Determination of $\kappa_1(T)$ and $\kappa_2(T)$ for Type-II Superconductors with Arbitrary Impurity Concentration*

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Formulas are derived for the constants κ_1 and κ_2 of superconducting alloys for arbitrary temperatures and impurity concentrations. Beginning with Gorkov's equations in matrix form, we calculate the free-energy density up to fourth order in $|\Delta|$, using Abrikosov's assumption of a fluxoid lattice. The impurities are treated by the usual averaging technique, retaining only s and p scattering, and a special technique is developed to treat exactly the nonvanishing commutator of different components of the gauge-invariant derivative. The results contain the already known limiting cases. Intermediate values are obtained by performing machine calculations using the general formulas developed here. It is found that the values of κ_1 and κ_2 drop relatively strongly even for small impurity concentration, and also depend unexpectedly strongly on the ratio of s and p scattering. The recent result of Caroli, Cyrot, and de Gennes is confirmed according to which κ_2 is approximately equal to κ_1 in the dirty limit.

I. INTRODUCTION

THE magnetic behavior of type-II superconductors was first explained theoretically in Abrikosov's fundamental paper[1] using the Ginzburg-Landau equations. These were later shown by Gorkov[2] to follow from the BCS theory of superconductivity for temperatures near T_c. Subsequently, a considerable amount of work has been done either to derive generalized equations of the Ginzburg-Landau type[3] and, using these, to re-do some of Abrikosov's calculations,[4,5] thus extending the theoretical description to temperatures somewhat lower than T_c, or to proceed more directly from the Gorkov equations for the gap parameter Δ to obtain various results in several limiting cases.[6–11]

Formulas have been obtained for the upper critical field $B_{C2}(T)$ or the corresponding parameter $\kappa_1(T)$ by Gorkov[6] for the "pure" limit; by Shapoval,[7] by Maki,[7] and by de Gennes[7] for the "dirty" limit; by Tewordt[4] for T near T_c; by Helfand and Werthamer[8] for the general case (taking into account only s-wave scattering of the impurities); and by the author[9] (taking into account s- and p-wave scattering of the impurities).

Formulas for the magnetization slope $\partial M/\partial B$ at $B_{C2}(T)$ or the corresponding parameter $\kappa_2(T)$ have been derived by Maki and Tsuzuki[10] for the clean limit, by Neumann and Tewordt near T_c,[5] and by Maki[11] and Caroli, Cyrot, and de Gennes[12] in the dirty limit. As was pointed out by Caroli, Cyrot, and de Gennes,[12] however, Maki's[11] $\kappa_2(T)$ is incorrect, whereas in Ref. 12 the p scattering of the impurities is omitted, leading to the substitution of the transport lifetime τ_{tr} by the s scattering lifetime τ in the otherwise correct formula.

In the present paper, we give calculations of κ_1 and κ_2 for all temperatures and impurity concentrations, taking into account s- and p-wave scattering of the impurities, i.e., considering κ_1 and κ_2 as functions of T, τ, and τ_{tr}. Our results contain the already known limiting cases and give predictions for the intermediate regions of impurity concentrations, thus connecting the "pure" and "dirty" limits.

No new physical idea is involved in the calculations. The achievement is purely mathematical and rests mainly on two new methods: the exact treatment of the nonvanishing commutator of different components of the gauge-invariant derivative, and the introduction of a special set of quasi-doubly-periodic functions by means of which the space variation of the gap parameter and its powers is easily described for every simple bravais lattice of fluxoids.

The results are the following: Compared with all simple periodic arrays of vortices, the triangular array has the lowest free energy for all T, τ, τ_{tr} at B_{C2}. At a given T, κ_1 and κ_2 mainly depend on τ_{tr}, which dependence is already contained in the Ginzburg-Landau $\kappa(\tau_{tr})$. We consider then the ratios κ_1/κ and κ_2/κ as functions of T/T_c $(0<T/T_c<1)$, $\xi/l_{tr}=1/2\pi T_c\tau_{tr}$ $(0<\xi/l_{tr}<\infty)$, and of $l_{tr}/l=\tau_{tr}/\tau$ $(1<l_{tr}/l<2)$. Both κ_1/κ and κ_2/κ are always greater than 1, and $\kappa_2>\kappa_1$

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1 A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957) [English transl.: Soviet Phys.—JETP 5, 1144 (1957)].

2 L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959); 37, 1407 (1959) [English transls.: Soviet Phys.—JETP 9, 1364 (1959); 10, 998 (1960)].

3 N. R. Werthamer, Phys. Rev. 132, 663 (1963); L. Tewordt, *ibid.* 132, 595 (1963); Z. Physik 180, 385 (1964); T. Tsusuki, Progr. Theoret. Phys. (Kyoto) 31, 388 (1964); G. Eilenberger, Z. Physik 182, 427 (1965).

4 L. Tewordt, Z. Physik 184, 319 (1965).

5 L. Neumann and L. Tewordt, Z. Physik 191, 73 (1966).

6 L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 37, 833 (1959) [English transl.: Soviet Phys.—JETP 10, 593 (1960)].

7 E. A. Shapoval, Zh. Eksperim. i Teor. Fiz. 41, 877 (1961) [English transl.: Soviet Phys.—JETP 14, 628 (1962)]; K. Maki, Physics 1, 21 (1964); P. G. de Gennes, Phys. Condensed Matter 3, 79 (1965).

8 E. Helfand and N. R. Werthamer, Phys. Rev. Letters 13, 686 (1964); Phys. Rev. 147, 288 (1966).

9 G. Eilenberger, Z. Physik 190, 142 (1966).

10 K. Maki and T. Tsuzuki, Phys. Rev. 139, A868 (1965).

11 K. Maki, Physics 1, 21 (1964).

12 C. Caroli, M. Cyrot, and R. G. de Gennes, Solid State Commun. 4, 17 (1966).

except in the dirty limit, where they become equal within 1% accuracy.

The dependence on l_{tr}/l is not negligible, κ's with higher l_{tr}/l always being smaller. For intermediate ξ/l_{tr} the κ_ν/κ drop as much as 15% when l_{tr}/l goes from 1 to 2.

The dependence on ξ/l_{tr} appears for rather small impurity concentrations. For $\xi/l_{tr}=1/20$ the κ have already been found to deviate a few percent from their behavior in the pure limit. Also this dependence is not monotonic; with l_{tr}/l held fixed and $T=0$, κ_1/κ has a more or less sharp minimum at $\xi/l_{tr}\approx 1$; for the reasonable value $l_{tr}/l=1.5$, it goes down as far as 1.13, compared with 1.26 in the pure and 1.20 in the dirty limit.

Various graphs of the κ_ν/κ as a function of the different variables are given in Sec. 8.

II. OUTLINE OF THE PROGRAM

The difference of the thermodynamic potentials per unit volume Ω/V of a specimen in the superconducting and normal state in an external homogeneous magnetic field $\mathbf{B}_e=\mathrm{rot}\mathbf{A}_e(\mathbf{r})$ at temperature T can be written in the form

$$g=\frac{\Omega_s-\Omega_n}{V}=\alpha^2\frac{1}{V}\int d^31d^32\,K_2(T,[\mathbf{A}(\mathbf{r})],2,1)\psi^*(2)\psi(1)$$
$$+\alpha^4\frac{1}{V}\int d^31d^32d^33d^34\,K_4(T,[\mathbf{A}(\mathbf{r})],4,3,2,1)$$
$$\times\psi^*(4)\psi(3)\psi^*(2)\psi(1)$$
$$+\frac{1}{V}\int d^31\frac{1}{8\pi}(\mathrm{rot}\mathbf{A}(1)-\mathrm{rot}\mathbf{A}_e(1))^2+O(\alpha^6),\qquad(2.1)$$

where $\alpha\psi(\mathbf{r})=\Delta(\mathbf{r})$ is the gap parameter, $\psi(\mathbf{r})$ being normalized according to

$$\int|\psi(\mathbf{r})|^2d^3r/V=1,$$

and $\mathbf{B}(\mathbf{r})=\mathrm{rot}\mathbf{A}(\mathbf{r})$ is the microscopic magnetic field inside the specimen.

Both $\alpha\psi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ have to be determined by minimizing g. In particular, $\mathbf{B}(\mathbf{r})$ has to be determined by the equation

$$\frac{1}{4\pi}\,\mathrm{rot}\mathbf{B}(\mathbf{r})=-\alpha^2\frac{1}{V}\int d^31d^32\frac{\delta}{\delta\mathbf{A}(\mathbf{r})}K_2(T,[\mathbf{A}(\mathbf{r})],2,1)$$
$$\times\psi^*(2)\psi(1)+O(\alpha^4),\qquad(2.2)$$

which, however, leaves an additive integration constant open. Since we want to escape the boundary problem, we split $\mathbf{B}(\mathbf{r})$ into a constant part $\mathbf{B}_i=\bar{B}\hat{z}=\mathrm{rot}\mathbf{A}_i(\mathbf{r})$ and a varying part $\alpha^2\mathbf{B}_0(\mathbf{r})$ with zero average flux, as was done in Ref. 13:

$$\mathbf{B}(\mathbf{r})=\mathbf{B}_i+\alpha^2\mathbf{B}_0(\mathbf{r});\quad\frac{1}{V}\int\mathbf{B}_0(\mathbf{r})d^3r=0.\qquad(2.3)$$

Equation (2.1) is then also minimized with respect to $\mathbf{B}_i$, whereas $\mathbf{B}_0(\mathbf{r})$ is uniquely determined by (2.2) and (2.3).

The upper critical field $B_{C2}(T)$ is the limiting external field strength below which a negative g is possible with an infinitesimal α and certain functions $\psi(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, whereas above $B_{C2}(T)$ the free-energy density g is positive with an infinitesimal α, whatever $\psi(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. Since the magnetic term in (2.1) gives at most a contribution of order α^4, B_{C2} is that external field strength B_e for which the lowest eigenvalue E_0 of the equation

$$\frac{1}{V}\int d^31\,K_2(T,[\mathbf{A}_e(\mathbf{r})],2,1)\psi(1)=E(T,B_e)\psi(2)\qquad(2.4)$$

vanishes. The solutions of (2.4) turn out to be identical with the well-known eigenfunctions of the Schrödinger equation for a particle of charge $2e$ in the magnetic field B_e. The eigenvalues E also have the same infinite degeneracy as for that Schrödinger equation.

In Sec. 4 we determine the lowest eigenvalue $E_0(T,B)$. From this $B_{C2}(T)$ is determined and

$$\kappa_1(T)=B_{C2}(T)/\sqrt{2}B_c(T)$$

[where $B_c(T)$ is the thermodynamical critical field calculated and shown in the graphs of Sec. 8].

For external field strengths just below B_{C2} we use Eqs. (2.1), (2.2), and (2.3) and get

$$g=\alpha^2\frac{1}{V}\int d^31d^32\,K_2(T,[\mathbf{A}_i(\mathbf{r})],2,1)\psi^*(2)\psi(1)$$
$$+\alpha^4\frac{1}{V}\int d^31d^32d^33d^34\,K_4(T,[\mathbf{A}_i(\mathbf{r})],4,3,2,1)$$
$$\times\psi^*(4)\psi(3)\psi^*(2)\psi(1)$$
$$-\alpha^4\frac{1}{V}\int d^31\frac{1}{8\pi}\mathbf{B}_0^2(1)+\frac{1}{8\pi}(\mathbf{B}_e-\mathbf{B}_i)^2+O(\alpha^6).\qquad(2.5)$$

The optimal $\psi(\mathbf{r})$ which minimizes Eq. (2.5) still satisfies (2.4) with $E=E_0(T,\bar{B})$ as long as we neglect contributions to g of order α^6 or higher. However, the fourth-order term now mixes the different eigenfunctions belonging to E_0, and whereas in principle the minimum condition should tell us which of these functions is to be used, this is not feasible in practice.

Instead, we shall assume with Abrikosov,[1] that we have a plane bravais lattice of simple fluxoids which

[13] G. Eilenberger, Z. Physik **180**, 32 (1965).

means that $|\psi(\mathbf{r})|^2$ is doubly periodic in the x-y plane with unit-cell area $\pi c/e\bar{B}$, and is independent of z (z axis parallel to $\mathbf{B}_e$). It is no surprise, then, that we always find the triangular lattice to have the lowest g as is the case for calculations with the Ginzburg-Landau equation.[13,14]

Given a certain lattice according to the conditions stated above, $\psi(\mathbf{r})$ is completely determined by (2.4) with $E=E_0(\bar{B},T)$. We get then, instead of (2.5) (with obvious abbreviations),

$$g=\alpha^2E_0(T,\bar{B})+\alpha^4\{K_4,T,\bar{B}\}-\alpha^4\{B_0^2,T,\bar{B}\}+(\mathbf{B}_e-\mathbf{B}_i)^2/8\pi+O(\alpha^6).$$

From

$$\frac{\partial g}{\partial\alpha}=0,\quad \frac{\partial g}{\partial\bar{B}}=0$$

follows

$$\alpha^2=-\frac{1}{2}\frac{E_0(T,\bar{B})}{\{K_4,T,\bar{B}\}-\{B_0^2,T,\bar{B}\}}+O(\alpha^4),$$

$$g=-\frac{1}{4}\frac{E_0^2(T,\bar{B})}{\{K_4,T,\bar{B}\}-\{B_0^2,T,\bar{B}\}}+\frac{1}{8\pi}(B_e-\bar{B})^2+O(\alpha^6),$$

$$M=\frac{1}{4\pi}(\bar{B}-B_e)=-\frac{1}{2}\frac{E_0(T,\bar{B})\partial E_0(T,\bar{B})/\partial B}{\{K_4,T,\bar{B}\}-\{B_0^2,T,\bar{B}\}}+O(\alpha^4)$$

$$=-\frac{1}{2}\frac{[\partial E_0(T,B_{c2})/\partial B]^2(B_e-B_{c2})}{\{K_4,T,B_{c2}\}-\{B_0^2,T,B_{c2}\}-2\pi[\partial E_0(T,B_{c2})/\partial B]^2}+O(\alpha^4),$$

$$-\frac{1}{\chi}=-\left(4\pi\frac{\partial M}{\partial B_e}\right)^{-1}=\frac{\{K_4,T,B_{c2}\}-\{B_0^2,T,B_{c2}\}}{2\pi[\partial E_0(T,B_{c2})/\partial B]^2}-1+O(\alpha^4). \tag{2.6}$$

In Abrikosov's theory[1] the formula corresponding to (2.6) is

$$-1/\chi=(2\kappa^2-1)I_0+O(\alpha^4);\quad I_0=\int|\psi(\mathbf{r})|^4d^3r/V.$$

In our more general case, it turns out in Secs. 6 and 7 that we can write

$$\frac{\{K_4,T,B_{c2}\}}{2\pi[\partial E_0(T,B_{c2})/\partial B]^2}=2\kappa_2^2I_0+\lambda_2I_2+\lambda_4I_4+\cdots,$$

$$\frac{\{B_0^2,T,B_{c2}\}}{2\pi[\partial E_0(T,B_{c2})/\partial B]^2}+1=\eta I_0+\zeta_2I_2+\zeta_4I_4+\cdots,$$

where I_2, I_4, etc. are integrals similar to I_0 containing higher eigenfunctions of Eq. (2.4). The parameters λ_ν, ζ_ν, however, are small compared with $2\kappa_2^2$ and η, respectively, the total corrections to the first term not exceeding 1%. They may thus be neglected for any practical purpose. The deviation of η from 1 is also negligible for most cases, but if $2\kappa_2^2$ is near 1 it may be of influence for small T and small impurity concentration, where it reaches 1.12.

Formulas for κ_2 and η are derived in Secs. 6 and 7 and plotted in Sec. 8.

III. DESCRIPTION IN TERMS OF GREEN'S FUNCTION

To determine the kernels K_2 and K_4 we use the notation and some of the formulas developed in Ref. 9. It is not convenient here, however, to use formula (1.6) of Ref. 9, which is appropriate only if one wants g in powers of the derivatives of Δ.

Instead, since the functional derivative of g with respect to $\Delta^*(r)$ and $\Delta(r)$ yields

$$\Delta(\mathbf{r})/\lambda-iF(0,\mathbf{r},\mathbf{r})\quad\text{and}\quad\Delta^*(\mathbf{r})/\lambda+iF^+(0,\mathbf{r},\mathbf{r}),$$

[14] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. 133, A1227 (1964).

respectively, we can use for the free-energy term in g:

$$\frac{1}{V}\int d^3r\left\{\frac{|\Delta(\mathbf{r})|^2}{\lambda}+\frac{1}{\beta}\sum_{l=-\infty}^{+\infty}\int\frac{d^3k}{(2\pi)^3}\operatorname{Tr}[\tfrac{1}{2}\hat{\Delta}(\mathbf{r})(\hat{G}_1(\omega_l,\mathbf{r},\mathbf{k})+\tfrac{1}{2}\hat{G}_3(\omega_l,\mathbf{r},\mathbf{k})+\cdots)]\right\},\tag{3.1}$$

where $\hat{G}_1$, $\hat{G}_3$, etc. are the contributions of 1st, 3rd, etc. power of $|\Delta|$ to $\hat{G}(\omega_l,\mathbf{r},\mathbf{k})$, which is connected to the Green's-function matrix averaged over all impurity positions:

$$\langle\hat{G}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')\rangle_{\text{imp.}}=(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\hat{G}(\omega,\mathbf{r},\mathbf{k}).$$

$\hat{G}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')$ is defined by the equation

$$\begin{pmatrix} i\omega+\zeta+iv_F\hat{k}\,\partial_r & 0\\ 0 & -i\omega+\zeta+iv_F\hat{k}\,\partial_r\end{pmatrix}\hat{G}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')-\int\frac{d^3q}{(2\pi)^3}V(\mathbf{k}-\mathbf{q})\hat{G}(\omega,\mathbf{r},\mathbf{q},\mathbf{k}')=\hat{E}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')+\hat{\Delta}(\mathbf{r})\hat{G}(\omega,\mathbf{r},\mathbf{q},\mathbf{k}'),\tag{3.2}$$

$$\hat{\Delta}(\mathbf{r})=\begin{pmatrix}0 & -\Delta(\mathbf{r})\\ \Delta^*(\mathbf{r}) & 0\end{pmatrix},\quad \hat{E}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

To solve (3.2) we define the following *operators*, which are supposed to work on functions $\Delta(\mathbf{r})$, $\Delta^*(\mathbf{r})$:

$$\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')=\begin{bmatrix}\dfrac{1}{i\omega+\zeta+iv_F\hat{k}\,\partial_r} & 0\\ 0 & \dfrac{1}{-i\omega+\zeta+iv_F\hat{k}\partial_r}\end{bmatrix}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'),$$

$$\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')=\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')+\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}_1')V(\mathbf{k}_1'-\mathbf{k}_1)\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_1,\mathbf{k}')+\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}_1')V(\mathbf{k}_1'-\mathbf{k}_1)\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_1,\mathbf{k}_2')V(\mathbf{k}_2'-\mathbf{k}_2)\hat{G}_{00}^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_2,\mathbf{k}')+\cdots,\tag{3.3}$$

and

$$\hat{G}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')=\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')+\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}_1)\hat{\Delta}(\mathbf{r})\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_1,\mathbf{k}')+\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}_1)\hat{\Delta}(\mathbf{r})\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_1,\mathbf{k}_2)\hat{\Delta}(\mathbf{r})\hat{G}_0^{\text{op}}(\omega,\mathbf{r},\mathbf{k}_2,\mathbf{k}')+\cdots,\tag{3.4}$$

where (3.3) and (3.4) are meant to include integration $\int d^3k/(2\pi)^3$ over repeated k variables.

Obviously, the application of $\hat{G}^{\text{op}}$ on $\hat{E}$ solves (3.2), i.e.,

$$\hat{G}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')=\hat{G}^{\text{op}}(\omega,\mathbf{r},\mathbf{k},\mathbf{k}')\hat{E}.$$

Using these representations, the trace in (3.1) can now be evaluated up to fourth power of $|\Delta|$. We get in the pure case:

$$K_2(T,[\mathbf{A}_i(\mathbf{r})],2,1)=\delta^3(1-2)\left(\frac{1}{\lambda}+\frac{1}{\beta}\sum_{l=-\infty}^{+\infty}\int\frac{d^3k}{(2\pi)^3}\frac{-1}{i\omega+\zeta+iv_F\hat{k}\,\partial_1}\times\frac{1}{-i\omega+\zeta}\right),\tag{3.5}$$

and

$$K_4(T,[\mathbf{A}_i(\mathbf{r})],4,3,2,1)=\delta^3(1-2)\delta^3(1-3)\delta^3(1-4)\times\frac{1}{\beta}\sum_{l=-\infty}^{+\infty}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\frac{1}{i\omega_l+\zeta-iv_F\hat{k}\,\partial_4}\frac{1}{-i\omega_l+\zeta+iv_F\hat{k}(\partial_1+\partial_2)}\frac{1}{i\omega_l+\zeta+iv_F\hat{k}\,\partial_1}\frac{1}{-i\omega_l+\zeta}.\tag{3.6}$$

Since Eqs. (3.5) and (3.6) are finally to be integrated over the r variables, we made use of the fact that differentiation with respect to $\mathbf{r}$ may be shifted by partial integration with respect to $\mathbf{r}$.

In the impure case, we have to average over the positions of the impurities in the scattering potentials V in (3.3). Doing this by the standard technique[15] requires pairwise "contraction" of factors V without "crossing" of impurity lines in the corresponding graphs.

[15] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

The totality of those contractions, where both factors V are contained in the same $\hat{G}_p^{\text{op}}$, yields the replacement of all ω by $\omega+1/2\tau$. In the following we therefore shall use ω_l always in the meaning

$$\omega_l=(\pi/\beta)(2l+1)+1/2\tau.$$

Contractions connecting factors V in different operators $\hat{G}_0^{\text{op}}$ give rise to vertex corrections, which are treated in the next section.

IV. THE QUADRATIC TERM AND ITS VERTEX CORRECTIONS

The term quadratic in $|\Delta|$ has already been treated very briefly in Ref. 9. We give a more extended treatment here to explain the method of handling the noncommuting components of the operator $\mathbf{\partial}_r$, which is used throughout the following.

The operator $\mathbf{\partial}_r/i$, when working on Δ, has the components

$$\left(\frac{\partial}{i\partial x}+2y\frac{e}{c}\bar{B},\ \frac{\partial}{i\partial y},\ \frac{\partial}{i\partial z}\right),$$

taking the magnetic field in the z direction. Since $\Delta(r)$ is assumed not to depend on z, we have in polar coordinates

$$-iv_F\hat{k}\,\mathbf{\partial}_r=v_F(e\bar{B}/c)^{1/2}\times(\cos\vartheta)(e^{i\varphi}F_++e^{-i\varphi}F_-)=v_F(e\bar{B}/c)^{1/2}\hat{k}\mathbf{F}_r,$$

with

$$F_+=\frac{1}{2(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{i\partial x}-\frac{\partial}{\partial y}+2y\frac{e}{c}\bar{B}\right);$$
$$F_-=\frac{1}{2(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{i\partial x}+\frac{\partial}{\partial y}+2y\frac{e}{c}\bar{B}\right).\qquad(4.1)$$

When working on $\Delta^*(\mathbf{r})$, we have instead

$$+iv_F\hat{k}\,\mathbf{\partial}_r=v_F(e\bar{B}/c)^{1/2}(\cos\vartheta)\times(e^{-i\varphi}F_+^*+e^{i\varphi}F_-^*)=v_F(e\bar{B}/c)^{1/2}\hat{k}\mathbf{F}_r^*,$$

with

$$F_+^*=\frac{-1}{2(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{i\partial x}+\frac{\partial}{\partial y}-2y\frac{e}{c}\bar{B}\right);$$
$$F_-^*=\frac{-1}{2(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{i\partial x}-\frac{\partial}{\partial y}-2y\frac{e}{c}\bar{B}\right).\qquad(4.2)$$

We have

$$[F_-,F_+]=[F_-^*,F_+^*]=1.$$

It is convenient in the following to measure all energies, i.e., ζ, $1/\beta$, $1/\tau$, in units $v_F(e\bar{B}/c)^{1/2}$. The right-hand side of Eq. (3.5) becomes then (with N being the density of states at the Fermi surface):

$$\delta^3(1-2)\left(\frac{1}{\lambda}-\frac{1}{\beta}\sum_{l=-\infty}^{+\infty}\pi N\times\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\int\frac{d^2\hat{k}}{4\pi}\frac{1}{i\omega_l+\zeta-\hat{k}\mathbf{F}_1}\times\frac{1}{-i\omega+\zeta}\right).$$

If we carry out the φ part of the integration over all directions $\hat{k}$ in this expression, the result contains F_+ and F_- only as powers of F_+F_-; i.e., the functions $\psi_0(\mathbf{r})$ with $F_-\psi_0(\mathbf{r})=0$ and the functions $F_+{}^n\psi_0(\mathbf{r})/(n!)^{1/2}$ are the eigenfunctions of the kernel K_2. It is not difficult to see that this is also true after insertion of vertex corrections for the impure case. In the pure case it is also possible to show that the eigenvalues increase with increasing n; in the impure case this is likely, but it is difficult to prove since the vertex corrections become complicated for high n. The vertex correction to Eq. (4.2) consists of an operator O which has to be inserted into the integrand between the two factors which are the remainders of two $\hat{G}_0^{\text{op}}$. Here O has to account for all possible configurations of impurity lines connecting the two G_0 after the averaging process, and therefore obeys the integral equation

$$O(\hat{k}\mathbf{F}_r)=1+\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\int\frac{d^2\hat{k}_1}{4\pi}\left(\frac{1}{\tau}+\frac{3\hat{k}\hat{k}_1}{\tau_1}\right)\times\frac{1}{i\omega+\zeta-\hat{k}_1\mathbf{F}_r}O(\hat{k}_1\mathbf{F}_r)\frac{1}{-i\omega+\zeta},\qquad(4.3)$$

which contains s and p scattering of the impurities. Since the kernel in Eq. (4.3) is linear in $\hat{k}$ the operator O must be so and Eq. (4.3) therefore is solved with the ansatz

$$O(\hat{k}\mathbf{F}_r)=(1+A(2\omega)(2\omega+i\hat{k}\mathbf{F}_r))/D(2\omega).\qquad(4.4)$$

Inserting (4.4) in (4.3), the ζ integration can be done immediately. The only problem left is the proper evaluation of the integral

$$\int\frac{d^2\hat{k}}{4\pi}\left(\frac{1}{\tau}+\frac{3\hat{k}\hat{k}_1}{\tau_1}\right)\frac{1}{2\omega+i\hat{k}_1\mathbf{F}_r},$$

which is not completely trivial, because the two components of $F=e^{i\varphi}F_++e^{-i\varphi}F_-$ do not commute. We may write, however,

$$\frac{1}{2\omega+i\hat{k}\mathbf{F}_r}=\int_0^\infty d\rho\,\exp[-2\omega\rho-\tfrac{1}{2}\rho^2\cos^2\vartheta]\times\exp[-i\rho(\cos\vartheta)\,e^{i\varphi}F_+]\exp[-i\rho(\cos\vartheta)e^{-i\varphi}F_-],$$

having used

$$e^{A+B}=e^Ae^Be^{-[A,B]/2}.$$

Now F_- can be equated to zero, since we want to apply the operator only on functions ψ_0 with $F_-\psi_0=0$. The idea of considering operator identities which are valid only if applied to functions ψ_0 is the basic trick that makes the derivations of this paper possible. It is always applied in the following without explicit mention.

The $\hat{k}$ integration in Eq. (4.3) can now be done and we get the result

$$A(2\omega)=-\frac{3}{2\tau_1}(1-2\omega f(2\omega)),$$
$$D(2\omega)=1-\frac{1}{\tau}f(2\omega)+\left(2\omega-\frac{1}{\tau}\right)A(2\omega), \quad (4.5)$$

where

$$f(x)=\int_0^\infty d\rho\, e^{-x\rho}\int_0^{\pi/2} d\vartheta(\cos\vartheta)\exp[-\tfrac{1}{2}\rho^2\cos^2\vartheta]. \quad (4.6)$$

The function $f(x)$, its properties, and related functions are treated in the Appendix.

Similarly, the integration in K_2 can be done, resulting in an identity

$$K_2(T,[\mathbf{A}_i(\mathbf{r})],2,1)=\delta^3(1-2)E_0(T,\bar{B}),$$

valid only if applied to a function ψ_0; i.e., E_0 is the eigenvalue of K_2 belonging to ψ_0. We get

$$E_0(T,\bar{B})=N\left(\ln\frac{T}{T_c}+\sum_{l=0}^{\infty}\frac{1-2\omega_l f(2\omega_l)}{(l+\frac{1}{2})D(2\omega_l)}\right). \quad (4.7)$$

In Eq. (4.7), ω_l is still measured in units $v_F(e\bar{B}/c)^{1/2}$. Equated to zero, the right-hand side thus gives an implicit equation for B_{C2}. The result coincides with the results of Gorkov[6] for the pure limit, of Maki[10] for the dirty limit, and of Helfand and Werthamer[8] for $1/\tau_1=0$.

Directly below T_c we get from Eq. (4.7)

$$E_0(T,\bar{B})=N\left(\frac{T-T_c}{T_c}+\bar{B}\frac{(e/c)v_F^2}{3\pi^2T_c^2}\times\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2(2l+1+1/2\pi T_c\tau_{\rm tr})}\right).$$

With

$$B_c=2\pi T_c\left(\frac{8\pi N}{7\zeta(3)}\right)^{1/2}\frac{T_c-T}{T_c}$$

Gorkov's[2] result follows:

$$K(\tau_{\rm tr})=\frac{3\pi T_c}{8(e/c)v_F^2}\left(\frac{7\zeta(3)}{\pi N}\right)^{1/2}\times\left(\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2(2l+1+1/2\pi T_c\tau_{\rm tr})}\right)^{-1}. \quad (4.8)$$

V. THE PERIODIC EIGENFUNCTIONS OF K_2 WITH EIGENVALUE E_0

Next, we consider the periodic eigenfunctions of K_2, which shall be used in the following sections. Because we assume independence of z, all variables will be restricted to the x-y-plane in this section.

We assume the primitive lattice vectors

$$\mathbf{r}_{\rm I}=(x_{\rm I},0);\quad \mathbf{r}_{\rm II}=(x_{\rm II},y_{\rm II}) \quad (5.1)$$

which are restricted by the flux-quantization condition

$$2\pi/x_{\rm I}y_{\rm II}=(2e/c)\bar{B}.$$

Let

$$\psi_0(\mathbf{r}|0)=\left(\frac{2y_{\rm II}}{x_{\rm I}}\right)^{1/4}\sum_{p=-\infty}^{+\infty}\exp\frac{2\pi}{x_{\rm I}y_{\rm II}}\left\{-\frac{(y+py_{\rm II})^2}{2}+ipy_{\rm II}\left(x+\frac{p}{2}x_{\rm II}\right)\right\} \quad (5.2)$$

$$=\left(\frac{2y_{\rm II}}{x_{\rm I}}\right)^{1/4}\exp\left(-\frac{\pi y^2}{x_{\rm I}y_{\rm II}}\right)\vartheta_3\left(\frac{\pi}{x_{\rm I}}(x+iy)\middle|\frac{1}{x_{\rm I}}(x_{\rm II}+iy_{\rm II})\right) \quad (5.3)$$

in the notation of Whittaker-Watson.[16] Also let

$$\psi_0(\mathbf{r}|\mathbf{r}_0)=\exp\left(\frac{2\pi i}{x_{\rm I}y_{\rm II}}y_0x\right)\psi_0(\mathbf{r}+\mathbf{r}_0|0), \quad (5.4)$$

and

$$\psi_n(\mathbf{r}|\mathbf{r}_0)=\frac{1}{(n!)^{1/2}}F_+^n\psi_0(\mathbf{r}|\mathbf{r}_0). \quad (5.5)$$

The set of functions $\psi_n(\mathbf{r}|\mathbf{r}_0)$ has the following important properties:

(a) $F_-\psi_0(\mathbf{r}|\mathbf{r}_0)=0$, i.e., the $\psi_n(\mathbf{r}|\mathbf{r}_0)$ are eigenfunctions of the Schrödinger equation for a particle with charge $2e$ in the magnetic field $\bar{B}$ and are thus eigenfunctions of K_2.

(b) The functions are doubly periodic in the following sense:

$$\psi_n(\mathbf{r}+\mathbf{r}_{\rm I}|\mathbf{r}_0)=\exp\left(2\pi i\frac{y_0}{y_{\rm II}}\right)\psi_n(\mathbf{r}|\mathbf{r}_0),$$
$$\psi_n(\mathbf{r}+\mathbf{r}_{\rm II}|\mathbf{r}_0)=\psi_n(\mathbf{r}|\mathbf{r}_0)\times\exp\left(-2\pi i\frac{x_0}{x_{\rm I}}-\frac{x_{\rm II}y_0}{x_{\rm I}y_{\rm II}}+\frac{x}{x_{\rm I}}+\frac{x_{\rm II}}{2x_{\rm I}}\right). \quad (5.6)$$

(c) The set of functions $\psi_n(\mathbf{r}|\mathbf{r}_0)$ [$n=0, 1, 2, \cdots$; r_0 contained in lattice cell] forms a complete, orthogonal system of functions in the x-y-plane with

$$\int\psi_n^*(\mathbf{r}|\mathbf{r}_0)\psi_m(\mathbf{r}|\mathbf{r}_0')d^2r=x_{\rm I}y_{\rm II}\delta_{nm}\delta^2(\mathbf{r}_0-\mathbf{r}_0'). \quad (5.7)$$

[16] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1952).

(d) There is no other set of functions with the properties a, b, c. This follows with Eq. (5.3) from the uniqueness theorem on doubly periodic analytic functions.

(e) The functions have inversion symmetry

$$\psi_n(\mathbf{r}|\mathbf{r}_0)=(-1)^n\psi_n(-\mathbf{r}|-\mathbf{r}_0). \tag{5.8}$$

(f) It follows from Eqs. (5.8) and (5.6) that

$$\psi_n\left(\frac{\mathbf{r}_{\mathrm{I}}}{2}+\frac{\mathbf{r}_{\mathrm{II}}}{2}\middle|0\right)=0, \quad \text{for even } n$$

and

$$\psi_n(0|0)=\psi_n\left(\frac{\mathbf{r}_{\mathrm{I}}}{2}\middle|0\right)=\psi_n\left(\frac{\mathbf{r}_{\mathrm{II}}}{2}\middle|0\right)=0, \quad \text{for odd } n.$$

Because $\psi_0(\mathbf{r}|\mathbf{r}_0)$ differs from $\psi_0(\mathbf{r}|0)$ only by a shift of coordinates and a corresponding gauge factor, we may assume in the following that

$$\Delta(\mathbf{r})=\alpha\psi_0(\mathbf{r}|0).$$

In the fourth-order terms we have to deal with products

$$\varphi\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\frac{\mathbf{r}_1-\mathbf{r}_2}{2}\right)=\psi_0(\mathbf{r}_1|0)\psi_0(\mathbf{r}_2|0).$$

It is obvious, that

$$_2F_{r-}\varphi(\mathbf{r},\mathbf{r}')=0, \quad _2F_{r'-}\varphi(\mathbf{r},\mathbf{r}')=0,$$

where $_2F_-$ is defined like F_- but with $2\bar{B}$ instead of $\bar{B}$. Hence $\varphi(\mathbf{r},\mathbf{r}')$ can be expressed in terms of the functions $_2\psi_0(\mathbf{r}|\mathbf{r}_0)$ which are defined like $\psi_0(\mathbf{r}|\mathbf{r}_0)$, but with a lattice cell spanned by the vectors $\mathbf{r}_{\mathrm{I}}/2$ and $\mathbf{r}_{\mathrm{II}}$, corresponding to a doubled magnetic field. Using the periodicity, symmetry, uniqueness, and normalization of the functions involved, we get the important result which will be used in the following:

$$\psi_0(\mathbf{r}_1|0)\psi_0(\mathbf{r}_2|0)=a\,{}_2\psi_0\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\middle|0\right){}_2\psi_0\left(\frac{\mathbf{r}_1-\mathbf{r}_2}{2}\middle|0\right)$$
$$+b\,{}_2\psi_0\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\middle|\frac{\mathbf{r}_{\mathrm{II}}}{2}\right){}_2\psi_0\left(\frac{\mathbf{r}_1-\mathbf{r}_2}{2}\middle|\frac{\mathbf{r}_{\mathrm{II}}}{2}\right) \tag{5.9}$$

with $|a|^2=|b|^2=\frac{1}{2}$.

VI. CALCULATION OF THE MAGNETIC FIELD TERM

The expression for $[\delta/\delta\mathbf{A}_i(\mathbf{r})]K_2$ follows in the pure case by differentiating Eq. (3.5) with respect to $\mathbf{A}_i(\mathbf{r})$. We get $[1/\beta,\ \omega_l$ always in units $v_F(e\bar{B}/c)^{1/2}]$:

$$\frac{1}{4\pi}\operatorname{rot}\mathbf{B}_0(\mathbf{r})=\lim_{1=2=r}\frac{1}{(e\bar{B}/c)^{1/2}}\frac{1}{\beta}\sum_{l=-\infty}^{+\infty}\int\frac{d^3k}{(2\pi)^3}\left(-2\frac{e}{c}\hat{k}\right)\frac{1}{i\omega_l+\zeta-\hat{k}\mathbf{F}_1}\times\frac{1}{-i\omega_l+\zeta}\times\frac{1}{i\omega_l+\zeta-\hat{k}\mathbf{F}_2^*}\psi^*(2)\psi(1). \tag{6.1}$$

In the impure case we have to introduce similar corrections as in Sec. 4. The corrected expression can be represented by the graph of Fig. 1.

The integration of Eq. (6.1) will be demonstrated only for the pure case. It is done in the same way for the impure case except that more terms have to be put together. In Eq. (6.1) the ζ part of the k integration can easily be done, leaving the expression

$$\frac{1}{(e\bar{B}/c)^{1/2}}\operatorname{rot}\mathbf{B}_0(\mathbf{r})=\frac{32\pi N}{\bar{B}\beta}\sum_{l=0}^{\infty}\int\frac{d^2\hat{k}}{4\pi}\lim_{1=2=r}\frac{\hat{k}}{\hat{k}(\mathbf{F}_1-\mathbf{F}_2^*)}\left(\frac{1}{2\omega_l+i\hat{k}\mathbf{F}_2^*}-\frac{1}{2\omega_l+i\hat{k}\mathbf{F}_1}\right)\psi^*(2)\psi(1). \tag{6.2}$$

The integral in this expression, we call it $\mathbf{U}$, can be written

$$\mathbf{U}=\lim_{1=2=r}\int_0^{\pi/2}\cos\vartheta d\vartheta\int_0^{2\pi}\frac{d\varphi}{4\pi}\int_0^{\infty}d\rho\exp(-2\omega_l\rho-\tfrac{1}{2}\rho^2\cos^2\vartheta)\{\exp[-i\rho(\cos\vartheta)e^{-i\varphi}(F_{2+}^*-F_{1-})]$$
$$-\exp[-i\rho(\cos\vartheta)e^{i\varphi}(F_{1+}-F_{2-}^*)]\}\frac{\hat{k}}{(\cos\vartheta)\{e^{i\varphi}(F_{1+}-F_{2-}^*)+e^{-i\varphi}(F_{1-}-F_{2+}^*)\}}\psi_0^*(2)\psi_0(1). \tag{6.3}$$

The operators $F_{1+}-F_{2-}^*$ and $F_{1-}-F_{2+}^*$ commute with each other; if we imagine $\psi^*(2)\psi(1)$ being decomposed into its Fourier components, they act in the limit $1=2$ as pure c numbers for the single component. Suppose that $|F_{1-}-F_{2+}^*|$ acting on a certain component gives a c number larger than $|F_{1+}-F_{2-}^*|$ does (the final result being independent of which one is larger), then the last operator in Eq. (6.3) can be written as

$$\frac{\hat{k}e^{i\varphi}}{\cos\vartheta}\sum_{\nu=0}^{\infty}(ie^{i\varphi})^{2\nu}\frac{(F_{1+}-F_{2-}^*)^\nu}{(F_{1-}-F_{2+}^*)^{\nu+1}}.$$

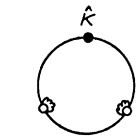

FIG. 1. The open circles with the curly lines around them stand for $O(\hat{k}\mathbf{F}_1)$ and $O(\hat{k}\mathbf{F}_2^*)$, respectively; the closed circle stands for the factor $2(e/c)\hat{k}$.

The φ integration in U_x-iU_y can now be done, yielding

$$U_x-iU_y=\lim_{1=2=r}\int_0^{\pi/2}\cos\vartheta d\vartheta\int_0^\infty d\rho\exp(-2\omega_l\rho-\tfrac{1}{2}\rho^2\cos^2\vartheta)\sum_{\nu=1}^\infty\frac{(\rho\cos\vartheta)^{2\nu}}{(2\nu)!}(F_{1+}-F_{2-}^*)^\nu(F_{1-}-F_{2+})^{\nu-1}\psi_0^*(2)\psi_0(1).$$

This expression has a common factor $F_{1+}-F_{2-}^*$, which becomes

$$-\frac{1}{(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)$$

in the limit $1=2=r$. On the left-hand side of Eq. (6.2), however, the sum of the x component and $(-i)$ times the y component amounts to

$$-\frac{1}{(e\bar{B}/c)^{1/2}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)B_0(r).$$

Omitting the common operator on both sides thus integrates Eq. (6.2).

Including the vertex corrections, we get the general expression

$$B_0(r)=\frac{16\pi^2N}{\bar{B}\beta}\sum_{l=0}^\infty\frac{1}{D^2(2\omega_l)}\Bigg\{[\tfrac{1}{2}f_1(2\omega_l)+(1-2\omega_lf(2\omega_l))A(2\omega_l)](|\psi_0(\mathbf{r})|^2-1)+\lim_{1=2=r}\sum_{\nu=2}^\infty\frac{f_\nu(2\omega_l)}{(2\nu)!}$$
$$\times((F_{1+}-F_{2-}^*)(F_{1-}-F_{2+}^*))^{\nu-1}\psi_0^*(2)\psi_0(1)\Bigg\},\quad(6.4)$$

where

$$f_\nu(x)=\int_0^\infty d\rho\,e^{-x\rho}\int_0^{\pi/2}d\vartheta\,\rho^\nu(\cos\vartheta)^{2\nu+1}e^{-\frac{1}{2}\rho^2\cos^2\vartheta}.$$

The functions $f_\nu(x)$ are related to $f(x)$ and discussed in the Appendix. The 1 beside $|\psi(\mathbf{r})|^2$ in Eq. (6.4) is the integration constant chosen such as to satisfy Eq. (2.3).

The main contribution to Eq. (6.4) comes from the first term; this term reduces to the result of Maki and Tsuzuki[10] (formula 22) for the pure limit and to the result of Maki[11] (formula 17) for the dirty limit. The evaluation of

$$\{B_0^2,T,\bar{B}\}=\frac{1}{V}\int d^3r\frac{1}{8\pi}\mathbf{B}_0^2(\mathbf{r})$$

is done by means of the identity (5.9). After some transformations one arrives at the result

$$\frac{1}{V}\int d^3r\lim_{\substack{1=2=\mathbf{r}\\3=4=\mathbf{r}}}[(F_{3+}-F_{4-}^*)(F_{3-}-F_{4+}^*)]^\mu$$
$$\times[(F_{1+}-F_{2-}^*)(F_{1-}-F_{2+})]^\nu\psi_0^*(4)\psi_0(3)\psi_0^*(2)\psi_0(1)=\frac{p!}{2^p}\sum_{q=0}^{[p/2]}\binom{p}{2q}I_{2q},\quad(6.5)$$

where $p=\nu+\mu$ and

$$I_q=\tfrac{1}{2}(|{}_2\psi_q(0|0)|^2+|{}_2\psi_q(0|\mathbf{r}_{\mathrm{II}}/2)|^2).\quad(6.6)$$

For odd q we have $I_q=0$ because of the symmetry properties of the ψ_q.

We get finally

$$\{B_0^2,T,\bar{B}\}=\frac{e^28\pi N^2v_F^2}{c^2(2\pi T)^4}(g_1^2(I_0-1)+g_1g_2I_0+(g_1g_3+\tfrac{1}{2}g_2^2)(I_0+I_2)+\cdots),\quad(6.7)$$

with

$$g_1=\left(\frac{2\pi}{\beta}\right)^3\sum_{l=0}^\infty\frac{1}{D^2(2\omega_l)}\{\tfrac{1}{2}f_1(2\omega_l)+(1-2\omega_lf(2\omega_l))A(2\omega_l)\};\quad g_\nu=\left(\frac{2\pi}{\beta}\right)^3\sum_{l=0}^\infty\frac{f_\nu(2\omega_l)}{(2\nu)!D^2(2\omega_l)},\quad\nu=2,3,\cdots\quad(6.8)$$

where $1/\beta$, ω_l are still in units of $v_F(e\bar{B}/c)^{1/2}$. Moreover, it is easily checked from Eqs. (4.7) and (6.8) that

$$\frac{\partial}{\partial B}E_0(T,\bar{B})=\frac{2eNv_F}{c(2\pi T)^2}g_1. \tag{6.9}$$

Thus we get for the quantities (2.8)

$$\eta=1+\frac{g_2}{g_1}+\left(\frac{g_3}{g_1}+\frac{1}{2}\left(\frac{g_2}{g_1}\right)^2\right)+\cdots,\quad \zeta_2=\left(\frac{g_3}{g_1}+\frac{1}{2}\left(\frac{g_2}{g_1}\right)^2\right)+\cdots,\quad \zeta_4=\cdots. \tag{6.10}$$

The machine calculations of these quantities show that the neglected quantities are very small.

VII. CALCULATION OF $\{K_4,T,\bar{B}\}$

From Eq. (3.6) we have in the pure case

$$\{K_4,T,\bar{B}\}=\frac{2\pi N}{v_F^2(e/c)\bar{B}\beta}\sum_{l=0}^{\infty}\frac{1}{V}\int d^3r\int\frac{d^2\hat{k}}{4\pi}\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\lim_{\substack{1=2=r\\3=4=r}}\frac{1}{i\omega_l+\zeta-\hat{k}\mathbf{F}_4{}^*}$$

$$\times\frac{1}{-i\omega_l+\zeta-\hat{k}(\mathbf{F}_1-\mathbf{F}_2{}^*)}\;\frac{1}{i\omega_l+\zeta-\hat{k}\mathbf{F}_1}\;\frac{1}{-i\omega_l+\zeta}\psi_0{}^*(4)\psi_0(3)\psi_0{}^*(2)\psi_0(1).$$

In the presence of impurities, we have again to introduce vertex corrections, which gives rise to three terms corresponding to the graphs of Fig. 2. These graphs correspond to those used by Gorkov,[2] the difference lying in the fact that we are dealing with the noncommuting components of $\mathbf{F}$.

The evaluation of these expressions is straightforward (although extremely cumbersome) using the methods described in the previous sections. The result is

$$\{K_4,T,\bar{B}\}=\frac{2N}{(2\pi T)^2}\left(\frac{2\pi}{\beta}\right)^3\sum_{l=0}^{\infty}\frac{1}{D^4(2\omega_l)}\{S_1+S_2+S_3+S_4+S_5+S_6\} \tag{7.1}$$

with the six contributions

$$S_1=\frac{f''}{2!}I_0+\frac{f_1''}{4!}2I_0+\frac{f_2''}{6!}\left(\frac{11}{2}I_0+\tfrac{1}{2}I_2\right)+\frac{f_3''}{8!}(20I_0+4I_2)+\cdots,$$

$$S_2=-\frac{1}{\tau}\left\{(f')^2I_0+2f'\frac{f_1'I_0}{3!2}+\left(2f'\frac{f_2'}{5!}+\left(\frac{f_1'}{3!}\right)^2\right)\frac{I_0+I_2}{2}+\left(2f'\frac{f_3'}{7!}+2\frac{f_1'f_2'}{3!5!}\right)\frac{I_0+3I_2}{4/3}+\cdots\right\},$$

$$S_3=\frac{3}{\tau_1}\left\{\left(\frac{f_1}{2!}\right)^2\frac{I_0}{2}+2\frac{f_1f_2(I_0+I_2)}{2!4!2}+\left(2\frac{f_1f_3}{2!6!}+\left(\frac{f_2}{4!}\right)^2\right)\frac{I_0+3I_2}{4/3}+\cdots\right\},$$

$$S_4=-2A\left(D+1-\frac{1}{\tau}f\right)\left\{f'I_0+\frac{f_1'I_0}{3!2}+\frac{f_2'(I_0+I_2)}{5!2}+\frac{f_3'(I_0+3I_2)}{7!4/3}+\cdots\right\},$$

$$S_5=A^2\{(3f+A)D+f(1-(1/\tau)f)\}I_0,$$

$$S_6=-2A^2\left\{\frac{f_1I_0}{2!2}+\frac{f_2(I_0+I_2)}{4!2}+\frac{f_3(I_0+3I_2)}{6!4/3}+\cdots\right\}.$$

The functions A, D, and f_ν are the functions considered in the previous sections; they depend on the argument $2\omega_l$. In each of the sums, the terms are ordered according to the powers of the "derivatives" F involved in the calculation, the higher orders thus coming from higher "nonlocality" of the kernel K_4.

As the machine calculations have shown, the main contributions come from the first *two* terms in each of the S_ν, which cancel partially. A smaller contribution comes from the terms $[(11/2)f_2''/6!+20f_3''/8!]I_0$ of the sum S_1; they contribute some 10% of the total expression for small T and small impurity concentration.

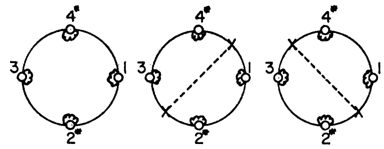

FIG. 2. The open circles with index $\nu^{(*)}$ stand for $O(kF_\nu{}^*)$; the dashed lines represent a single impurity line.

This corresponds to the fact that S_1 is the only sum that survives in the clean limit; it contains most of the nonlocality of the kernel K_4. All the other terms contribute less than 1% and may thus be discarded; in particular, the integral I_2, whose dependence on the lattice structure differs slightly from that of I_0, has no practical importance. The whole effect of the lattice structure is thus contained in the value of I_0 which is known to have its lowest value, 1.16, for a triangular lattice.[13,14]

S_1 is also responsible for the logarithmic divergence of κ_2 for small T and $1/\tau$ since the $f_\nu''(x)$ are proportional to $1/x$ for small x, as is shown in the Appendix (whereas D is finite). When

$$2\omega_l=\left(2\pi T(2l+1)+\frac{1}{\tau}\right)\Big/ v_F(e\bar{B}/c)^{1/2}\gg 1\,,$$

i.e., either in the dirty limit or for $T\approx T_c$, the asymptotic formulas given in the Appendix may be used. We then get

$$\{K_4,T,\bar{B}\}=I_0\frac{2N}{(2\pi T)}\sum_{l=0}^{\infty}\left(2l+1+\frac{v_F{}^2e\bar{B}/c}{3\pi T}\tau_{\mathrm{tr}}\right)^{-3}. \quad (7.2)$$

For $1/2\pi T\tau_{\mathrm{tr}}\gg 1$ this coincides with the recent result of Caroli, Cyrot, and de Gennes,[12] except that they got $1/\tau$ instead of $1/\tau_{\mathrm{tr}}=1/\tau-1/\tau_1$. It contains contributions from the first two terms in each of the sums S_ν. If we would take only the first term of each S_ν into account, we would arrive at Maki's result[11] (formula 5).

For $T\approx T_c$, on the other hand, (7.2) reduces to $\frac{7}{8}\zeta(3)(N/2\pi^2T_c{}^2)I_0$, corresponding to the coefficient of the nonlinear term in Gorkov's derivation of the Ginzburg-Landau equation[2]; so we get

$$\frac{\{K_4,T_c,\sigma\}}{4\pi((\partial/\partial B)E_0(T_c,0))^2}=\kappa^2(\tau_{\mathrm{tr}}) \quad (7.3)$$

with the same $\kappa^2(\tau_{\mathrm{tr}})$ as in (4.7).

VIII. RESULTS OF THE MACHINE CALCULATIONS

Using the results of the preceding sections, we did machine calculations of η, κ_1, and κ_2 as functions of three

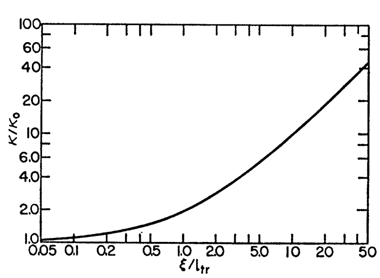

FIG. 3. κ/κ_0 as a function of ξ/l_{tr}. κ_0 is the value of κ in the pure limit. In the dirty limit κ is proportional to ξ/l_{tr}.

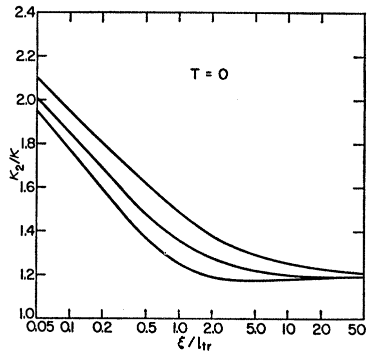

FIG. 4. κ_2/κ as a function of ξ/l_{tr} at $T=0$. The upper curve is for $l_{\mathrm{tr}}/l=1$, the lower curve is for $l_{\mathrm{tr}}/l=2$, and the curve in between is for $l_{\mathrm{tr}}/l=1.5$. The asymptotic behavior in the pure limit is like $(\ln l_{\mathrm{tr}}/\xi)^{1/2}$.

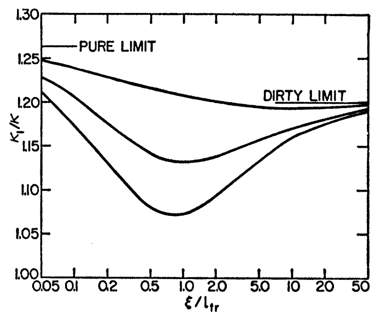

FIG. 5. κ_1/κ as a function of ξ/l_{tr} at $T=0$.

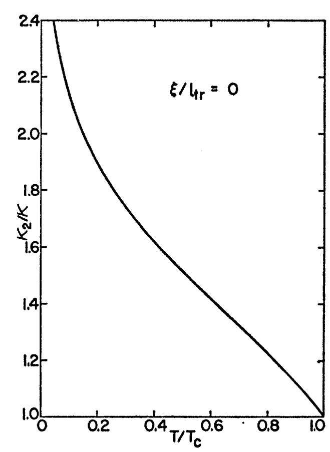

FIG. 6. κ_2/κ as a function of reduced temperature T/T_c in the pure limit.

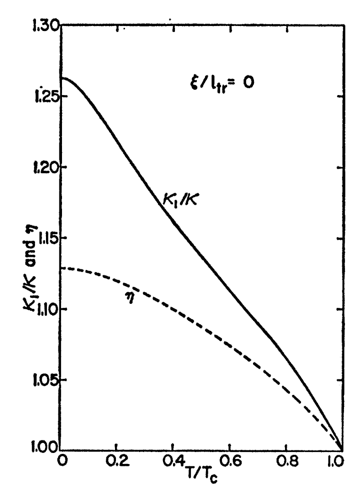

FIG. 7. κ_1/κ and η as functions of reduced temperature T/T_c in the pure limit.

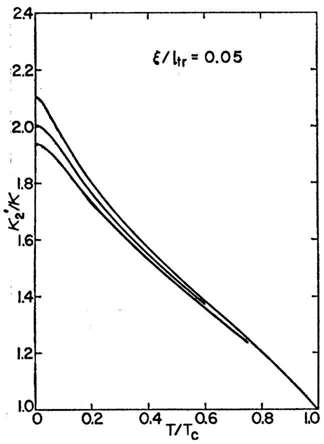

FIG. 8. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=0.05$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

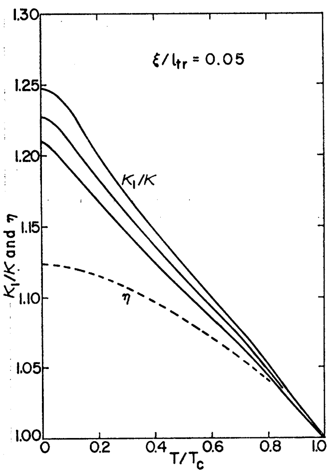

FIG. 9. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{tr}=0.05$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

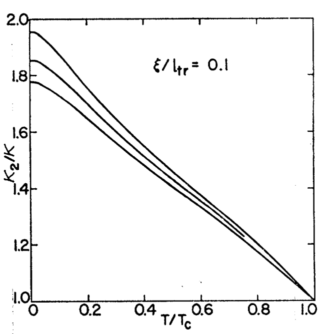

FIG. 10. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=0.1$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

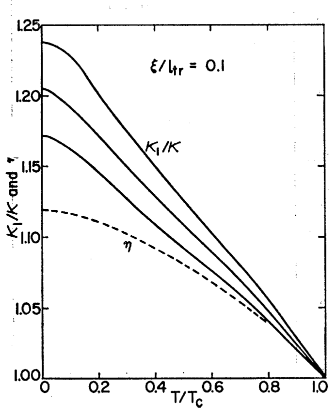

FIG. 11. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{tr}=0.1$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

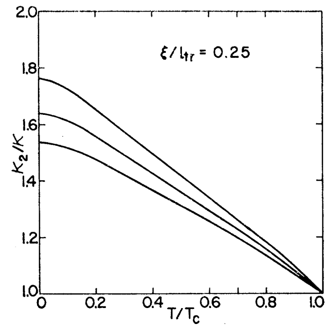

FIG. 12. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=0.25$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

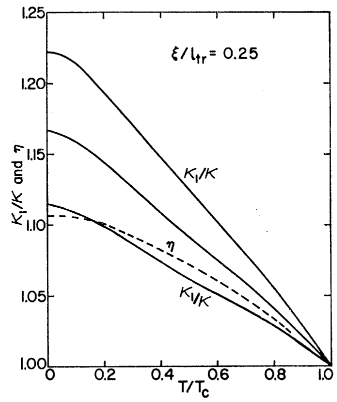

FIG. 13. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{tr}=0.25$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

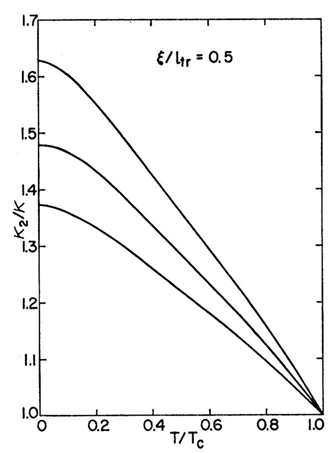

FIG. 14. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=0.5$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

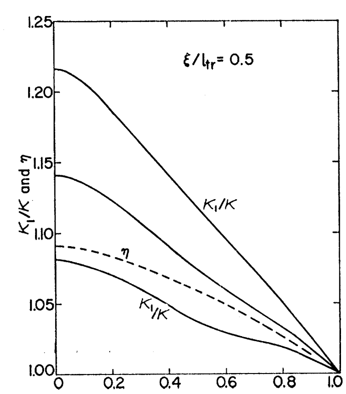

FIG. 15. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{tr}=0.5$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

variables. Figure 3 shows the value of $\kappa/\kappa_{\text{pure}}$ at T_c as a function of ξ/l_{tr} according to Eq. (4.8). Figure 4 and 5 show κ_1/κ and κ_2/κ, respectively, at $T=0$ as functions of ξ/l_{tr}. We calculated these quantities for $l_{\text{tr}}/l=1$ (upper curve), $l_{\text{tr}}/l=1.5$ (middle curve), and $l_{\text{tr}}/l=2$ (lower curve). Any reasonable scattering potential should give ratios l_{tr}/l within this range, the most reasonable ratio being about 1.5.

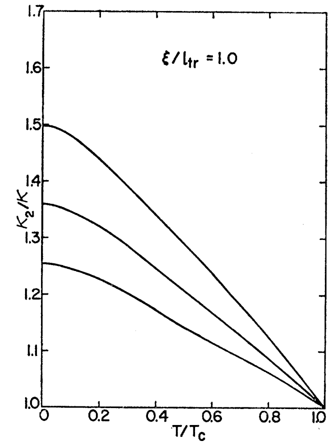

FIG. 16. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=1.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

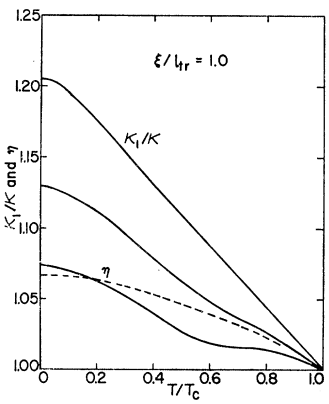

FIG. 17. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=1.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

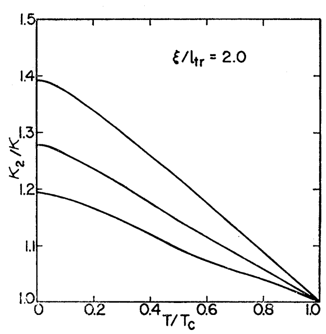

FIG. 18. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=2.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

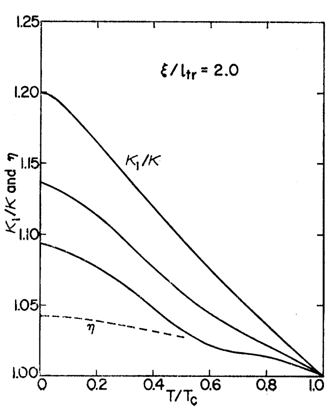

FIG. 19. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=2.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

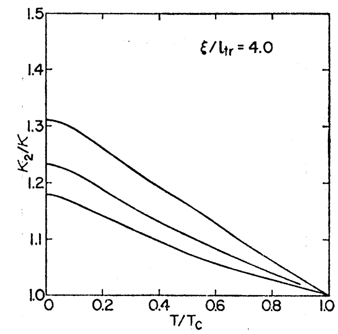

FIG. 20. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=4.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

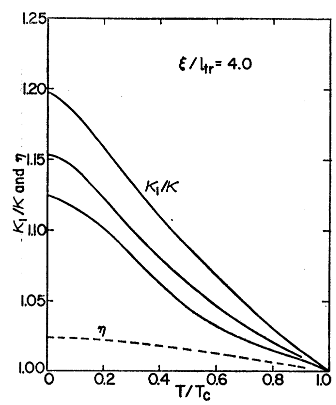

FIG. 21. κ_1/κ and η as functions of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=4.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

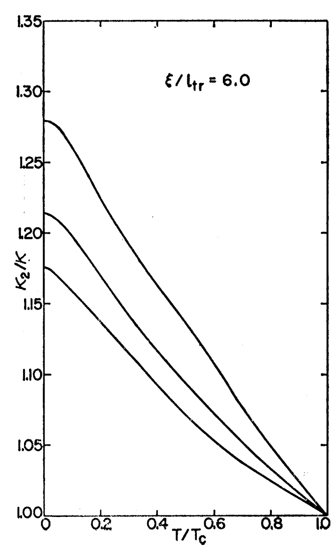

FIG. 22. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{\text{tr}}=6.0$. The upper curve is for $l_{\text{tr}}/l=1$. The lower curve is for $l_{\text{tr}}/l=2$. The curve in between is for $l_{\text{tr}}/l=1.5$.

The comparatively large influence of even little scattering is obvious for $\xi/l_{tr}=1/20$ (depending on l_{tr}/l), where κ_1/κ is already lowered by some percent and κ_2/κ has not yet reached its asymptotic behavior which is proprtional to $[\ln(l_{tr}/\xi)]^{1/2}$. Also the minimum of κ_1/κ at about $\xi/l_{tr}\approx 1$ and its dependence on l_{tr}/l is clearly exhibited by Fig. 5. Figures 6 to 29 show κ_2/κ,

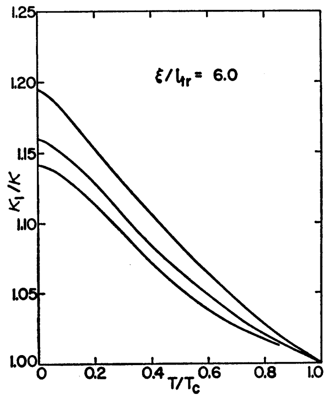

FIG. 23. κ_1/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=6.0$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

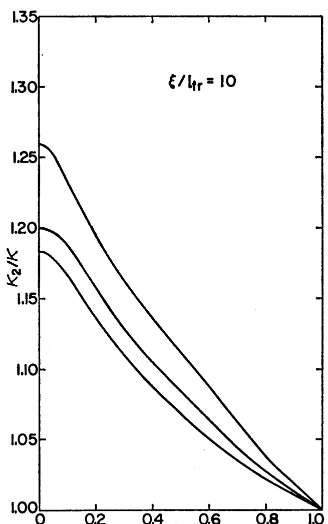

FIG. 24. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=10$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

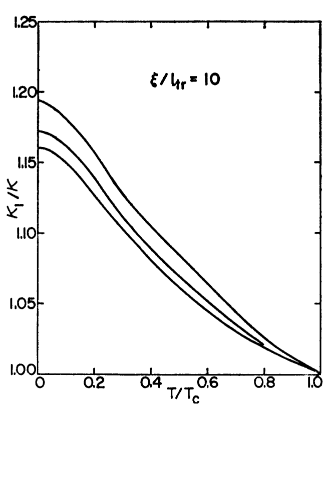

FIG. 25. κ_1/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=10$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

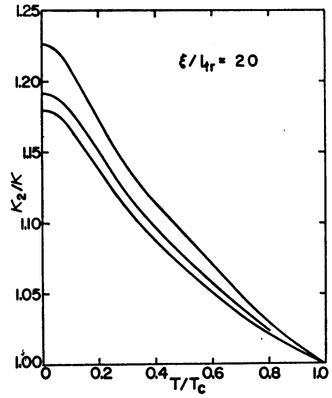

FIG. 26. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=20$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

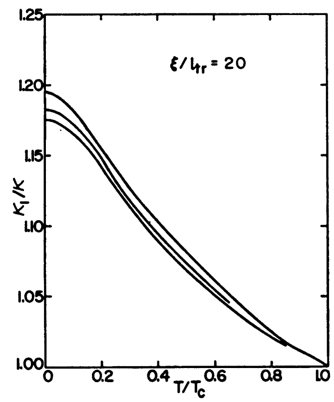

FIG. 27. κ_1/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=20$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

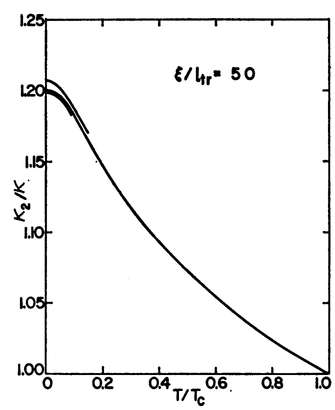

FIG. 28. κ_2/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=50$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

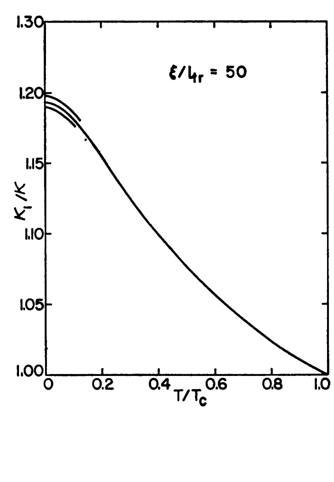

FIG. 29. κ_1/κ as a function of the reduced temperature T/T_c for $\xi/l_{tr}=50$. The upper curve is for $l_{tr}/l=1$. The lower curve is for $l_{tr}/l=2$. The curve in between is for $l_{tr}/l=1.5$.

κ_1/κ, and η as functions of T for various values of ξ/l_{tr}. Again, in each diagram the upper curve corresponds to $l_{tr}/l=1$, the middle curve to $l_{tr}/l=1.5$, and the lower curve to $l_{tr}/l=2$. The curves $\eta(T)$ are given only if $\eta(0)$ exceeds 1.02. They are for $l_{tr}/l=1.5$, the deviations from this for other values of l_{tr}/l being very small.

It should be noticed that the scales are not the same in all the diagrams for κ_2/κ. Although equal scales would have simplified comparison, their adoption seemed not practical because of the high values of κ_2/κ for low impurity concentrations. We compared our results also with the values of $(\partial/\partial T)(\kappa_1/\kappa)$ and $(\partial/\partial T)(\kappa_2/\kappa)$ at T_c, given by Tewordt[4] and Neumann and Tewordt,[5] respectively. Since we calculated κ_1/κ and κ_2/κ at intervals $\Delta T/T_c=0.05$ on the abscissa, only an approximate comparison was possible. The agreement was best for the curves with $l_{tr}/l=1$, our values at $T/T_c=0.95$ always lying slightly lower than those calculated by the linear approximation with the slopes given in Refs. 4 and 5.

ACKNOWLEDGMENTS

I am indebted to Professor V. Ambegaokar for reading and commenting on the manuscript and to Dr. E. Helfand and Dr. N. R. Werthamer for an extensive discussion, which also led to the detection of a minor error in the numerical calculation.

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APPENDIX: THE FUNCTION $f(x)$ AND RELATED FUNCTIONS

The central function for all numerical applications of the formulas given in the previous sections was

$$f(x)=\int_0^\infty d\rho\, e^{-\rho x}\int_0^{\pi/2} d\vartheta\,(\cos\vartheta)\,\exp(-\tfrac{1}{2}\rho^2\cos^2\vartheta)\,. \quad \text{(A1)}$$

It has the two further integral representations:

$$f(x)=\int_0^\infty d\rho\, e^{-\frac{1}{2}\rho^2}\arctan\frac{\rho}{x} \quad \text{(A2)}$$

(in this form it appears in the paper of Helfand and Werthamer[8]) and

$$f(x)=\frac{1}{(2\pi)^{1/2}}\int_1^\infty d\rho\frac{\ln[(\rho+1)/(\rho-1)]}{(\rho^2-1)^{1/2}}\times\exp\left(\frac{-x^2}{2(\rho^2-1)}\right). \quad \text{(A3)}$$

It satisfies the inhomogeneous differential equation

$$\left(x\frac{d^2}{dx^2}-x^2\frac{d}{dx}\right)f(x)=1 \quad \text{(A4)}$$

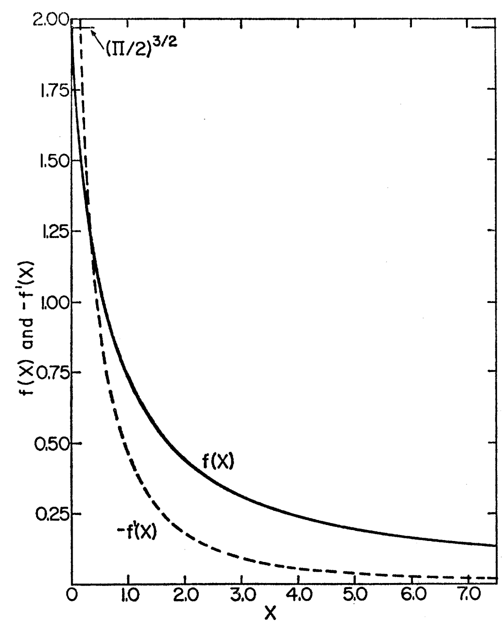

FIG. 30. The functions $f(x)$ and $-f'(x)$; the latter diverges logarithmically at $x=0$.

which may serve to prove the identity of (A1), (A2), and (A3), and which we used extensively to express the higher derivatives of f by f and f' during the machine calculations.

For small x the function f is calculated most easily by using the expansion

$$f(x)=\left(\frac{\pi}{2}\right)^{1/3}+\sum_{\nu=0}^{\infty}\frac{\ln x+\frac{1}{2}(C-\ln 2-a_\nu)}{(2\nu+1)2^\nu\nu!}x^{2\nu+1} \quad \text{(A5)}$$

with the Euler constant $C=0.5992$ and

$$a_\nu=\frac{2}{2\nu+1}+\sum_{n=1}^{\nu}\frac{1}{n},$$

and for large x by using the asymptotic expansion

$$f(x)=\sum_{\nu=0}^{\infty}\frac{(-2)^\nu\nu!}{2\nu+1}x^{-(2\nu+1)}\,. \quad \text{(A6)}$$

In Fig. 30 we have plotted $f(x)$ and $-f'(x)$ which have been used to calculate the related functions $f_\nu^{(\mu)}(x)$ discussed below. These functions, defined by

$$f_\nu(x)=\int_0^\infty d\rho\, e^{-x\rho}\int_0^{\pi/2} d\vartheta\,(\cos\vartheta)\times(\rho\cos^2\vartheta)^\nu\exp(-\tfrac{1}{2}\rho^2\cos^2\vartheta) \quad \text{(A7)}$$

and their derivatives, can be reduced firstly to sums of derivatives of $f(x)$ using integration by parts in (A7) and secondly to expressions containing only the functions $f(x)$ and $f'(x)$ and polynomials of x using (A4).

TABLE I. The first few functions $(-1)^{\mu}f_{\nu}^{(\mu)}/(2\nu+\mu)!$.

ν	μ	$x\rightarrow 0$	All x	x large
0	0	$\left(\frac{\pi}{2}\right)^{3/2}$	$f(x)$	$\frac{1}{x}-\frac{2}{3x^3}+\frac{8}{5x^5}-\frac{48}{7x^7}+\frac{384}{9x^9}-\cdots$
1	0	$\frac{1}{2}\left(\frac{\pi}{2}\right)^{3/2}$	$\frac{1}{2!}(f(x)+xf'(x))$	$\frac{2}{3x^3}-\frac{16}{5x^5}+\frac{144}{7x^7}-\frac{512}{9x^9}+\cdots$
2	0	$\frac{1}{8}\left(\frac{\pi}{2}\right)^{3/2}$	$\frac{1}{4!}(3f(x)+(5+x^2)xf'(x)+x)$	$\frac{8}{15x^{15}}-\frac{48}{7x^7}+\frac{256}{3x^9}-\cdots$
3	0	$\frac{1}{48}\left(\frac{\pi}{2}\right)^{3/2}$	$\frac{1}{6!}(15f(x)+(33+13x^2+x^4)xf'(x)+11x+x^3)$	$\frac{16}{35x^7}-\frac{512}{9x^9}+\cdots$
4	0	$\frac{1}{384}\left(\frac{\pi}{2}\right)^{3/2}$	$\frac{1}{8!}(105f(x)+(279+163x^2+25x^4+x^6)xf'(x)+121+23x^3+x^5)$	$\frac{128}{315x^9}-\cdots$
0	1	$\frac{1}{2}\left(\ln\frac{x^2}{2}+C\right)$	$-f'(x)$	$\frac{1}{x^2}-\frac{2}{x^4}+\frac{8}{x^6}-\frac{48}{x^8}+\cdots$
1	1	$\frac{1}{6}\left(\ln\frac{x^2}{2}+C\right)$	$-\frac{1}{3!}((2+x^2)f'(x)+1)$	$\frac{2}{3x^4}-\frac{16}{3x^6}+\frac{48}{x^8}-\cdots$
2	1	$\frac{1}{30}\left(\ln\frac{x^2}{2}+C\right)+\frac{1}{20}$	$-\frac{1}{5!}((8+8x^2+x^4)f'(x)+6+x^2)$	$\frac{8}{15x^6}-\frac{48}{35x^8}+\cdots$
3	1	$\frac{1}{210}\left(\ln\frac{x^2}{2}+C\right)+\frac{11}{1260}$	$-\frac{1}{7!}((48+72x^2+18x^4+x^6)f'(x)+44+16x^2+x^4)$	$\frac{16}{35x^8}-\cdots$
0	2	$\frac{1}{2x}$	$\frac{1}{2!}\left(xf'(x)+\frac{1}{x}\right)$	$\frac{1}{x^3}-\frac{4}{x^5}+\frac{24}{x^7}-\frac{192}{x^9}+\cdots$
1	2	$\frac{1}{12x}$	$\frac{1}{4!}\left((4+x^2)xf'(x)+x+\frac{2}{x}\right)$	$\frac{2}{3x^5}-\frac{8}{x^7}+\frac{96}{x^9}-\cdots$
2	2	$\frac{1}{90x}$	$\frac{1}{6!}\left((24+12x^2+x^4)xf'(x)+x^3+10x+\frac{8}{x}\right)$	$\frac{8}{15x^7}-\frac{64}{5x^9}+\cdots$
3	2	$\frac{1}{840x}$	$\frac{1}{8!}\left((192+144x^2+24x^4+x^6)xf'(x)+x^5+22x^3+104x+\frac{48}{x}\right)$	$\frac{16}{35x^9}-\cdots$

Table I gives the functions $(-1)^{\mu}f_{\nu}^{(\mu)}/(2\nu+\mu)!$ represented as the first term of their power series, as linear combination of the functions $f(x)$ and $f'(x)$, and as the first few terms of their asymptotic series, which is given by

$$\frac{(-1)^{\mu}f_{\nu}^{(\mu)}}{(2\nu+\mu)!}=\sum_{p=0}^{\infty}\frac{(-1)^{p}(2(\nu+p)+\mu)!(p+\nu)!2^{\nu}}{(2(p+\nu)+1)!!(2\nu+\mu)!p!}x^{-[2(p+\nu)+\mu+1]}.$$

The second and last columns of the table show how these quantities, which make up the terms of the S_{ν}, become smaller with increasing order ν.