If we assume that the line shape is approximately Gaussian, (12) indicates that $\tau^{-1}\rightarrow 0$ as $H_1^2\rightarrow 2\Delta^2$ since the line-shape factor falls off more rapidly than the other factors. However, this result rests on the spin-temperature assumption embodied in (4), which may not be justified in this region where the secular term $D'$ is small. In its dependence on $H_1$, $\tau^{-1}$ shows an interesting transition, being proportional to $H_1^2$ for small $H_1$ but approaching zero for large $H_1$, which reflects the well-known fact that when $H_1\gg H_L$ the Zeeman and spin-spin systems are decoupled.

Although for simplicity we have treated the case of pure dipolar interaction, (10) applies also if exchange interactions of the form $A=\sum_{j>k} A_{jk}\mathbf{I}_j\cdot\mathbf{I}_k$ are present. The main difference is that $A$ should be added to $D$ and $D'$. The condition $\Delta'\gg H_L$ for the neglect of $D''$ because of its nonsecular character is also affected by the contribution of the exchange interactions to $H_L$.

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# Photon-Counting Distributions of Modulated Laser Beams

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The first explicit experimental verification is given of Mandel's formula relating the photon-counting distribution of a radiation field to its intensity fluctuations. This was done by measuring the counting distributions obtained from a He–Ne gas-laser beam having superimposed square- and sine-wave modulations of various depths and frequencies.

THE photon-counting distribution of a radiation field is related theoretically to the intensity fluctuations of the field by a well-known formula first derived by Mandel.[1] Previously this formula has been used to explain certain observed photon-counting distributions from theoretical models of the radiation field and its intensity fluctuations, e.g., to explain in terms of narrow-band Gaussian light the geometric counting distribution observed in appropriate conditions from the beam of a gas laser operating just below threshold.[2] No explicit experimental verification of the formula has so far been given. In this article we present experimental data which for the first time provides such a verification. These data were obtained by measuring the counting distributions obtained from a He–Ne gas-laser beam having superimposed square- and sine-wave modulations of various depths and frequencies.

For our purposes, it is convenient to write Mandel's formula for the probability $p(n,T)$ of $n$ photoelectrons being emitted in a sampling time $T$ as

$$p(n,T)=\int_0^\infty \frac{(r_T T)^n}{n!}e^{-r_T T}P(r_T)dr_T\,, \tag{1}$$

where

$$r_T(t)=\frac{1}{T}\int_t^{t+T}[r_0+\epsilon f(\tau)]d\tau \tag{2}$$

is the effective photoelectron emission rate over a sampling time $T$ at $t$ and $P(r_T)$ is the probability distribution of this quantity for random samples. The instantaneous emission rate is expressed as the sum of a constant rate $r_0$ plus a modulation whose form and frequency are determined by $f(\tau)$ and whose depth is characterized by $\epsilon$. By introducing the central moments $\mu_s$ of the distribution $P(r_T)$, i.e.,

$$\mu_s=\int_0^\infty (r_T-r_0)^s P(r_T)dr_T\,, \tag{3}$$

Eq. (1) can be expressed as a series in $\mu_s/r_0^s$, viz.,

$$p(n,T)=\frac{\bar{n}_0^n}{n!}e^{-\bar{n}_0}+\sum_{k=0}^{\infty}(-1)^k\frac{\bar{n}_0^{n+k}}{n!k!}\sum_{s=2}^{n+k}{}^{n+k}C_s\frac{\mu_s}{r_0^s}\,, \tag{4}$$

where $\bar{n}_0=r_0T$. This expression clearly shows the Poisson form of $p(n,T)$ in the absence of modulation.

Table I. The relationship between $\mu_2/r_0^2$ and $(\epsilon/r_0)^2$ for various modulation forms for the case of the modulation varying slowly with respect to the sampling time.

| Modulation | $(\mu_2/r_0^2)/(\epsilon/r_0)^2$ |
|---|---|
| Square wave with amplitude $\epsilon$ | 1 |
| Sine wave with amplitude $\epsilon$ | $\frac{1}{2}$ |
| Sawtooth with amplitude $\epsilon$ | $\frac{1}{3}$ |
| Gaussian noise with standard deviation $\epsilon$ | 1 |

[1] L. Mandel, Proc. Phys. Soc. (London) 72, 1037 (1959).
[2] C. Freed and H. Haus, Phys. Rev. Letters 15, 943 (1965); A. W. Smith and J. A. Armstrong, Phys. Letters 19, 650 (1966).

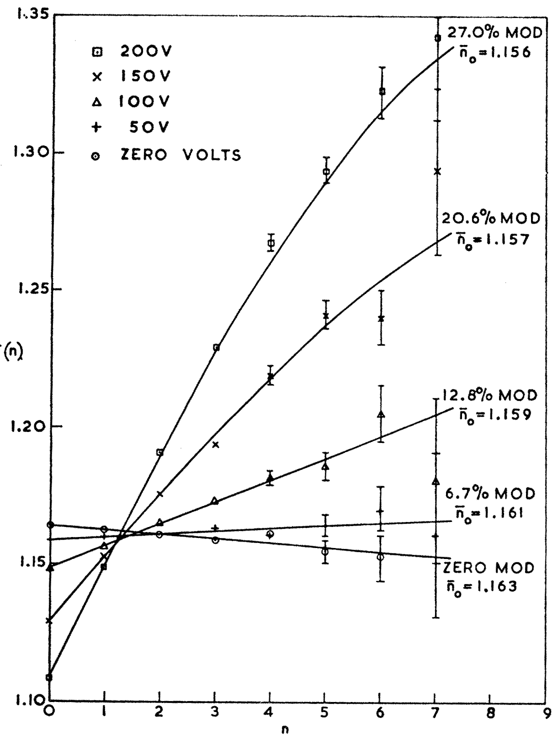


FIG. 1. The experimentally determined values of $F(n)$ as a function of $n$ for various 50-cps sine-wave voltages on the KDP modulator. The solid lines show the theoretical fit obtained to the data using a common dead time of 6.5 nsec and values of $\bar{n}_0$ and modulation coefficient shown beside each curve.

The result (4) is particularly useful when we are dealing with small modulations, for $\mu_s/r_0{}^s \sim (\epsilon/r_0)^s$, so that (4) is effectively a power-series expansion in the modulation coefficient $\epsilon/r_0$. The form of the modulation and its frequency relative to the sampling time determine the proportionality constant connecting $\mu_s/r_0{}^s$ and $(\epsilon/r_0)^s$; some examples are given in Table I.

It is convenient in studying the modulation effects on $p(n,T)$ to remove the ever-present Poisson term by working with the associated quantity[3] $F(n) \equiv (n+1)p(n+1, T)/p(n,T)$ which has the constant value $\bar{n}_0$ in the absence of modulation. It follows from (4) that for small enough modulations $F(n)$ has the simple form

$$F(n)=\bar{n}_0\left[\left(1-\bar{n}_0\frac{\mu_2}{r_0{}^2}\right)+\frac{\mu_2}{r_0{}^2}n\right], \tag{5}$$

which shows a linear rise of $F(n)$ with $n$ at a rate proportional to the square of the modulation coefficient.

Our experimental measurements were made using the output of a Spectra Physics 119 He–Ne gas laser operating well above threshold. The laser beam was passed through a suitably arranged potassium dihydrogen phosphate (KDP) electro-optic modulator system by means of which intensity fluctuations of variable size, shape, and frequency could be superimposed on to the laser beam by applying the appropriate voltage modulations to the KDP. The associated photon-counting distribution was measured using a photomultiplier and the recording system previously described.[3] A sampling time of $10^{-5}$ sec was used to help minimize dead-time effects[4] and exactly $10^8$ samples were taken for each distribution. Figure 1 shows our results for the laser beam both unmodulated and modulated with a slow 50-cps sine-wave voltage of increasing

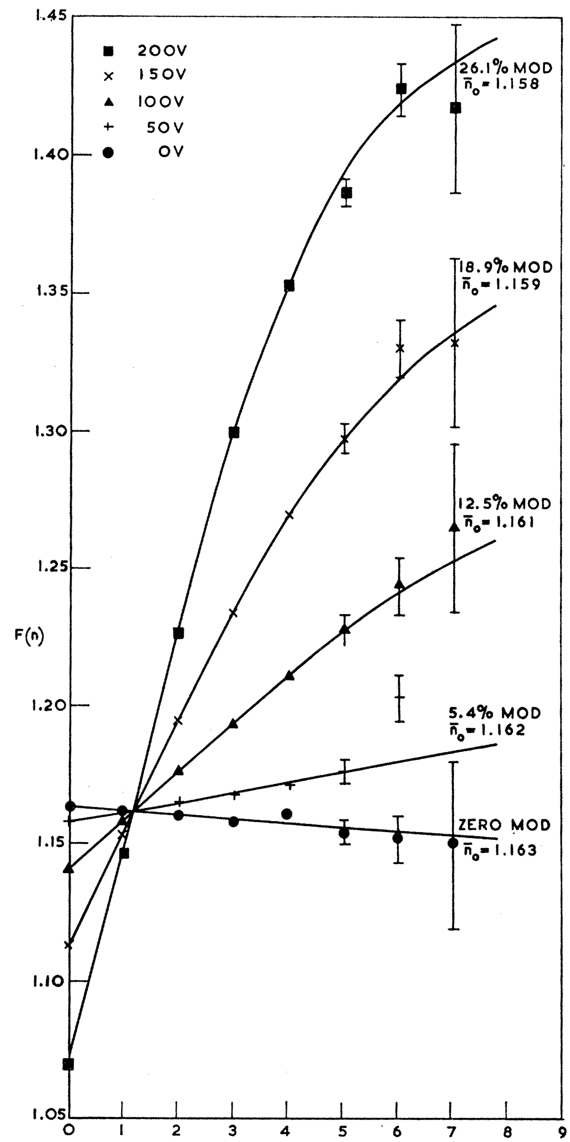


FIG. 2. The experimentally determined values of $F(n)$ as a function of $n$ for various 50-cps square-wave voltages on the KDP modulator. The solid lines show the theoretical fit obtained to the data using a common dead time of 6.5 nsec and values of $\bar{n}_0$ and modulation coefficient shown beside each curve.

[3] F. A. Johnson, T. P. McLean, and E. R. Pike, *Proceedings of the International Conference on the Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), p. 706.

[4] F. A. Johnson, R. Jones, T. P. McLean, and E. R. Pike, Phys. Rev. Letters **16**, 589 (1966).

magnitude applied to the KDP modulator. The experimental data show the expected linear dependence of $F(n)$ upon $n$ for small modulations with a slope increasing with modulation depth. The small negative slope of the straight line defined by the data for zero modulation arises from dead-time effects in our recording system[4] and can be shown to arise from an effective dead time of 6.5 nsec.[5] For the slow modulations with which we are dealing, this dead-time effect can be incorporated accurately into the analysis of our data.[4] The solid lines in Fig. 1 show the excellent fit which can be obtained to the experimental data using (4) for a sine-wave modulation appropriately corrected for dead time, choosing in each case values only for $\bar{n}_0$ and the modulation depth.

Figure 2 shows similar data for a 50-cps square-wave modulation. The fit of theory to experiment is again excellent. The two modulation coefficients at each voltage, obtained by fitting the theory to the square- and sine-wave data show a common linear dependence on voltage as expected to within the estimated 5% accuracy of the modulation.[6]

For a sine-wave modulation of arbitrary frequency $f$, the instantaneous photoemission rate can be written as $r_0+\epsilon\sin(2\pi ft+\theta)$ and (2) gives

$$r_T(t)=r_0+\epsilon\left(\frac{\sin\pi fT}{\pi fT}\right)\sin(2\pi ft+\pi fT+\theta). \tag{6}$$

We see therefore that the effective emission rate $r_T$ is also sine-wave modulated but with a modulation coefficient $|\epsilon\sin\pi fT/r_0\pi fT|$ which is a function of the product of the modulation frequency $f$ and the sampling time $T$. For small modulations, $F(n)$ should, therefore, be linearly related to $n$ with a slope equal to

$$\bar{n}_0\frac{\mu_2}{r_0^2}=\tfrac{1}{2}\bar{n}_0\left(\frac{\epsilon}{r_0}\right)^2\left(\frac{\sin\pi fT}{\pi fT}\right)^2. \tag{7}$$

[5] The actual dead time was 12 nsec and a negative slope corresponding to this value could be obtained by working at very low efficiency and with very short sampling time. In conditions outside these limits we compensate the lower negative slope by use of an "effective" dead time which includes single after-pulsing effects.

[6] Peak-to-peak voltages were set by displaying the wave form on an oscilloscope screen. Neither the sine or the square wave were ideal due to quadratic and higher terms introduced by the electro-optic system in the sine wave and the finite rise time of the square wave.

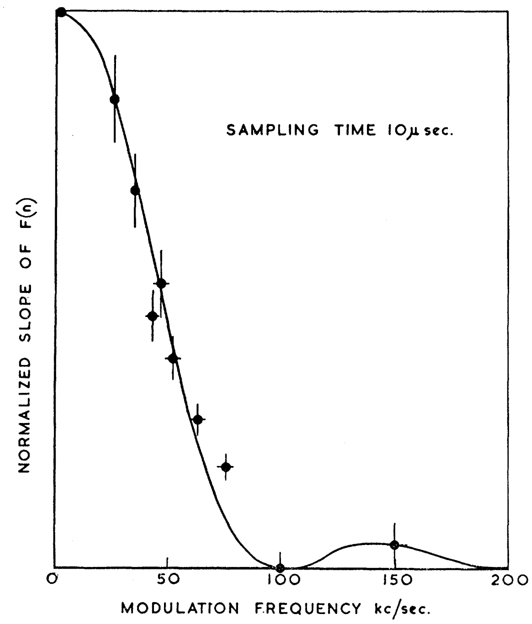


FIG. 3. The slope of $F(n)$ versus $n$ as a function of sine-wave modulation frequency. The slope values are measured relative to the value for a slow 50-cps modulation and all runs were taken using the same value of $\bar{n}_0$. The solid line shows the theoretically predicted variation as given by Eq. (7).

The zeros of this slope occur at frequencies for which the sampling time contains an integral number of periods of the modulation; the modulation effect then averages to zero in $r_T$ and is effectively absent as far as the counting distribution is concerned. Figure 3 shows the experimentally determined slopes of $F(n)$ for 100-V sine-wave modulation on the KDP modulator as a function of the modulation frequency. The slopes are measured[7] relative to that for the slow 50 cps modulation for which the $(\sin\pi fT/\pi fT)^2$ factor in (7) is effectively unity. The solid line shows the frequency dependence of this slope as predicted by (7); it is in satisfactory agreement with the experimental points.

[7] Dead-time effects have not been explicitly taken into account in this treatment of the data, since no adequate theory of the effects currently exists for this situation in which the radiation intensity is varying over a sampling time. However, the effects are small in this case and they should be adequately dealt with by measuring all shapes relative to that for a slow modulation.