Magnetization of Ellipsoidal Superconductors

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A well-known result which can be rigorously deduced from Maxwell's equations and potential theory is that the magnetization $\mathbf{M}$ of any homogeneous, isotropic ellipsoid is related to the local field $\mathbf{B}$ and the external applied magnetic field $\mathbf{H}_0$ by $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}=\mathbf{H}_0-4\pi\mathsf{N}\cdot\mathbf{M}(\mathbf{H})$. The configuration matrix N, whose elements are the demagnetization coefficients, depends only on the shape of the ellipsoid. In particular, this relation provides a means for computing the magnetization $\mathbf{M}$ (as a function of $\mathbf{H}_0$), if it is known for the special case of an infinitely long cylinder whose axis is aligned with $\mathbf{H}_0$. This transformation has been commonly applied to systems for which the $\mathbf{M}(\mathbf{H})$ relationship is linear: $\mathbf{M}=\chi\mathbf{H}$, where χ is a constant. We find, more generally, when the ellipsoid is homogeneous and isotropic, that the transformation depends only on the assumption that $\mathbf{M}$ is a smooth, single-valued, and otherwise arbitrary function of the local field $\mathbf{B}$. We apply the above result to the well-known magnetization functions for superconductors and find for spheroids whose symmetry axis is parallel to $\mathbf{H}_0$:

$$4\pi\mathbf{M}=-\mathbf{H}_0/(1-n) \quad \text{(Meissner state)}$$

and

$$4\pi\mathbf{M}=-[(H_{c2}-H_0)/(\gamma+n)H_0]\mathbf{H}_0 \quad \text{(mixed state)},$$

where $\gamma=(2\kappa_2^2-1)\beta$, and n is the element of N associated with the symmetry direction. We also compute the torque exerted on a superconducting spheroid whose symmetry axis is not aligned with $\mathbf{H}_0$.

INTRODUCTION

In the performance of an experiment on a magnetized sample one often needs to be concerned about the effect of the sample shape on its magnetization and on the magnetic field in its vicinity. A familiar example is the simple case of a substance whose induced magnetization varies linearly with an applied external field. If a long thin rod aligned parallel to a uniform applied field H_{0z} is found to magnetize according to

$$M_z=\chi H_{0z}, \tag{1}$$

where χ is a constant, then it is known that for a specimen of the same substance but of any ellipsoidal shape the magnetization will be given by

$$M_z=\chi H_{0z}/(1+4\pi n_z\chi), \tag{2}$$

where χ has the same value as in (1) and the shape of the ellipsoid is characterized by the geometrical factor n_z, the so-called demagnetization coefficient. The validity of Eq. (2) is understood to depend on the fact that the specimen magnetizes uniformly.

In the following, we consider bodies for which the dependence of M on H_0 is in general not linear[1] as in Eq. (1) and investigate the problem of describing the magnetization of such a body of arbitrary ellipsoidal shape, thus obtaining expressions analogous to (1) and (2) for the general nonlinear case. In pursuing this goal, we first reproduce for the sake of clarity and definiteness the classical boundary-value approach to the solution of the macroscopic field distribution arising from a magnetized body. In the final section we apply the general results to the computation of the magnetization of variously shaped superconductors and consider the effect on the torque and specific heat.

[1] We do not include ferromagnetics, however, and limit our attention to systems for which a well-defined "reversible" magnetization is induced by an externally applied field.

THE CLASSICAL FORMULATION

We consider that the quasistatic magnetization of a body may be attributed to a distribution of its magnetization currents $\mathbf{j}(\mathbf{r})$ which are related to the body's magnetization field $\mathbf{H}_s$ according to the Maxwell (time averaged) equations

$$\nabla\times\mathbf{H}_s=(4\pi/c)\mathbf{j}(\mathbf{r}), \quad \nabla\cdot\mathbf{H}_s=0. \tag{3}$$

We define the "magnetization" associated with $\mathbf{H}_s$ and $\mathbf{j}$ as

$$c\nabla\times\mathbf{M}=\mathbf{j}(\mathbf{r}), \tag{4}$$

where $\mathbf{M}$ is to be finite at all points inside the body and vanish outside though $\mathbf{H}_s$ need not.

Thus, in the presence of an applied external field $\mathbf{H}_0$, the total local field $\mathbf{B}=\mathbf{H}_0+\mathbf{H}_s$ satisfies the relations

$$\nabla\times\mathbf{B}=4\pi\nabla\times\mathbf{M}, \quad \nabla\cdot\mathbf{B}=0. \tag{5}$$

We will find it more convenient, as usual, to introduce the fictitious field $\mathbf{H}$ defined by $\mathbf{H}\equiv\mathbf{B}-4\pi\mathbf{M}$. (6)

From (5) $\mathbf{H}$ satisfies

$$\nabla\times\mathbf{H}=0, \quad \nabla\cdot\mathbf{H}=-4\pi\nabla\cdot\mathbf{M}. \tag{7}$$

The boundary conditions which lead to the desired macroscopic solutions are that the tangential and normal components of $\mathbf{H}$ and $\mathbf{B}$, respectively, shall be continuous across the surface of the specimen. To complete the formulation of the problem it is necessary to specify the way in which the magnetization $\mathbf{M}$ depends on the local field $\mathbf{B}$. In the following, we summarize the results of such a formulation; details of the analysis are outlined in the Appendix.

We consider only homogeneous isotropic substances so that the dependence of $\mathbf{M}$ on $\mathbf{B}$ may be given by a scalar function $\mathbf{M}=f(\mathbf{B})$. It can be shown that if this function is single-valued and has a continuous derivative, then one is led to a constitutive relation $\mathbf{M}=g(\mathbf{H})$

for which the effects of geometry do not enter explicitly. These effects are handled implicitly through the dependence of **H** on the shape of the body. Moreover, for ellipsoidal samples the field **H** and therefore also **M** turn out to be uniform inside the specimen. One then obtains the well-known result[2] (see Appendix),

$$H_i = H_{0i} - 4\pi \sum_j n_{ij} M_j(H) \quad (8)$$

which relates the fields **H** and **M** to the externally applied field $\mathbf{H}_0$. In (8), the i's and j's refer to the Cartesian components of the vectors and the elements of the matrix $\{n_{ij}\} \equiv \mathbf{N}$ are the well-known geometrical factors[3] called "demagnetization coefficients" which prescribe the shape of the ellipsoid.

It should be emphasized that since the constitutive relation $g(\mathbf{H})$ does not explicitly involve the geometry of the specimen (that is, the n_{ij}), this function can be completely elucidated for any convenient geometry and the dependence on configuration accounted for through the field **H**. The functional dependence may be discovered empirically. In this connection, we note that outside the specimen the field produced by the magnetized body approximates at distant points the field of a dipole whose moment is defined by

$$\mathfrak{M} \equiv \frac{1}{2c} \int_S (\mathbf{r} \times \mathbf{j}) dv = \frac{1}{2} \int_S (\mathbf{r} \times \nabla \times \mathbf{M}) dv = \int_S \mathbf{M} dv, \quad (9)$$

where the integrals are to be taken over the volume of the specimen S and the last equality results from application of Gauss' theorem. As most magnetometric experimental procedures involve measurements which depend on the field in the vicinity of the specimen, it is conventional to interpret the experiment as a measurement of the magnetic moment $\mathfrak{M}$ and hence of the $\mathbf{M}(\mathbf{H}_0)$ dependence. In particular, if one can determine the dependence $\mathbf{M}_0(\mathbf{H}_0)$ of **M** on $\mathbf{H}_0$ for the special case of a cylinder long compared to its lateral dimensions, for which case $\mathbf{H} = \mathbf{H}_0$,[4] then the constitutive relation $\mathbf{M}_\mathrm{N}$ for an arbitrary ellipsoid characterized by the configuration matrix **N** is simply given, using (8) by

$$\mathbf{M}_\mathrm{N}(\mathbf{H}_0) = \mathbf{M}_0[\mathbf{H}(\mathbf{H}_0, \mathbf{N})]. \quad (10)$$

In the next section we apply the transformation (10) to the magnetization functions of superconductors.

[2] Equation (8) is derived in numerous standard texts by invoking the assumption of uniform magnetization: e.g., W. F. Brown, *Magnetostatic Principles in Ferromagnetism* (North-Holland Publishing Company, Amsterdam, 1962). In the present case, we prove (see Appendix) that it follows directly from the well-behaved nature of the magnetization function $f(\mathbf{B})$.

[3] For tabulated values see Brown (Ref. 2) and E. C. Stoner, Phil. Mag. **36**, 803 (1945).

[4] More precisely, we mean an exceedingly prolate spheroid (eccentricity approaching unity) with $\mathbf{H}_0$ parallel to the axis of revolution. The only element of **N** which enters Eq. (8) in this case is that one multiplying the component of **M** parallel to $\mathbf{H}_0$. Since this component, $n_{||}$, is zero and **M** is parallel to $\mathbf{H}_0$, the sum vanishes.

APPLICATION TO SUPERCONDUCTORS

Magnetization

In applying the foregoing results to superconductors[5] we restrict the present discussion to bulk specimens. That is, all of the dimensions of the specimen are much larger than the weak-field penetration depth. This restriction is necessary primarily because the magnetization becomes size-dependent as well as shape-dependent when the sample approaches this characteristic dimension.

Since insofar as their magnetization is concerned, type-I superconductors may be treated as a special case of type-II, it will be sufficient to consider the latter. The analytical expressions for $\mathbf{M}(\mathbf{H}_0)$ for the geometry $n_{||} = 0$[4,6] are known for two cases: (a) the Meissner state for which

$$4\pi\mathbf{M} = -\mathbf{H}_0 \quad (11)$$

and (b) the mixed state in the vicinity of the transition field H_{c2} where the magnetization is given by the Abrikosov formula

$$4\pi\mathbf{M} = -[(H_{c2} - H_0)/\gamma H_0]\mathbf{H}_0. \quad (12)$$

Here $\gamma \equiv (2\kappa_2^2 - 1)\beta$, where β is a numerical constant of order unity, and κ_2 is the Abrikosov-Maki parameter, the value of which may be the objective of a magnetization measurement. As implied by the restriction $n_{||} = 0$, these formulas are expected to apply to an exceedingly prolate spheroid which is aligned with the applied field $\mathbf{H}_0$.

In the following, we restrict the discussion to spheroids and choose cylindrically symmetric coordinates with the cylinder (z) axis parallel to the axis of rotation of the spheroid. In this representation the matrix **N** is diagonal with only two distinct nonzero elements $n_{zz} \equiv n$ and $n_{xx} = n_{yy} = (1-n)/2$, where n is the demagnetization coefficient corresponding to the z axis of the spheroid.

To compute the magnetization for the case where $n \neq 0$ according to the relation (10), we replace $\mathbf{H}_0$ by **H**

[5] At this writing, we have learned that very recently Kulik has independently proposed the application of Eq. (8) to the magnetization of type II superconductors without, however, a detailed examination of the circumstances under which the procedure is valid. I. O. Kulik, JETP Pis'ma v Redaktisyu **3**, 395 (1966) [English transl.: JETP Letters **3**, 259 (1966)].

[6] The $n_{||} = 0$ geometry has particular significance in regard to thermodynamic considerations, as it is only for this case that $\mathbf{B} = \mathbf{H} = \mathbf{H}_0$ at *all* points exterior to the specimen. Thus, the volume integrals describing the free energies may be correctly terminated at the bounding surface of the specimen in the sense that contributions of the form

$$\int_{V'} dV H_0^2/8\pi$$

from the space outside of the sample may be regarded as subtractive constants. For this reason, it is characteristic of macroscopic thermodynamic theories to prescribe $\mathbf{M}(\mathbf{H}_0)$ for the $n_{||} = 0$ case (e.g., the Ginzburg-Landau theory of superconductivity), while the opposite situation [**M** given as $\mathbf{M}(\mathbf{B})$] is apt to be given in a microscopic discussion (e.g., the Lifschitz-Kosevitch theory for the de Haas–van Alphen effect).

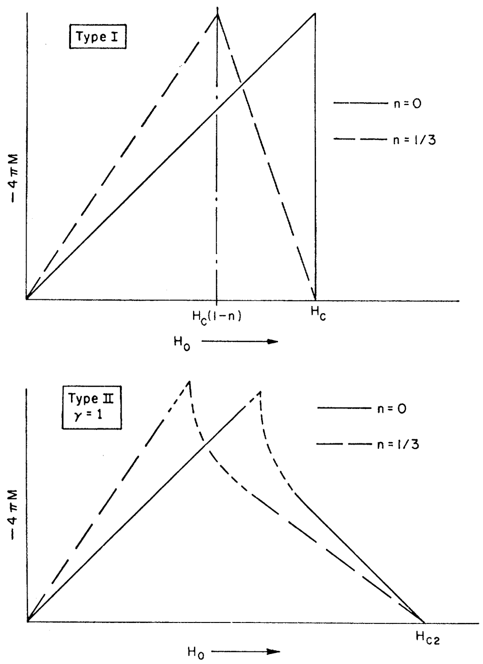

FIG. 1. The magnetization of a type-I and a type-II superconductor for the case of a very long thin rod aligned with the applied field ($n=0$), and for the case of a sphere ($n=\frac{1}{3}$). The curves have been chosen so that the two types of superconductor have the same thermodynamic critical field H_c. For the type-II superconductor in this case, the Abrikosov-Maki parameter κ_2 has been chosen so that $\gamma\equiv(2\kappa_2^2-1)\beta=1$. The slopes of the straight lines indicated by the long dashes are deduced from the present simple theory to be $1/(1-n)$ for the Meissner state, and $-1/(\gamma+n)$ for the intermediate ($\gamma=0$) and mixed states. The latter result for the mixed state has been deduced independently by Maki from the Ginzburg-Landau theory for the case of a sphere.

in Eqs. (11) and (12) obtaining

$$4\pi\mathbf{M}=-\mathbf{H} \tag{13}$$

for the Meissner state, and

$$4\pi\mathbf{M}=-[(H_{c2}-H)/\gamma H]\mathbf{H} \tag{14}$$

for the mixed state.

We consider first a specimen whose symmetry axis is aligned parallel to $\mathbf{H}_0$. For this case, the vector notation may be dropped since $\mathbf{M}$, $\mathbf{H}$, and $\mathbf{H}_0$ are parallel. Combining (8) and (13), one finds easily for the Meissner state

$$H=H_0-4\pi nM=H_0+nH\,, \tag{15}$$

so that

$$H=H_0/(1-n)$$

and

$$4\pi M=-H=-H_0/(1-n) \tag{16}$$

which is the well-known result. [Equation (16) is, of course, just Eq. (2) with $\chi=-1/4\pi$.] For the mixed state, we obtain a generalization of Abrikosov's formula, Eq. (12), by combining (8) and (14):

$$H=(\gamma H_0+nH_{c2})/(\gamma+n)\,. \tag{17}$$

Hence

$$-4\pi M=(H_{c2}-H_0)/(\gamma+n)\,. \tag{18}$$

When $n=0$, Eq. (18) reduces to the familiar Abrikosov formula as expected.[7] One notes that for superconductors for which κ_2 is only slightly greater than $1/\sqrt{2}$, i.e., $\gamma\sim+0$, the effect of n on the slope of the magnetization curve is appreciable and must be taken into account in order to deduce the correct value of κ_2 from the experimental slope.

The results embodied in Eqs. (16) through (18) extend immediately to type-I superconductors by simply replacing H_{c2} by H_c, and letting $\gamma\rightarrow 0$. From (15) we see that H reaches the value H_c at $H_0=H_c\times(1-n)$, and from (17) $H=H_c$ for all values $H_c(1-n)\leq H_0\leq H_c$. In other words, we have the well-known result that the magnetization obeys Eq. (16) for $0\leq H_0\leq(1-n)H_c$ and then decreases linearly according to

$$-4\pi M=(H_c-H_0)/n\,,\quad [\text{intermediate state } H_c(1-n)\leq H_0\leq H_c]. \tag{19}$$

These results and the corresponding results for a type-II superconductor for which $\gamma=1$ are illustrated in Fig. 1. For the sake of comparison the curves have been chosen so that H_c is the same for both the type-I and the type-II cases; that is, the areas enclosed by each of the four curves is the same.[8] From the fact that H_c is invariant with geometry, and thus that the enclosed area is similarly invariant, one can deduce that the slope of the magnetization curve in the intermediate state is $1/n$. For the type-II case, however, Eq. (18) cannot be derived by appealing to conservation of area.

From the experimental point of view, the usefulness of the foregoing formulas lies in the fact that they allow some liberty in the choice of specimen shape. As long as the specimen shape approximates an ellipsoid an effective value,[9] of n is *determined* experimentally by the Meissner-state magnetization. This value may then be used in fitting formula (18) to the mixed-state magnetization. In this way, the effect of shape is accounted for in determining the value of γ and hence κ_2.

Torque

When the symmetry axis of the specimen is *not* aligned with H_0, one expects in general that the magnetization will not be parallel with the applied field. As a result, the sample will experience a torque which may be expressed by

$$\tau=(M_{\perp}H_{0\parallel}-M_{\parallel}H_{0\perp})V\,, \tag{20}$$

[7] Maki has derived Eq. (18) for a sphere ($n=\frac{1}{3}$) by direct calculation from the Ginzburg-Landau theory (private communication).

[8] The area $=\int_0^\infty MdH_0\equiv H_c^2/8\pi=G_n(H=0)-G_s(H=0)$, i.e., the free-energy density difference between the normal and superconducting states in *zero* field. Hence the area is independent of the shape.

[9] This is discussed by Brown, Ref. 2, pp. 52, 53, 83, and 84.

where V is the volume of the spheroid and the subscripts $\|$ and $\perp$, as before, refer to components parallel and perpendiculaar to the z axis. For the Meissner state of spheroidal specimens, the components $M_{\|}$ and $M_{\perp}$ may be found by straightforward simultaneous solution of the vector equations (8) and (13). One obtains

$$4\pi M_{\|}=-H_{0\|}/(1-n_{\|})=-H_0\cos\psi/(1-n),$$
$$4\pi M_{\perp}=-H_{0\perp}/(1-n_{\perp})=-2H_0\sin\psi/(1+n), \quad (21)$$

and

$$4\pi\tau=V[(3n-1)/2(1-n^2)]H_0^2\sin2\psi, \quad (22)$$

where ψ is the angle between the symmetry axis and $\mathbf{H}_0$, and V is the volume of the spheroid. For the mixed state of a type-II (or the intermediate state of a type-I) superconductor we may obtain a solution as follows. Let the magnetization $\mathbf{M}$ make an angle φ with the symmetry axis of the spheroid. Then the equations for the components of $\mathbf{M}$ may be expressed as

$$4\pi M_{\|}=h\cos\varphi;\quad 4\pi M_{\perp}=h\sin\varphi, \quad (23)$$

where

$$h\equiv(H-H_{c2})/\gamma.$$

Using (23) and solving (10) and (14) we obtain

$$\cos\varphi\equiv\cos(\psi+\epsilon)=H_0\cos\psi/[(\gamma+n)h+H_{c2}], \quad (24)$$

and

$$\sin\varphi\equiv\sin(\psi+\epsilon)=H_0\sin\psi/\{[\gamma+(1-n)/2]h+H_{c2}\}, \quad (25)$$

where ϵ is the angle between the magnetization vector and the applied field vector. Elimination of φ between these equalities leads to a quartic equation for h. However, a very good approximate solution results from expanding $\sin\varphi$ and $\cos\varphi$ about $\epsilon\approx0$. One obtains to $O(\epsilon^2)$ the quadratic for h:

$$\frac{\cos^2\psi}{(\gamma+n)h+H_{c2}}+\frac{\sin^2\psi}{[\gamma+(1-n)/2]h+H_{c2}}=\frac{1}{H_0}[1+O(\epsilon^2)]. \quad (26)$$

To see that ϵ is in fact small for cases of practical interest we combine (24) and (25) to obtain

$$\sin\epsilon=\tfrac{1}{4}H_0\sin2\psi\times\left\{\frac{h(3n-1)}{[[\gamma+(1-n)/2]h+H_{c2}][(\gamma+n)h+H_{c2}]}\right\}. \quad (27)$$

Clearly $\sin\epsilon$ is maximized for $\gamma\rightarrow0$ and $\psi=\pi/4$. For these values (26) becomes

$$n[(1-n)/2]h^2+[(1+n)/4][2H_{c2}-H_0]h+H_{c2}(H_{c2}-H_0)=0 \quad (28)$$

and (27) takes the form

$$\sin\epsilon=(3n-1)h/[h(1+n)+4H_{c2}]. \quad (29)$$

Next we note that Abrikosov's formula (12) and hence the subsequent analysis is valid only where $H_{c2}-H_0$ is small compared to H_{c2}. In this limit, (28) and (29) give

$$|\sin\epsilon|\approx\left|\frac{3n-1}{n+1}\right|\frac{H_{c2}-H_0}{H_{c2}}. \quad (30)$$

Thus since $0\leq n\leq1$ we see that ϵ is indeed small for all configurations, so that we may take (26) to be applicable for all cases of practical interest.

The general expression for the torque follows from (20) and is given by

$$4\pi\tau=V(3n-1)\left(\frac{H_0^2\sin2\psi}{4}\right)\times\frac{h^2}{[(\gamma+n)h+H_{c2}][[\gamma+(1-n)/2]h+H_{c2}]}. \quad (31)$$

A special case of interest is that of a disk-shaped foil which is thin compared to its lateral dimensions. The foil is well approximated by an exceedingly oblate spheroid in which case one finds[10]

$$n\approx1-\pi t/2a, \quad (32)$$

where t is the thickness and a is the "radius" of the foil. If $\psi\approx\pi/2$, the foil is aligned nearly parallel to H_0 and one finds

$$\text{(Meissner state)}\quad 4\pi\tau=2H_0^2[(\pi/2)-\psi]a^3, \quad (33)$$

where we have approximated the area of the foil by πa^2. Therefore, one has the interesting result that the torque on a type-I superconductor is proportional to a fictitious volume[10] $V/(1-n)\approx2a^3$.

Since the sample makes a finite angle with H_0, it is likely that the intermediate state will set in slightly below H_c. The torque at this point *changes sign* and becomes

$$4\pi\tau=-4(H_c-H_0)H_0(\pi/2-\psi)a^3. \quad (34)$$

If $\gamma\neq0$, the foil will undergo a transition to the *mixed phase* at an intermediate field, and in the vicinity of H_{c2}, Eqs. (13) to (26) give

$$4\pi\tau=\frac{V(H_{c2}-H_0)^2H_0(\pi/2-\psi)}{\gamma[(\gamma+1)H_0-H_{c2}]}. \quad (35)$$

Note that the denominator of (35) is positive for H_0 near H_{c2} because $\gamma>0$. Since the torque is positive initially in the Meissner state, the torque on a type-II superconductor will either exhibit two sign changes or none at all, depending on whether the value of H_0 at which (35) changes sign is close enough to H_{c2} so that the Abrikosov formula will apply. Incidently, for the mixed state, we see that the torque is proportional to the true volume.

[10] L. D. Landau and E. M. Lifschitz, *Electrodynamics of Continuous Media* (Pergamon Press, Inc., New York, 1960).

A foil situated almost perpendicular to the applied field, on the other hand, is expected to enter the intermediate state at extremely low fields. One finds

$$4\pi\tau(\gamma=0)=V(H_c-H_0)^2H_0\psi/H_c\,, \tag{36}$$

$$4\pi\tau(\gamma\neq 0)=V\frac{(H_{c2}-H_0)^2H_0\psi}{(\gamma+1)(\gamma H_0+H_{c2})}\,. \tag{37}$$

We emphasize that these results pertain to a superconductor for which there is no superconducting sheath. It is possible to derive from the Ginzburg-Landau theory analytic expressions[11] for $M(H_0)$ for a semi-infinite foil in the vicinity of H_{c3}, and thus one might compute the behavior of the magnetization and torque of a finite sample in the manner described in the foregoing. However, the bulk of the evidence thus far indicates that for real finite specimens, the sheath exhibits characteristics of multiply connected superconductors and thus behaves in a characteristically *irreversible* manner[12–16] unlike the predictions of the present theory. Our results are probably comparable with experiment only where the sheath has been suppressed by some means, e.g., by plating with a normal metal (Cu, Cr, etc.).

Specific Heat

Another case in which the shape of the specimen may effect the interpretation of experimental data is in the measurement of the specific heat in an applied field near the transition to the normal state. Elementary thermodynamic considerations lead to the conclusion that the discontinuity in the specific heat at the transition point is given by

$$(C_s-C_n)_{T_cH}=T_c(H_0)\{(\chi_s-\chi_n)_{H_{\rm crit}}(dH_{\rm crit}/dT)^2{}_{T_c(H_0)} + (M_s-M_n)_{H_{\rm crit}}(d^2H_{\rm crit}/dT^2)_{T_c(H_0)}\}\,, \tag{38}$$

where $T_c(H_0)$ is the transition temperature in the field H_0, $H_{\rm crit}$ is either H_{c2} or H_c, M is the magnetization, and χ stands for MdH_0. For the transition at H_{c2}, $M_n-M_s=0$, so one has

$$\frac{\Delta C_{ns}}{T_c(H_0)}=\left(\frac{dH_{c2}}{dT}\right)^2_{T_c(H_0)}\frac{1}{4\pi(\gamma+n)}\,. \tag{39}$$

Thus the jump in the specific heat depends on the shape of the sample. As is well known, this formula does not apply when $H_0=0$ for the fundamental reason that the penetration depth tends to ∞ at the zero-field transition temperature T_{c0}. One regains instead the Rutgers formula

$$\frac{\Delta C}{T_{c0}}=\frac{1}{4\pi}(dH_c/dT)^2\,. \tag{40}$$

For a type-I superconductor for which $\gamma=0$, the specific-heat jump when measured in a nonzero field should be

$$\frac{\Delta C}{T_c(H_0)}=\frac{1}{4\pi n}(dH_c/dT)^2{}_{T_c(H_0)} \tag{41}$$

which is the well-known result.[17]

SUMMARY AND DISCUSSION

We have considered substances whose magnetization exhibits a well-defined relationship with an applied magnetic field and described the manner in which a sample's geometry affects this magnetization. We have sought to establish with some rigor what minimal assumptions are required to arrive at the fundamental relationship, Eq. (8), between $\mathbf{M}$ and $\mathbf{H}_0$. Whereas this relationship has been conventionally deduced by assuming uniform magnetization, in the present case we have found that it can be derived from Maxwell's equation if:

(1) there is a smooth single-valued functional dependence of the magnetization $\mathbf{M}$ on the local field $\mathbf{B}$;

(2) the medium is isotropic and homogeneous so that the functional dependence is a scalar one; and, as usual,

(3) macroscopic boundary conditions on $\mathbf{B}$ and $\mathbf{H}$ are imposed.

It should be pointed out that assumption (2), which forces $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ to be parallel in all cases, can be generalized to a vector functional dependence and the present procedure followed in otherwise the same manner. This would allow for anisotropy as required for crystalline substances though the analysis would be quite messy for anything but the simplest cases.

Our deduction that the magnetization is spatially uniform stems from assumption (1) where we have $\mathbf{M}$ depending only implicitly on position through the local field vector $\mathbf{B}$, that is, $\mathbf{M}(\mathbf{r})=\mathbf{M}[\mathbf{B}(\mathbf{r})]$. This would be the case if a strictly local electrodynamics [i.e., $\mathbf{j}(\mathbf{r})=c\nabla\times\mathbf{M}\propto\mathbf{A}(\mathbf{r})$, where $\mathbf{B}=\nabla\times\mathbf{A}$] were employed in arriving at $\mathbf{M}(\mathbf{B})$. On the other hand, a fairly simple example of an implicitly nonlocal dependence is the Ginzburg-Landau theory for which $j(\mathbf{r})\propto|\psi(\mathbf{r})|^2\mathbf{A}(\mathbf{r})$. Here the nonlocality can be seen from the fact that $\psi(\mathbf{r})$ is the solution of a differential equation depending on $\mathbf{A}(\mathbf{r})$ and hence is expressible as the solution of an integral equation involving $\mathbf{A}(\mathbf{r}')$. In a case of this kind, one might expect $\mathbf{M}$ to depend explicitly on $\mathbf{r}$ as well as

[11] M. J. Zuckerman, Phys. Letters 13, 277 (1964); J. A. Cape (unpublished).

[12] J. D. Livingston and H. W. Schadler, Progr. Mater. Sci. 12, 185 (1964).

[13] E. J. Sandiford and D. G. Schweitzer, Phys. Letters 13, 98 (1964).

[14] H. J. Fink, Phys. Letters 19, 364 (1965).

[15] H. J. Fink, Phys. Rev. Letters 16, 447 (1966).

[16] L. J. Barnes and H. J. Fink, Phys. Letters 20, 583 (1966).

[17] Type-I behavior such as predicted by Eq. (40) has been discussed in detail by D. Shoenberg [*Superconductivity* (Cambridge University Press, New York, 1960)].

on $\mathbf{B}(\mathbf{r})$. It is well known, of course, that the Ginzburg-Landau theory predicts a nonuniform magnetization for the mixed state and intermediate state of superconductors. Nevertheless, in the present analysis we have applied the transformation to the mixed-state magnetization of type-II superconductors and deduced the same result, Eq. (18), as that derived by much more tedious means from the Ginzburg-Landau theory by Maki in the particular case of a sphere. This is possible because the sample dimensions are large compared to the penetration depth and hence also to the scale of the microscopic structure.

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APPENDIX: DERIVATION OF THE TRANSFORMATION

We note that because of Eq. (7), if $\mathbf{M}$ is given explicitly as a function of an arbitrary point $\mathbf{r}$ within the magnetized specimen, then the $\mathbf{H}$ field will be uniquely determined if we specify its value at infinity to be $\mathbf{H}_0$. This unique field moreover satisfies the usual integral equation

$$\mathbf{H}(\mathbf{r})=\mathbf{H}_0-\nabla\int_{S^*}dV'\,R^{-1}\nabla'\cdot\mathbf{M}(\mathbf{r}')\,, \qquad \text{(A1)}$$

where

$$R\equiv|\mathbf{r}-\mathbf{r}'|\,,$$

and the integration is performed over a region S^* which contains all source points $\mathbf{r}'$ in the specimen S.

Suppose, however, that the magnetization $\mathbf{M}$ is known as a single-valued function $f(\mathbf{B})$ of the local field $\mathbf{B}$. Then from the implicit function theorem the function $h(\mathbf{M},\mathbf{H})$ defined by

$$h(\mathbf{M},\mathbf{H})\equiv\mathbf{M}-f(\mathbf{H}+4\pi\mathbf{M})=0 \qquad \text{(A2)}$$

can be solved for $\mathbf{M}$, say $\mathbf{M}=g(\mathbf{H})$, as long as h has continuous derivatives with respect to both its arguments and $\partial h/\partial M\neq 0$ throughout S. This implies that the originally selected relation f must be continuously differentiable and must satisfy $f'\not\equiv 1/4\pi$ (f' denoting differentiation of f with respect to its single argument).

When these conditions are satisfied, (A1) takes the form

$$\mathbf{H}(\mathbf{r})=\mathbf{H}_0-\nabla\int_S dV'\,R^{-1}\nabla'\cdot g[\mathbf{H}(\mathbf{r}')]\,. \qquad \text{(A3)}$$

It is not hard to show that when $g'(\mathbf{H})$ satisfies a Lipschitz condition over S^* [which it will when $f'(\mathbf{B})$ does], then the integral equation (A3) has a unique solution $\mathbf{H}(\mathbf{H}_0)$ in the class of continuously differentiable vector functions which approach $\mathbf{H}_0$ for large $\mathbf{r}$. But consider the case of uniform $\mathbf{H}$. Then $\mathbf{M}=g[\mathbf{H}(\mathbf{r})]=g(\mathbf{H})$ is also uniform and hence

$$\mathbf{H}=\mathbf{H}_0-\nabla\int_S dV'\,R^{-1}\nabla'\cdot\mathbf{M}=\mathbf{H}_0+\left(\nabla\nabla\int_S R^{-1}dV'\right)\cdot\mathbf{M} \qquad \text{(A4)}$$

where $\nabla\nabla$ is to be interpreted as a dyadic product and $\hat{n}$ is a unit outward normal. Now if S is an ellipsoid it can be shown that the configuration matrix $\mathbf{N}$ given by

$$\mathbf{N}\equiv-\frac{1}{4\pi}\nabla\nabla\int_S R^{-1}dV' \qquad \text{(A5)}$$

is independent of $\mathbf{r}$ inside or on S. Thus we find

$$\mathbf{H}=\mathbf{H}_0-4\pi\mathbf{N}\cdot\mathbf{M} \qquad \text{(A6)}$$

which together with the constitutive relation $\mathbf{M}=g(\mathbf{H})$ determines $\mathbf{H}=\mathbf{H}(\mathbf{H}_0)$ and thus $\mathbf{M}=g[\mathbf{H}(\mathbf{H}_0)]$ as functions of $\mathbf{H}_0$. Since the solution to (A3) is unique, (A6) must be this solution.

Thus, we conclude that subject only to the stated smoothness conditions on $\mathbf{M}=f(\mathbf{B})$, the fields $\mathbf{M}$ and $\mathbf{H}$ are uniform inside ellipsoidal specimens and satisfy (A6). The latter may be written in its well-known component form

$$H_i=H_{0i}-4\pi n_{ij}M_j \qquad \text{(A7)}$$

which we have made use of in the text.