intensity measurements will be necessary. Since local instantaneous measurement of the field is not required, a classical analysis of the measurement procedure will be given. A full quantum-mechanical analysis would perhaps be more satisfactory, but as the reader will see, this would entail the solution of the many-body interaction of field and matter in a nonlinear dielectric; a problem that has not yet been treated in the literature.[7]

The measurement procedure for $L_{ijkl}{}^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\tau_2,\tau_3,\tau_4)$ will be limited to quasimonochromatic fields that have a narrow angular spread about a particular direction. As a result of the small angular spread assumption, the vector nature of the electric field will not be considered explicitly and $E_i(\mathbf{x},t)$ will be replaced by the scalar function $V(\mathbf{x},t)$, Further, since the field will be assumed to be quasimonochromatic, we will set $\tau_2=\tau_3=\tau_4=0$. From the analysis it should be clear how one may extend the procedure and measure higher order coherence functions subject to the same restrictions.

In the final section of this paper we will show that it should be possible to measure the fourth-order coherence function in the laboratory for laser light using presently obtainable intensity levels. As a special case, this measurement could be used to measure the contracted moment considered by Hanbury Brown and Twiss.

2. PROCEDURE FOR MEASUREMENT OF FOURTH-ORDER COHERENCE FUNCTIONS

2.1 Definitions and the Scalar Approximation

In general we would like to consider the coherence function

$$L_{ijkl}{}^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\tau_2,\tau_3,\tau_4) =\langle E_i(\mathbf{x}_1,t)E_j(\mathbf{x}_2,\,t+\tau_2)E_k{}^*(\mathbf{x}_3,\,t+\tau_3)E_l{}^*(\mathbf{x}_4,\,t+\tau_4)\rangle.$$

The measurement procedure we will describe, however, is limited to fields propagating with a narrow angular spread about a principal direction. In this case the divergence condition associated with Maxwell's equations is satisfied to a good approximation, and we need consider only polarization effects in a plane perpendicular to the mean propagation direction. To simplify the discussion further and with no real loss of generality, we will consider the light polarized in a single direction. (By the use of rotators, polarization effects may easily by included.) We thus may replace the vector function $E_i(\mathbf{x},t)$ by the scalar function $V(\mathbf{x},t)$ and consider the coherence function

$$L^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\;\tau_2,\tau_3,\tau_4) =\langle V(\mathbf{x}_1,t)V(\mathbf{x}_2,\,t+\tau_2)V^*(\mathbf{x}_3,\,t+\tau_3)V^*(\mathbf{x}_4,\,t+\tau_4)\rangle.$$

[7] We are presently studying the Heisenberg equations of motion for a system of many-level atoms coupled to many modes of the electromagnetic field and hope to present a quantum analysis for this model in a future paper.

In addition to the restriction to fields with narrow angular spread, we will also restrict our attention to fields that are quasimonochromatic and hence may choose $\tau_2=\tau_3=\tau_4=0$. In the quasimonochromatic case the only important fourth-order coherence function is, as we point out in the next section, L^{42}. For completeness, however, we note that we may define coherence functions of the form $L^{4p}(p\leq 4)$, where p denotes the number of terms that are conjugated. For example

$$L^{40}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\;\tau_2,\tau_3,\tau_4) =\langle V(\mathbf{x}_1,t)V(\mathbf{x}_2,\,t+\tau_2)V(\mathbf{x}_3,\,t+\tau_3)V(\mathbf{x}_4,\,t+\tau_4)\rangle.$$

2.2 Outline of Experiment

To measure the four-point coherence function for radiation with small angular spread we set up a screen with four small pinholes. (Refer to Fig. 1.) As we just stated we call the polarized fields emerging from these holes, $V(\mathbf{x}_1,t),\,\cdots,\,V(\mathbf{x}_4,t)$; $V(\mathbf{x}_1,t)$ is an analytic signal. The holes at P_i in screen A are taken to have diameters large compared to the mean wavelength, $\bar{\lambda}=(c/\bar{\nu})$, but to be small enough so that the field does not vary significantly over the hole.

The fields are collimated by lenses when they emerge from the holes and by the use of mirrors V_1 and V_2 $[V_i\equiv V(\mathbf{x}_i,t)]$ are combined to yield $\alpha_1V_1+\alpha_2V_2$; and V_3 and V_4 are combined to yield $\alpha_3V_3+\alpha_4V_4$. ($\alpha_1V_1+\alpha_3V_3$ and $\alpha_2V_2+\alpha_4V_4$ could, for example, have been formed instead, of course, and we shall discuss this question of different pairing later.) $\alpha_1V_1+\alpha_2V_2$ and $\alpha_3V_3+\alpha_4V_4$ are then passed through nonlinear dielectrics.

V_i is taken in our analysis to be a quasimonochromatic signal with mean frequency $\bar{\nu}$. The nonlinear dielectric is chosen to produce from the quasimonochromatic signal $\alpha_1V_1+\alpha_2V_2$ (or V_3+V_4) a weak quasimonochromatic signal with mean frequency $2\bar{\nu}$. It also passes the original signal. The filters placed before the holes P_5 and P_6 filter out the original signals with mean frequency $\bar{\nu}$. The final stage of the procedure is to allow the various quasimonochromatic signals with mean frequency $2\bar{\nu}$ to pass through the holes P_5 and P_6 in ap-

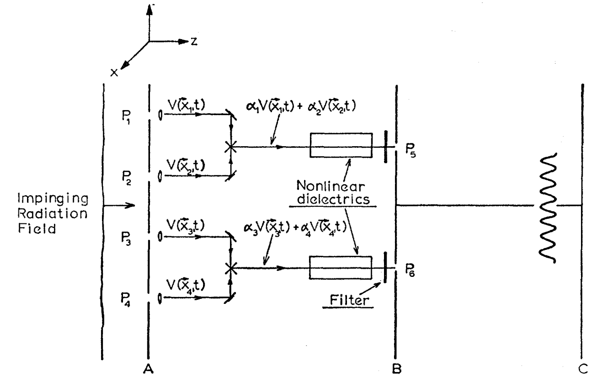

FIG. 1. Schematic diagram of experiment.

propriate combinations and produce an interference pattern on screen C. From the interference pattern on C, we determine the four-point coherence function, L^{42}

The coherence function we wish to measure is

$$L^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\,0,0,0) = \langle V(\mathbf{x}_1,t)V(\mathbf{x}_2,t)V^*(\mathbf{x}_3,t)V^*(\mathbf{x}_4,t)\rangle. \quad (1)$$

For a quasimonochromatic signal we may write

$$V(\mathbf{x},t)=\exp(-2\pi i\bar{\nu}t+i\bar{k}z)U(\mathbf{x},t)\,, \quad (2)$$

where $U(\mathbf{x},t)$ varies slowly in time, in times of the order $1/\bar{\nu}$. In this case $L^{4p}\approx 0$ if $p=1$ or 3. The quantities L^{40} and L^{44} may be shown to equal zero in general (see Beran and Corson).[8] Thus, only L^{42} need be considered. Further $\tau_2=\tau_3=\tau_4$ may be set equal to zero since their effect is only to produce oscillating terms of the form $\exp(-2\pi i\bar{\nu}\tau_i)$ if τ_i is chosen such that $\tau_i\ll(1/\Delta\nu)$. Here ($\Delta\nu$ is the characteristic frequency spread of the quasimonochromatic signal.) If the condition $\tau_i\ll(1/\Delta\nu)$, and an associated condition requiring that path length differences between $\mathbf{x}_i$ and $\mathbf{x}_j$ be much less than $c/\Delta\nu$, are not met, the quasimonochromatic approximation loses its utility. We consider here only such cases where it is appropriate to set $\tau_2=\tau_3=\tau_4=0$.

If Eq. (2) is substituted into Eq. (1), we have

$$L^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\,0,0,0) = \langle U(\mathbf{x}_1,t)U(\mathbf{x}_2,t)U^*(\mathbf{x}_3,t)U^*(\mathbf{x}_4,t)\rangle. \quad (3)$$

2.3 Propagation in a Nonlinear Dielectric

Consider now what happens when a signal passes through the nonlinear dielectric. The wave equation governing the propagation of polarized radiation within a dielectric may be written in the phenomenological form

$$\nabla^2 V^r=\frac{1}{C^2}\frac{\partial^2 V^r}{\partial t^2}+\frac{4\pi\partial^2 P}{C^2\partial t^2}\,, \quad (4)$$

where P is the polarization induced by the incident field. The polarization is made up of two parts for the dielectric we wish to consider; a linear part denoted by P^L, and a nonlinear part denoted by P^{NL}. (See Bloembergen.[9]) To posit a relation between P and V^r, we first take the Fourier transform of both sides of Eq. (4) We have then[10]

$$\nabla^2\hat{V}^r+k^2\hat{V}^r=-4\pi k^2\hat{P}\,, \quad (5)$$

where

$$\hat{V}^r(\nu)=\int_{-\infty}^{\infty}V^r(t)e^{2\pi i\nu t}dt\,,$$

$$\hat{P}(\nu)=\int_{-\infty}^{\infty}P(t)e^{2\pi i\nu t}dt\,,$$

and $k=2\pi\nu/c$. Since $V^r(t)$ is real, we have $\hat{V}^{*r}(\nu)=\hat{V}^r(-\nu)$. We now assume

$$\hat{P}(\nu)=\chi^L(\nu)\hat{V}^r(\nu)+\sum_{\nu'+\nu''=\nu}\chi^{NL}(\nu',\nu'';\nu'+\nu''=\nu)\hat{V}^r(\nu')\hat{V}^r(\nu'') \quad (6)$$
$$=\hat{P}^L+\hat{P}^{NL}.$$

We will consider only nonlinear dielectrics where χ^L and χ^{NL} are real. The summation is over all combinations such that $\nu'+\nu''=\nu$. Thus Eq. (5) becomes

$$\nabla^2\hat{V}^r(\nu)+k_D{}^2(\nu)\hat{V}^r(\nu)=-4\pi\sum_{\nu'+\nu''+\nu}k^2 \times\chi^{NL}(\nu',\nu'';\nu'+\nu''=\nu)\hat{V}^r(\nu')\hat{V}^r(\nu'')\,, \quad (7)$$

where

$$k_D{}^2(\nu)=k^2(\nu)(1+4\pi\chi^L(\nu)).$$

For a nonlinear dielectric, the term on the right-hand side of Eq. (7) is generally small compared to $k_D{}^2(\nu)\times\hat{V}^r(\nu)$. The effect of the nonlinearity may thus be found using a perturbation procedure. Neglecting the nonlinear term we find that the plane-wave solution of Eq. (7) in an infinite medium is

$$\hat{V}_0{}^r(\nu)=W(\nu)e^{ik_D(\nu)z}\,, \quad (8)$$

where $W(\nu)$ is an arbitrary complex function chosen so that $W^*(\nu)=W(-\nu)$. We note that $k_D(\nu)=-k_D(-\nu)$.

The first-order perturbation is accomplished by replacing $\hat{V}^r(\nu)$ in the right-hand side of Eq. (7) by $\hat{V}_0{}^r(\nu)$. This gives

$$\nabla^2\hat{V}^r(\nu)+k_D{}^2(\nu)\hat{V}^r(\nu)=-4\pi\sum_{\nu'+\nu''=\nu}k^2 \times\chi^{NL}(\nu',\nu'';\nu'+\nu''=\nu)\hat{V}_0{}^r(\nu')\hat{V}_0{}^r(\nu''). \quad (9)$$

The particular solution of Eq. (9) is

$$\hat{V}_p{}^r(\nu)=-4\pi\sum_{\nu'+\nu''=\nu}k^2\chi^{NL}(\nu',\nu'';\nu'+\nu''=\nu) \times\frac{W(\nu')W(\nu'')e^{i[k_D(\nu')+k_D(\nu'')]z}}{k_D{}^2(\nu'+\nu'')-(k_D(\nu')+k_D(\nu''))^2}. \quad (10)$$

Taking the inverse Fourier transform of $\hat{V}^r(\nu)$ we have, including the incident wave,

$$V^r(t)=\int_{-\infty}^{\infty}\hat{V}^r(\nu)e^{-2\pi i\nu t}d\nu$$
$$=\int_{-\infty}^{\infty}W(\nu)e^{ik_D(\nu)z-2\pi i\nu t}d\nu$$
$$-4\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k^2\chi^{NL}(\nu',\nu'',\nu'+\nu''=\nu)W(\nu')W(\nu'')}{k_D{}^2(\nu'+\nu'')-(k_D(\nu')+k_D(\nu''))^2}$$
$$\times\exp\{i[k_D(\nu')+k_D(\nu'')]z-2\pi i\nu t\}d\nu'd\nu''\,, \quad (11)$$

[8] M. Beran and P. Corson, J. Math. Phys. **6**, 271 (1965).

[9] N. Bloembergen, *Nonlinear Optics* (W. A. Benjamin, Inc., New York, 1965).

[10] This entire formalism may be extended to use the analytic signal representation and the interaction problem may be treated in the language of coherence theory. This is done by Beran and DeVelis, J. Opt. Soc. Am. **57**, 186 (1967).

where we have now changed

$$\int_{-\infty}^{\infty} d\nu \sum_{\nu'+\nu''=\nu}$$

to a double integral over ν' and ν''.

In the quasimonochromatic approximation, we demand that $W(\nu)$ be essentially nonzero in the following regions:

$$\bar{\nu}\pm\Delta\nu\,, \quad -\bar{\nu}\pm\Delta\nu\,, \tag{12}$$

where $|\Delta\nu/\bar{\nu}|\ll 1$. We now assume that over the range $\Delta\nu$,

$$\frac{\chi^{NL}(\nu',\,\nu'';\,\nu'+\nu'')}{k_D{}^2(\nu'+\nu'')-(k_D(\nu')+k_D(\nu''))^2} \approx \frac{\chi^{NL}(\bar{\nu},\bar{\nu};\,2\bar{\nu})}{k_D{}^2(2\bar{\nu})-(2k_D(\bar{\nu}))^2} \equiv \frac{B(\bar{\nu},\bar{\nu};\,2\bar{\nu})}{4\pi\bar{k}^2}\,, \tag{13}$$

and

$$\chi^L(\nu)\approx\chi^L(\bar{\nu}) \tag{14}$$

when in Eq. (13) ν' and ν'' have the same sign. With these assumptions, we may write

$$V^r(t)=V_0{}^r(t)+V^{rn0}(t)+V^{rn}(t)\,, \tag{15}$$

where $V_0{}^r(t)$ is a term with frequencies in the neighborhood of $\bar{\nu}$, $V^{rn0}(t)$ is a term with frequencies in the neighborhood of 0, and $V^{rn}(t)$ is a term with frequencies in the neighborhood of $2\bar{\nu}$. The form of $V^{rn}(t)$ is

$$V^{rn}(t)\approx -B(\bar{\nu},\bar{\nu};\,2\bar{\nu})2\,\mathrm{Re} \times\{\exp[i2(k_D(\bar{\nu})z-2\pi\bar{\nu}t)](U(t))^2\}\,, \tag{16}$$

where

$$U(t)=\int_0^{\infty} W(\bar{\nu}+\Delta\nu)e^{i(\Delta k_D(\bar{\nu})z-2\pi\Delta\nu t)}d(\Delta\nu)\,,$$

$$\Delta k_D(\bar{\nu})\equiv\Delta k[1+4\pi\chi^L(\bar{\nu})]^{1/2}.$$

To the approximation we have considered we may assume that a quasimonochromatic signal of mean ferquency $\bar{\nu}$,

$$V_0{}^r(t)=2\,\mathrm{Re}[e^{i[k_D(\bar{\nu})z-2\pi\bar{\nu}t]}U(t)]\,, \tag{17}$$

propagating in an infinite media generates a quasimonochromatic signal of average frequency $2\bar{\nu}$ and a spectrum of waves with average frequency near zero.

$V^{rn}(t)$ is not, however, the most general solution since Eq. (10) is only the particular solution for Eq. (9). The complete solution for frequencies in the neighborhood of $2\bar{\nu}$ requires an additional term of the form

$$\hat{V}_H{}^r(\nu)=A(\nu)e^{ik_D(\nu)z}\,, \tag{18}$$

where $A(\nu)$ must be determined by the boundary conditions. For the experiment we wish to perform, we assume that the nonlinear dielectric is an infinite slab bounded by two planes $z=z_1$ and $z=z_2$; $z_1<z_2$. The boundary condition is that $V(t)$ must be continuous at the plane $z=z_1$ (see Bloembergen).[9] This condition demands that there be a reflected wave at $z=z_1$. If for convenience we set $z_1=0$ and consider the case when

$$1-4k_D{}^2(\bar{\nu})/k_D{}^2(2\bar{\nu})\ll 1\,,$$

we may meet this condition if

$$A(\nu)=\sum_{\nu'+\nu''=\nu}\frac{4\pi^2\chi^{NL}(\nu',\,\nu'';\,\nu'+\nu''=\nu)W(\nu')W(\nu'')}{k_D{}^2(\nu'+\nu'')-[k_D(\nu')+k_D(\nu'')]^2}\,. \tag{19}$$

If $\hat{V}_H{}^r(\nu)+\hat{V}_P{}^r(\nu)$ replaces $\hat{V}_P{}^r(\nu)$ in Eq. (11), we have for $V^{rn}(t)$, replacing Eq. (16)

$$V^{rn}(t)=-B(\bar{\nu},\bar{\nu};\,2\bar{\nu})2\,\mathrm{Re}\{[1-e^{iz[k_D(2\bar{\nu})-2k_D(\bar{\nu})]}] \times e^{i2(k_D(\bar{\nu})z-2\pi\bar{\nu}t)}(U(t))^2\}\,. \tag{20}$$

To maximize $V^{rn}(t)$ we choose z_2 such that $[k_D(2\bar{\nu})-2k_D(\bar{\nu})]z_2=\pi$.[11] Assuming this is done, we have finally

$$V^{rn}(t)=-B(\bar{\nu},\bar{\nu};\,2\bar{\nu})4\mathrm{Re}[e^{i2[k_D(\bar{\nu})z_2-2\pi\bar{\nu}t]}(U(t))^2]\,. \tag{21}$$

For $z>z_2$ we have thus (assuming negligible reflection of the primary wave at z_2)

$$V(t)=a_1e^{i(\bar{k}z-2\pi\ t)}U(t)+\text{low-frequency terms} +a_2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}[U(t)]^2\,, \tag{22}$$

where

$$U(t)=\int_0^{\infty} W(\bar{\nu}+\Delta\nu)e^{i(\Delta kz-2\pi\Delta\nu t)}d(\Delta\nu)\,,$$

and

a_1 and a_2 are real constants. In this quasimonochromatic approximation $V(t)$ is the analytic signal associated with the real field when $z>z_2$. We shall discuss the order of magnitude of a_1 and a_2 in the last section of this paper .

2.4 Measurement of $L^{42}(x_1,x_2,x_3,x_4;\,0,0,0)$

Referring to Fig. 1, we see that the signal $\alpha_1V_1+\alpha_2V_2$ enters a nonlinear dielectric. The signal leaving the dielectric is

$$V(t)=a_1e^{i(\bar{k}z-2\pi\bar{\nu}t)}(\alpha_1U_1(t)+\alpha_2U_2(t)) +\text{low frequency terms} +a_2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}[\alpha_1U_1(t)+\alpha_2U_2(t)]^2\,. \tag{23}$$

The signal then passes through a filter which passes only the $2\bar{\nu}$ terms. The signal leaving P_5 is

$$V(P_5,t)=a_2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}[\alpha_1U_1(t)+\alpha_2U_2(t)]^2\,. \tag{24}$$

Similarly, the signal leaving P_6 is

$$V(P_6,t)=a_2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}[\alpha_3U_3(t)+\alpha_4U_4(t)]^2\,. \tag{25}$$

The Young's interference experiment performed between the planes B and C allow one to calculate, in principle, the complex quantity (see Beran and

[11] *Footnote added in proof*. In Beran and DeVelis [Ref. 10 Eq. (24)], this condition is incorrectly stated. It should read as presented here. Hence in Eq. (25) of that paper $\alpha=4$.

Parrent).[2]

$$
\begin{aligned}
\Gamma(P_5,P_6)=&\langle V(P_5,t)V^*(P_6,t)\rangle\\
=&a_2^2[\alpha_1^2\alpha_3^2\langle U_1U_3^*U_1U_3^*\rangle\\
&+2\alpha_1^2\alpha_3\alpha_4\langle U_1U_3^*U_1U_4^*\rangle\\
&+\alpha_1^2\alpha_4^2\langle U_1U_4^*U_1U_4^*\rangle\\
&+2\alpha_1\alpha_2\alpha_3^2\langle U_1U_3^*U_2U_3^*\rangle\\
&+4\alpha_1\alpha_2\alpha_3\alpha_4\langle U_1U_3^*U_2U_4^*\rangle\\
&+2\alpha_1\alpha_2\alpha_4^2\langle U_1U_4^*U_2U_4^*\rangle\\
&+\alpha_2^2\alpha_3^2\langle U_2U_3^*U_2U_3^*\rangle\\
&+2\alpha_2^2\alpha_3\alpha_4\langle U_2U_3^*U_2U_4^*\rangle\\
&+\alpha_2^2\alpha_4^2\langle U_2U_4^*U_2U_4^*\rangle]. \qquad (26)
\end{aligned}
$$

If the real and imaginary parts of $\Gamma(P_5,P_6)$ can be measured then by appropriately choosing the α_i any of the functions $\langle U_iU_j^*U_kU_l^*\rangle$ may be determined if nine (or less) experiments are performed. In particular, we may thus *in principle* find

$$L^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\,0,0,0)=\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle.$$

As a practical matter the measurement of the phase of $\Gamma(P_5,P_6)$ has only rarely been accomplished. With an eye toward a practical measurement, we will now outline a slightly different measurement which should allow the measurement of $|\langle U_1U_2U_3^*U_4^*\rangle|$ reasonably directly. Instead of using one nonlinear dielectric for each pair of points, we now use three. See Fig. 2.

The signal emerging from the first nonlinear dielectric, after filtering, is

$$a_2U_1^2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}.$$

The signal emerging from the second dielectric is

$$a_2(U_1^2+2U_1U_2+U_2^2)e^{i2(\bar{k}z-2\pi\bar{\nu}t)}$$

and the signal emerging from the third dielectric is

$$a_2U_2^2e^{i2(\bar{k}z-2\pi\bar{\nu}t)}.$$

If the path lengths M_1P_5 and M_3P_5 are chosen to differ from the path length M_2P_5 by the distance $\bar{\lambda}/2$, then the signal emerging from P_5 is

$$2a_2U_1U_2e^{2i(\bar{k}z-2\pi\bar{\nu}t)}.$$

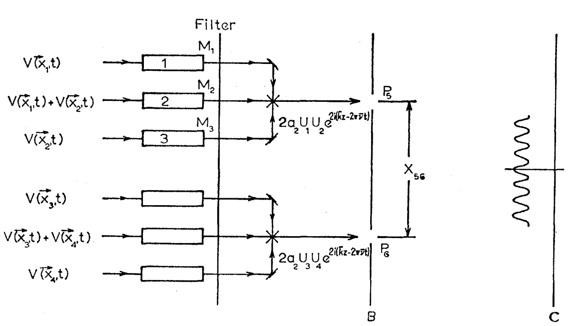

FIG. 2. Alternative experimental arrangement.

Similarly, the signal emerging from P_6 is

$$2a_2U_3U_4e^{2i(\bar{k}z-2\pi\bar{\nu}t)}.$$

Performing the interference experiment now yields directly

$$L^{42}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4;\,0,0,0)=\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle.$$

In practice we would ordinarily measure only the visibility of the fringes on plane C. This measurement yields a quantity proportional to

$$\gamma_{56}=\frac{|\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle|}{[\langle|U_1(t)|^2|U_2(t)|^2\rangle\langle|U_3(t)|^2|U_4(t)|^2\rangle]^{1/2}}.$$

To find $|\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle|$ from γ_{56}, we need only measure $\langle|U_1(t)|^2|U_2(t)|^2\rangle$ and $\langle|U_3(t)|^2|U_4(t)|^2\rangle$ the intensity of the radiation emerging from P_5 and P_6 respectively. This latter measurement gives, in fact, the same result as that obtained in a Hanbury Brown–Twiss experiment.

We note finally that in addition to determining

$$\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle,$$

we could also have found

$$\langle U_1(t)U_3(t)U_2^*(t)U_4^*(t)\rangle$$

or

$$\langle U_1(t)U_4(t)U_2^*(t)U_3^*(t)\rangle$$

by using different pairings of the radiations leaving screen A. In general, these functions are all expected to yield different values.

3. FEASIBILITY OF LABORATORY MEASUREMENT OF $|\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle|$

In this section we shall show, by an order of magnitude calculation, that $|\langle U_1(t)U_2(t)U_3^*(t)U_4^*(t)\rangle|$ may be measured in the laboratory using a high-power continuous laser. We shall begin by dipicting the most unfavorable situation and show how this may be improved upon.

We shall measure this function for radiation emerging from a steadystate laser delivering about 0.1 W/cm^2 in the red. We shall assume that the four holes pictured in Fig. 1 are each 0.01 cm^2 in area. Thus, the power passing through each hole is 10^{-3} W. We suppose that as a result of the splitting and reflection by the mirrors we lose a factor of 10 Thus, the power of the radiation entering the dielectrics in Fig. 2 is 10^{-4} W. We further assume that the collimator spreads the beam over an area of 0.1 cm^2 so that the flux is 10^{-3} W/cm^2.

The conversion of this primary power into the second harmonic is very low if no special care is taken. An order of magnitude estimate (Bloembergen[9] shows that $|P^{NL}|/P^L|$ is of the order of $\alpha|E|/|E_{\text{atomic}}|$, where α is a factor due to mismatch. $|E|$ is the magnitude of the electric field and $|E_{\text{atomic}}|$ is the magnitude of an atomic field. If we take $|E_{\text{atomic}}|=3\times10^8$ V/cm and

$|\alpha|=3\times10^2$ then $|P^{NL}|/|P^L|\approx10^{-13}$. Thus, the flux leaving the nonlinear dielectric is of the order of 10^{-16} W/cm^2. Assuming another factor of 10 loss resulting from reflection and recombination of the beams this means we have 10^{-17} W/cm^2 entering the hole P_4 (or P_6). For holes 0.01 cm^2 in area this gives an output of 10^{-19} W for each hole.

We need not, however, work with this low a power level. The factor, α, can be considerably increased by phase matching. To obtain better phase matching [essentially bringing $k_D^2(2\bar{\nu})-4k_D^2(\bar{\nu})$ closer to zero] it is only necessary to have the incoming wave impinge on the crystal at a slight angle. Since the primary and second harmonic waves then travel at slightly different angles in the crystal, it is possible to bring about a high degree of matching. In fact, Terhune, Maker, and Savage[12] have converted about 20% of the primary energy into the second harmonic.

The theoretical analysis given in the previous section is predicated on the fact that the energy in the second harmonic is small compared to the energy in the primary wave. Thus, we do not desire too high a degree of phase matching. For purpose of this experiment we will assume it possible to obtain sufficient phase matching to insure that $P^{NL}/P^L\approx10^{-10}$. In this case we would expect 10^{-16} W to emerge from the P_5 or P_6. For this power it is possible to determine the coherence between the radiations leaving P_5 and P_6 using photodetectors.

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Gravitational Field Equations for Sources with Axial Symmetry and Angular Momentum*

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The investigation of stationary axially symmetric gravity fields leads to a reduced system involving two field variables which describe the "Newtonian" and the "rotation" part of the metric. This paper presents a parametrization of this reduced problem which exhibits a previously unnoticed symmetry. Although the symmetry group [isomorphic to homogeneous Lorentz transformations on (2+1)-dimensional space] has a trivial action corresponding to unimodular linear transformations of the ϕt coordinate pair, its existence "explains" the existence of a very simple new Lagrangian for the reduced field equations, and the relatively simple form in which these equations (and the corresponding surface-independent flux integrals for mass and angular momentum) can now be written.

INTRODUCTION AND SUMMARY

PREVIOUS studies[1–5] of stationary vacuum solutions of Einstein's equations with axial symmetry have shown that the difficulties can be isolated in a reduced system involving only two independent coupled second-order equations in the two basic unknown functions entering the metric. In this paper we point out a previously unnoticed symmetry group [isomorphic to the homogenous Lorentz transformations in (2+1)-dimensional space] for this reduced problem. This symmetry governs the various ways in which the metric components can be expressed in terms of the two basic functions (field variables). In terms of two field variables α and β which we define, the reduced problem is summarized in a simple Lagrangian

$$\mathcal{L}=(\nabla\beta)^2-\cosh^2\beta\ (\nabla\alpha)^2,$$

involving only vector operations in flat Euclidean 3-space. For the corresponding field equations,

$$\nabla\cdot\mathbf{M}=0,$$

where

$$\mathbf{M}\equiv e^{-i\alpha}(\nabla\beta+\tfrac{1}{2}i\sinh2\beta\ \nabla\alpha),$$

only those solutions with axial symmetry are accepted. For solutions satisfying appropriate conditions which guarantee that the corresponding metric is asymptotically flat and nonsingular outside some bounded (source) region, the integral

$$\int_\Sigma(\mathbf{M}+\nabla\ln\rho)\cdot d\mathbf{S}=8\pi(m+iJ),$$

(where $\rho^2=x^2+y^2$) has the same value on every closed 2-surface Σ surrounding the source, and gives the mass m and total angular momentum J of the system.

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