Anisotropy and Temperature Dependence of the Upper Critical Field of Type-II Superconductors

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The anisotropic Ginzburg-Landau equation, originally proposed by Ginzburg and later derived from the microscopic theory by several workers, is unable to explain the anisotropy of H_{c2} observed in single crystals of type-II superconductors with cubic structure. The anisotropy is shown to arise from the first nonlocal corrections to the Ginzburg-Landau-Abrikosov-Gor'kov theory. Both the angular variation and the temperature dependence of the anisotropy as calculated here agree with that found recently in niobium. The temperature dependence of $H_{c2}(t)$ is derived for a polycrystalline but otherwise pure sample, and it is shown that Fermi-surface anisotropy increases the value of $H_{c2}(0)/[-H_{c2}'(1)]$ over that predicted for a spherical Fermi surface.

I. INTRODUCTION

THE Ginzburg-Landau-Abrikosov-Gor'kov (GLAG) theory[1–4] of superconductors in a magnetic field has been proved by numerous experiments in recent years to give an excellent qualitative account of type-II superconductivity, and a good quantitative account in a limited regime of temperatures near the zero-field critical temperature ($1-t\ll 1$). An extension of the GLAG theory to all temperatures has been accomplished by Maki[5] and by de Gennes *et al.*,[6] in the very short mean-free-path limit, again with considerable success. Also, the restriction to short mean free path has been removed for the calculation of the upper critical field H_{c2} by Helfand and Werthamer,[7] who took exact account of the intrinsic nonlocality.

One assumption common to all the work mentioned above is that of a *spherical* normal metal Fermi surface. This assumption implies that H_{c2} is isotropic, i.e., independent of orientation of the crystal with respect to the field. A generalization of the GLAG theory to incorporate anisotropy of the normal metal Fermi surface, and hence of H_{c2}, was already proposed phenomenologically in 1952 by Ginzburg.[8] The form he postulated was confirmed by a detailed derivation from the microscopic theory by Caroli, de Gennes, and Matricon[9] and by Gor'kov and Melik-Barkhudarov,[10] and then incorporated into the Maki-de Gennes[5,6] analysis by Tilley.[11] Anisotropy of H_{c2} has been observed[12,13] in single crystals of several type-II superconductors, in particular pure niobium, and interpreted in Ref. 12 as substantiating the anisotropic GLAG theory.[8–11]

However, we note that since Nb is a cubic material, the work of Refs. 9–11 in fact implies *no* anisotropy of H_{c2}. In this paper, we interpret the observed orientation dependence of H_{c2} as arising from nonlocal corrections[7] to the GLAG[1–4,9,10] and dirty limit[5,6,11] asymptotic expansions of the Gor'kov equations. We calculate the leading nonlocal correction and derive a formula predicting the angular dependence of H_{c2} and the variation of anisotropy with temperature and impurity concentration in the region of small nonlocality. The predicted formulas agree well with recent experiments.[12,13] Moreover, we derive a general expression for the anisotropy for arbitrary nonlocality, assuming that the anisotropy itself is small. We use this expression to derive a lower limit of H_{c2} for a polycrystalline sample.

II. CALCULATION OF ANISOTROPY: THE NEARLY LOCAL LIMIT

The determination of H_{c2} for a spherical Fermi surface has previously been shown[7] to reduce to the solution of the linear, homogeneous integral equation

$$\Delta(\mathbf{r})=VT\sum_{\nu}\int d^3r' K_\omega(\mathbf{r},\mathbf{r}')\Delta(\mathbf{r}'), \qquad (1)$$

where Δ is the pair wave function, $\omega=(2\nu+1)\pi T$ with ν an integer, and V is the interaction constant of the BCS model. The kernel K satisfies the integral equation

$$K_\omega(\mathbf{r},\mathbf{r}')=\int d^3r''\bar{G}_\omega(\mathbf{r},\mathbf{r}'')\bar{G}_{-\omega}(\mathbf{r},\mathbf{r}'')\times\{\delta^3(\mathbf{r}'',\mathbf{r}')+n|u|^2K_\omega(\mathbf{r}'',\mathbf{r}')\}, \qquad (2)$$

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where $n|u|^2$ is the product of impurity concentration and the square of the s-wave impurity scattering amplitude. The quantity $\bar{G}_\omega$ is the impurity-averaged one-electron Green's function for the normal metal, in the presence of a uniform magnetic field described by a vector potential $\mathbf{A}$:

$$\bar{G}_\omega(\mathbf{r},\mathbf{r}')=\int \frac{d^3p}{(2\pi)^3}\frac{\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]}{i\tilde{\omega}-\epsilon(\mathbf{p})} \times\exp\left[\frac{e}{ic}\int_{\mathbf{r}}^{\mathbf{r}'} d\mathbf{s}\cdot\mathbf{A}(\mathbf{s})\right]. \quad (3)$$

Here

$$\tilde{\omega}=\omega+n|u|^2\bar{G}_\omega(\mathbf{r},\mathbf{r})\equiv\omega+(\mathrm{sgn}\omega/2\tau), \quad (4)$$

where τ is the impurity scattering time; the $d\mathbf{s}$ path, semiclassically, is a straight line between the two endpoints, and for a spherical Fermi surface, $\epsilon(\mathbf{p})=(p^2/2m)-\mu$ with μ the chemical potential.

When the Fermi surface is allowed to be nonspherical, $\epsilon(\mathbf{p})$ becomes a more complicated function of $\mathbf{p}$, not usually specifiable analytically. Furthermore, it becomes significant to consider the deviations of the V and u potentials from strict δ functions in coordinate space and to include higher partial waves in their Fourier expansion. The generalization $V\rightarrow V(\mathbf{k},\mathbf{k}')$ and $\Delta(\mathbf{r})\rightarrow\Delta(\mathbf{r},\mathbf{k})$ is investigated in Ref. 10, while the generalization $|u|^2\rightarrow|u(\mathbf{k})|^2$ has been studied by Eilenberger.[14] In the following derivation, we ignore these last two generalizations, but after arriving at our final formulas we briefly discuss how they might be modified to include some of these effects.

Proceeding with the solution of Eqs. (1) and (2), we make use of the analysis of Ref. 7 to arrive at the implicit equation for the critical field:

$$1=VT\sum_\nu[S_\omega^{-1}-n|u|^2]^{-1}, \quad (5)$$

where

$$S_\omega=\int d^3R\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3} \times\frac{\exp[i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{R}]\langle\exp i\mathbf{R}\cdot\boldsymbol{\Pi}\rangle}{[i\tilde{\omega}-\epsilon(\mathbf{p}_1)][-i\tilde{\omega}-\epsilon(\mathbf{p}_2)]}, \quad (6)$$

and

$$\boldsymbol{\Pi}(\mathbf{r})\equiv-i\nabla_{\mathbf{r}}-2(e/c)\mathbf{A}(\mathbf{r}). \quad (7)$$

The brackets mean expectation value of the enclosed differential operator (on the variable $\mathbf{r}$) with respect to its lowest normalized eigenfunction, the calculation of which is the mathematical core of the problem in general. We have been able to make further progress only by making one or the other of two rather different simplifying assumptions. The first one we shall consider is that the system has a coherence distance sufficiently short that it is nearly local, and thus that it is permissible to expand $\exp(i\mathbf{R}\cdot\boldsymbol{\Pi})$ in powers of R, through order R^4. Then by choosing sum and difference coordinates for the momentum integrations and noting that values of the sum coordinates ($\mathbf{p}$) significant for the integration will be small compared to the Fermi momentum, we arrive at

$$S_\omega=\int d^3R\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{\exp(i\mathbf{p}\cdot\mathbf{R})}{[i\tilde{\omega}-\frac{1}{2}\mathbf{v}(\mathbf{q})\cdot\mathbf{p}]^2-\epsilon^2(\mathbf{q})} \times\langle 1-\tfrac{1}{2}(\mathbf{R}\cdot\boldsymbol{\Pi})^2+(1/24)(\mathbf{R}\cdot\boldsymbol{\Pi})^4+\cdots\rangle, \quad (8)$$

where the particle velocity $\mathbf{v}(\mathbf{q})=\nabla_{\mathbf{q}}\epsilon(\mathbf{q})$. The $\mathbf{R}$ integration is the first to be carried out, followed by the $\mathbf{p}$ integration which requires multiple integrations by parts. This gives

$$S_\omega=\int\frac{d^3q}{(2\pi)^3}\left\langle 1-\frac{1}{2}\left(\frac{1}{2}\mathbf{v}(\mathbf{q})\cdot\boldsymbol{\Pi}\frac{\partial}{\partial\tilde{\omega}}\right)^2 +\frac{1}{24}\left(\frac{1}{2}\mathbf{v}(\mathbf{q})\cdot\boldsymbol{\Pi}\frac{\partial}{\partial\tilde{\omega}}\right)^4+\cdots\right\rangle\frac{1}{\tilde{\omega}^2+\epsilon^2(\mathbf{q})}. \quad (9)$$

The q integral is broken into an angular integral and an energy integral,

$$\int d^3q/(2\pi)^3=\int d\hat{q}N(\hat{q})\int_{-\infty}^{\infty}d\epsilon(\hat{q}), \quad (10)$$

where $N(\hat{q})$ is the density of states on the Fermi surface in the direction $\hat{q}$. The ϵ integration can then be performed,

$$S_\omega=\frac{\pi}{|\tilde{\omega}|}\int d\hat{q}\,N(\hat{q}) \times\left\langle 1-\left(\frac{\mathbf{v}(\hat{q})\cdot\boldsymbol{\Pi}}{2\tilde{\omega}}\right)^2+\left(\frac{\mathbf{v}(\hat{q})\cdot\boldsymbol{\Pi}}{2\tilde{\omega}}\right)^4+\cdots\right\rangle. \quad (11)$$

Let us first consider the Π^2 term in this expansion. It is proportional to

$$\left[\int d\hat{q}N(\hat{q})\mathbf{v}_i(\hat{q})\mathbf{v}_j(\hat{q})\right],$$

which can be shown[9] to lead to the dc conductivity tensor σ_{ij}. In this way we recover the result of Tilley[11] based on the anisotropic Ginzburg-Landau equations.[8–10] The important point which was overlooked in Ref. 12 is that for a material with cubic symmetry, *any* second-rank tensor must be proportional to the unit tensor, i.e., $\sigma_{ij}=\sigma\delta_{ij}$, so that the quadratic term

$$\int d\hat{q}N(\hat{q})\langle(\mathbf{v}(\mathbf{q})\cdot\boldsymbol{\Pi})^2\rangle=\frac{1}{3}\int d\hat{q}N(\hat{q})v^2(\hat{q})\langle\Pi^2\rangle \quad (12)$$

[14] G. Eilenberger, Z. Physik **190**, 142 (1966); Phys. Rev. (to be published).

just involves the scalar $\langle \Pi^2 \rangle$ and *no* anisotropy in H_{c2} is obtained in this order.

For cubic symmetry, the anisotropy of H_{c2} thus arises from the Π^4 term, which is the first nonlocal correction[7,15] to the dirty limit and GLAG expressions. We shall evaluate the quantity $\langle (\mathbf{v}\cdot\mathbf{\Pi})^4 \rangle$ by assuming that the nonlocality is small, i.e., we shall use first-order perturbation theory. This means that we may evaluate the expectation value implied by the angular bracket by using the lowest eigenfunction of the unperturbed term Π^2. This is just the Gaussian harmonic-oscillator function used in Ref. 7, which gives

$$\langle (\mathbf{v}\cdot\mathbf{\Pi})^4 \rangle = (2eH/c)^2 \tfrac{3}{4} v_\perp{}^4, \tag{13}$$

where $\mathbf{v}_\perp$ is the part of $\mathbf{v}$ perpendicular to $\mathbf{H}$. Taking $\mathbf{H}$ as having direction cosines α, β, γ with respect to the cubic crystal axes, we arrive at

$$S_\omega = \frac{\pi \bar{N}}{|\tilde{\omega}|}\left[1 - \frac{1}{3}\frac{2eH}{c}\left(\frac{\bar{v}}{2\tilde{\omega}}\right)^2 + \left(\frac{2eH}{c}\right)^2\left(\frac{\bar{v}}{2\tilde{\omega}}\right)^4 \eta\right], \tag{14}$$

where we define

$$\bar{N} \equiv \int d\hat{q} N(\hat{q}), \tag{15}$$

$$\bar{v}^2 \equiv \int d\hat{q}[N(\hat{q})/\bar{N}] v^2(\hat{q}), \tag{16}$$

$$\eta \equiv \int d\hat{q}[N(\hat{q})/\bar{N}\bar{v}^4] v_\perp{}^4(\hat{q})$$

$$= \int d\hat{q}[N(\hat{q})/\bar{N}\bar{v}^4] \times [\tfrac{2}{5} v^4 + \tfrac{3}{4}(v^4 - 5v_x{}^4)(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 - \tfrac{1}{5})]. \tag{17}$$

Our assumption of small nonlocality is expressed more precisely by the condition

$$\epsilon \equiv (2eH/c)(\bar{v}/2\tilde{\omega})^2 \ll 1. \tag{18}$$

Substituting Eq. (14) into Eq. (5) and expanding in terms of this small quantity, and also introducing a convergence factor for the ν sum in the usual way, we obtain

$$\ln\frac{T_c}{T} = 2\pi T \sum_{\nu=-\infty}^{\infty}\left\{\frac{1}{2|\omega|} - \left[2|\omega| + \frac{1}{3}\frac{2eH}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right]^{-1} - \left(\frac{2eH}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right)^2 \frac{\eta - \frac{1}{9}}{2|\tilde{\omega}|}\left[2|\omega| + \frac{1}{3}\frac{2eH}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right]^{-2}\right\}. \tag{19}$$

This is an implicit equation for H as a function of T and of orientation. Again, since the nonlocality is small, its effect on determining $H(\alpha,\beta,\gamma)$ will be small. Thus if we define H_0 to be the field satisfying the local relation[5–7]

$$\ln\frac{T_c}{T} = F(H_0) \equiv 2\pi T \sum_{\nu=-\infty}^{\infty} \times\left\{\frac{1}{2|\omega|} - \left[2|\omega| + \frac{1}{3}\frac{2eH_0}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right]^{-1}\right\}, \tag{20}$$

then $H(\alpha,\beta,\gamma) - H_0$ will be small, and in the approximation to which we are working we find

$$H(\alpha,\beta,\gamma) = H_0 + \left[\frac{\partial F(H_0)}{\partial H_0}\right]^{-1} 2\pi T \sum_{\nu=-\infty}^{\infty}\left(\frac{2eH_0}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right)^2 \times \frac{\eta - \frac{1}{9}}{2|\tilde{\omega}|}\left[2|\omega| + \frac{1}{3}\frac{2eH_0}{c}\frac{\bar{v}^2}{2|\tilde{\omega}|}\right]^{-2}. \tag{21}$$

A slightly more convenient way to write Eq. (21) which focuses more directly on the orientation dependence of $H(\alpha,\beta,\gamma)$ is to subtract its average $\bar{H}$ over the directions (α,β,γ), which is the value of H_{c2} measured in a polycrystalline sample. This leads to

$$\frac{H(\alpha,\beta,\gamma) - \bar{H}}{\bar{H}} \approx \frac{H(\alpha,\beta,\gamma) - \bar{H}}{H_0}$$

$$= (9/4) A (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 - \tfrac{1}{5}) \frac{2eH_0\bar{v}^2}{c} \times \frac{\sum_\nu [2|\tilde{\omega}|]^{-3}\left[2|\omega| + \frac{1}{3}\left(\frac{2eH_0\bar{v}^2}{c}\right)(1/2|\tilde{\omega}|)\right]^{-2}}{\sum_\nu (2|\tilde{\omega}|)^{-1}\left[2|\omega| + \frac{1}{3}\left(\frac{2eH_0\bar{v}^2}{c}\right)(1/2|\tilde{\omega}|)\right]^{-2}}, \tag{22}$$

where

$$A = \int d\hat{q} \frac{N(\hat{q})}{\bar{N}}\left[\frac{v^4(\hat{q}) - 5v_x{}^4(\hat{q})}{\bar{v}^4}\right]$$

is a parameter characterizing the anisotropy of the Fermi surface (which vanishes for a spherical surface).

Equation (22) may be rewritten in the form

$$\frac{H(\alpha,\beta,\gamma) - \bar{H}}{\bar{H}} = (9/4) A (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 - \tfrac{1}{5}) \times G(\lambda, t) \tag{23}$$

in terms of the reduced temperature $t = T/T_c$ and the impurity-concentration parameter $\lambda \equiv (2\pi T_c \tau)^{-1}$. The function G depends only on λ and t, as can be seen by suitable transformation of variables on Eqs. (22) and (20). The expression for G is rather complicated and must be evaluated numerically in general. However, several important predictions may be made before

[15] The expansion of S_ω to this order for an isotropic material has also been obtained by L. Tewordt, Phys. Rev. **137**, A1745 (1965).

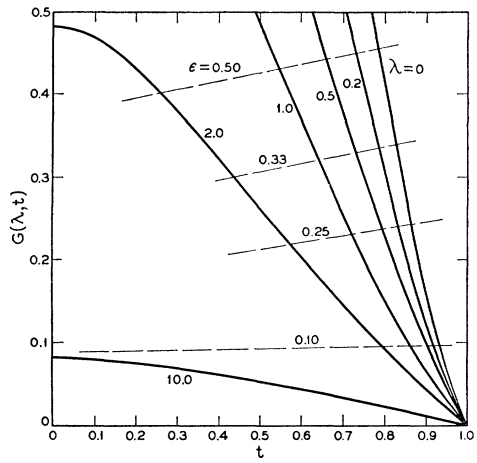

FIG. 1. The function G of Eq. (23) is plotted versus the reduced temperature t for various values of the impurity-concentration parameter λ. Dashed lines are drawn representing constant values of the expansion parameter ϵ [Eq. (18)].

discussing the numerical expressions. The first is that in the weakly nonlocal regime in which we are working, the angular dependence of H should be describable by the simple form $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2-\frac{1}{5}$. The second is that near the transition temperature $(1-t\ll1)$, where $H_0(t)\propto(1-t)$, the *relative* anisotropy $(H-\bar{H})/\bar{H}$ is also proportional to $1-t$ and hence vanishes at the transition. This is because the anisotropy in a cubic material comes from the first nonlocal correction to H and hence is higher order in $(1-t)$. In a noncubic material, the *relative* anisotropy is expected[11] to be temperature-independent. Finally, in the dirty limit $(\lambda\gg1)$, one may show that

$$(H-\bar{H})/\bar{H}\propto H_0(t)/\lambda^2.$$

Since H_0 is itself proportional to λ we find that $(H-\bar{H})/\bar{H}\propto\lambda^{-1}$. This conclusion again reflects the known fact[5,6] that nonlocal corrections become unimportant in the dirty limit.

More generally, we expect Eq. (23) to be valid whenever nonlocal corrections are sufficiently small, namely, whenever Eq. (18) is satisfied, $\epsilon\equiv[2eH_0(t)\bar{v}^2/c(2\pi T_c)^2]\times(t+\lambda)^{-2}\ll1$. This condition extends the domain of validity beyond the strict Ginzburg-Landau $(1-t\ll1)$ and dirty $(\lambda\gg1)$ limits. We have therefore calculated the function $G(\lambda,t)$ numerically and plotted the results in Fig. 1, using for $H_0(t)$ the exact result of Ref. 7 for a spherical Fermi surface. Since this value of H_0 differs significantly from the H_0 given by (20) only when $\epsilon\gtrsim1$ and is in fact less than the latter, we expect our computed $G(\lambda,t)$ to be a good representation in the region of small ϵ. We have indicated lines of constant ϵ on the figure, showing the range of validity of our expression. At low temperatures and for pure samples $(\epsilon\gtrsim1)$ we expect higher powers to come into the expansion in (8), which will lead to a different angular dependence, as well as a different function of λ and t.

As we stated earlier, we have made the simplifying assumption that the gap function depends only on the difference coordinate $\mathbf{r}$. As was shown in Ref. 10 in the Ginzburg-Landau regime, the generalization to a nonlocal $\Delta=\Delta(\mathbf{r},\mathbf{q})$ leads to an intrinsic gap-anisotropy function $\varphi^2(\hat{q})$ which multiplies the density of states $N(\hat{q})$. Since Eq. (1) for Δ is linear, we anticipate that the same will hold true in general; in any event the function φ^2 is a scalar function, so that we expect our arguments based on spatial symmetry to remain valid, with only minor modifications in the coefficients A and $G(\lambda,t)$.

III. THE LIMIT OF SMALL ANISOTROPY

The analysis thus far has been based on the assumption of small departures from locality. However, we are also able to make limited progress when the amount of variation of H_{c2} with orientation is only a small fraction of its mean value, irrespective of the degree of nonlocality. This means that we will again regard $(H(\alpha,\beta,\gamma)-\bar{H})/\bar{H}$ as $\ll1$, and hence we can once more compute to sufficient accuracy the ground-state expectation value $\langle\exp i\mathbf{R}\cdot\boldsymbol{\Pi}\rangle$ appearing in Eq. (6) by using the ground-state eigenfunctions of Π^2, which are appropriate for the (unperturbed) isotropic case. Thus

$$\langle\exp i\mathbf{R}\cdot\boldsymbol{\Pi}\rangle\cong\exp[-\tfrac{1}{4}(2eH/c)R_\perp^2]. \tag{24}$$

Substituting this back into Eq. (6), but not expanding in powers of R as before, the $\mathbf{R}$ integration may be performed, followed in this case by the $\epsilon(\hat{q})$ integration. This gives

$$S_\omega=\int\frac{d^3p}{(2\pi)^3}\delta(p_{\parallel})\frac{8\pi^2c}{2eH}\exp\left(-\frac{p_\perp^2c}{2eH}\right)\times\int d\hat{q}N(\hat{q})\frac{i\pi\,\mathrm{sgn}\omega}{i\tilde{\omega}-\frac{1}{2}\mathbf{v}(\hat{q})\cdot\mathbf{p}}. \tag{25}$$

Finally the integration over angles of $\mathbf{p}_\perp$ may be evaluated, with the result

$$S_\omega=\frac{\pi}{|\tilde{\omega}|}\int_0^\infty dw^2\exp(-w^2)\int d\hat{q}N(\hat{q})\times\left[1+\frac{2eH}{c}\left(\frac{v_\perp(\hat{q})w}{2\tilde{\omega}}\right)^2\right]^{-1/2}. \tag{26}$$

It is worth noting that if the Fermi surface were spherical, the remaining $\hat{q}$ integration could be carried out to yield

$$S_\omega=\frac{\pi\bar{N}}{|\tilde{\omega}|}I\left[\left(\frac{2eH}{c}\right)^{1/2}\frac{v_F}{2|\tilde{\omega}|}\right], \tag{27}$$

where we define the function

$$I(x)=\frac{1}{x}J(x)\equiv\int_0^\infty dw^2\exp(-w^2)(xw)^{-1}\tan^{-1}xw. \tag{28}$$

This is just the result of Ref. 7. For a nonspherical Fermi surface we must again invoke the assumption of small angular variation of H (no relative variation larger than 10% has yet been observed). We thus regard as small the difference between S_ω and $\bar{S}_\omega$, where $\bar{S}_\omega$ is defined as S_ω averaged over all field orientations, such as is appropriate for a polycrystalline sample:

$$\bar{S}_\omega=\frac{\pi\bar{N}}{|\tilde{\omega}|}\int d\hat{q}\frac{N(\hat{q})}{N}I\left[\left(\frac{2eH}{c}\right)^{1/2}\frac{v(\hat{q})}{2|\tilde{\omega}|}\right]. \quad (29)$$

We next substitute S_ω as given by Eq. (26) into Eq. (5), and expand to first order in $S_\omega-\bar{S}_\omega$ and $H-\bar{H}$, as before. The resulting expressions are too unwieldy to be useful except in the clean limit, $\tau^{-1}=0$:

$$\ln\frac{T_c}{T}=2\pi T\sum_{\nu=-\infty}^{\infty}\frac{1}{2|\omega|}\times\left[1-\int d\hat{q}\frac{N(\hat{q})}{\bar{N}}I\left(\left(\frac{2e\bar{H}}{c}\right)^{1/2}\frac{v(\hat{q})}{2|\omega|}\right)\right], \quad (30)$$

$$\frac{H-\bar{H}}{\bar{H}}=\left\{\sum_\nu\frac{1}{2|\omega|}\int_0^\infty dw^2\exp(-w^2)\int d\hat{q}\frac{N(\hat{q})}{\bar{N}}\times\left[\left(1+\frac{2e\bar{H}}{c}\left(\frac{v_1(\hat{q})w}{2\omega}\right)^2\right)^{-1/2}-I\left(\left(\frac{2e\bar{H}}{c}\right)^{1/2}\frac{v(\hat{q})}{2|\omega|}\right)\right]\right\}\times\left\{\sum_\nu\frac{1}{2|\omega|}\int d\hat{q}\frac{N(\hat{q})}{\bar{N}}\left(\frac{2e\bar{H}}{c}\right)^{1/2}\frac{v(\hat{q})}{2|\omega|}\times\frac{1}{2}I'\left(\left(\frac{2e\bar{H}}{c}\right)^{1/2}\frac{v(\hat{q})}{2|\omega|}\right)\right\}^{-1}. \quad (31)$$

Equation (31) gives the orientation dependence of H for a single-crystal sample, but the dependence is seen to be a very complicated one in general and we have not been able to extract any useful conclusions from it. On the other hand, an interesting prediction concerning the temperature dependence of $\bar{H}$, appropriate for a polycrystalline sample, can be drawn from Eq. (30). Both the $T\to 0$ and $T\to T_c$ limits of Eq. (30) can be taken using the methods of Eqs. (31)–(35) of Ref. 7, so that

$$\frac{2e\bar{H}(t)}{c}\frac{\bar{v}^2}{(2\pi T_c)^2}=\frac{12}{7\zeta(3)}(1-t), \quad 1-t\ll 1$$

$$=\frac{e^2}{4\gamma}\exp\left[-\int d\hat{q}\frac{N(\hat{q})}{\bar{N}}\ln\frac{v^2(\hat{q})}{\bar{v}^2}\right], \quad t=0. \quad (32)$$

Thus

$$\frac{\bar{H}(0)}{-\bar{H}'(1)}=0.7273\exp\left[-\int d\hat{q}\frac{N(\hat{q})}{\bar{N}}\ln\frac{v^2(\hat{q})}{\bar{v}^2}\right], \quad (33)$$

where the numerical coefficient 0.7273 is just that already deduced[7,16] for a spherical Fermi surface.

However, the remaining factor on the right-hand side of Eq. (33) can be shown[17] always to be ≥ 1 for an arbitrary Fermi surface. Hence the $t=0$ value of H_{c2} for a pure material, relative to its value near $t=1$, will always be larger for a sample with a nonspherical Fermi surface than would be expected on the basis of a calculation[7] with a spherical Fermi surface. Although we are unable to make a quantitative estimate of the enhancement factor in Eq. (33), the direction of the effect is in qualitative agreement with that observed experimentally[18] in pure niobium. Thus Fermi-surface anisotropy seems to be a promising, although not conclusive, explanation for the observed deviation[18] of $H_{c2}(t)$ from that predicted earlier[7] for a pure type-II superconductor.

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We wish to thank E. Fawcett, P. Meincke, and W. Reed for bringing this problem to our attention.

[16] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 833 (1959) [English transl.: Soviet Phys.—JETP **10**, 593 (1960)].

[17] This is a consequence of a theorem analogous to Peierls's theorem [R. Peierls, Phys. Rev. **54**, 918 (1938); see H. Falk Physica **29**, 1114 (1963)], which states that for any function f such that $f''\leq 0$, $\langle f(x)\rangle\leq f(\langle x\rangle)$, where the bracket denotes averages over a positive weight function. We apply this theorem to the logarithm function, the average being taken over the positive weight $N(\hat{q})/\bar{N}$, to show that $\langle\ln(v^2)\rangle\leq\ln(\langle v^2\rangle)$, which implies $\exp[-\langle\ln v^2\rangle+\ln\langle v^2\rangle]\geq 1$.

[18] T. McConville and B. Serin, Phys. Rev. **140**, A1169 (1965); D. K. Finnemore, T. F. Stromberg, and C. A. Swenson, *ibid.* **149**, 231 (1966).