Low-Lying Superfluid States in a Rotating Annulus*

ALEXANDER L. FETTER
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
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The low-lying states of rotating liquid He II in an annulus ($R_1 < r < R_2$) are studied with the model of a classical inviscid fluid. An exact hydrodynamic solution is obtained with the method of images for a system consisting of rectilinear vortices with circulation κ combined with circulation Γ_1 about the inner cylinder. The energy and angular momentum are calculated, both for an arbitrary configuration of vortices and for the particular configuration of a symmetric ring of l vortices. If Γ_1 is treated as a variational parameter, the critical angular velocity Ω_0 for the appearance of vortices in a narrow annulus is $(\kappa/\pi d^2)\ln(2d/\pi a)$, where d is the width of the annulus and a is the radius of the vortex core. For $\Omega < \Omega_0$, the equilibrium state is an irrotational (vortex-free) flow with quantized circulation $n\kappa (n=1, 2\cdots)$; these levels are equally spaced, and a given quantum state represents the lowest free energy only in a narrow angular-velocity interval of $\kappa/2\pi R^2$, where R is the mean radius of the annulus. The maximum quantum number of irrotational circulation is $2\pi R^2\Omega_0/\kappa = 2(R/d)^2\ln(2d/\pi a) \gg 1$. For $\Omega > \Omega_0$, the vortices lie on the circumference of a ring midway between the walls, and the number of vortices increases rapidly with Ω. If Γ_1 is constrained to vanish identically, the critical angular velocity Ω_c for the appearance of vortices in a narrow annulus is of order $\kappa/2\pi Rd$; this is equivalent to Feynman's critical velocity $v_c = O(\hbar/md)$ for singly quantized vortices with $\kappa = h/m$. In the opposite limit of a wide annulus ($R_1 \ll R_2$), the equilibrium state is shown to agree with Vinen's earlier calculations.

I. INTRODUCTION

THE classical theory of vortices was rejuvenated by the suggestion of quantized circulation in liquid He II.[1,2] Since the superfluid flow is necessarily irrotational,[2] the circulation around a contour C can be nonzero only if C encloses either an internal boundary or a singularity in the flow pattern. In liquid He II, such a singularity represents a vortex with circulation $\kappa = h/m$, where h is Planck's constant and m is the mass of a helium atom.[3] The only allowed states in a simply connected domain are various configurations of vortices. In a multiply connected domain, additional states are possible, consisting of circulation about the internal boundaries combined with vortices in the bulk of the fluid. The circulation Γ_α about the αth boundary is also quantized in units of h/m;[4] in contrast to the vortices in the fluid, the quantum number $n_\alpha = \Gamma_\alpha m/h$ can be much larger than one.[5]

Rectilinear vortices in liquid He II are usually associated with rotation of the superfluid. At a fixed angular velocity Ω, the equilibrium configuration of vortices and circulation can be found only from a detailed calculation for the particular container geometry in question. The simplest example is a cylinder of radius R, where the critical angular velocity for the appearance of vortices with circulation κ is $(\kappa/2\pi R^2) \times\ln(R/a)$, a being the radius of the vortex core.[6] A related problem of interest is the equilibrium state in an annular region $R_1 < r < R_2$. Experiments[7] indicate that rotating He II in a narrow annulus remains free of vortices for much higher angular velocities than in a simply connected cylinder; furthermore, the fluid flow at these angular velocities appears to be irrotational with a large circulation $\Gamma_1 \gg \kappa = h/m$ about the inner cylinder.[5] In order to show that these experimental results are natural consequences of the Onsager-Feynman vortex hypothesis, we here present a hydrodynamic calculation of the possible low-lying states of liquid He II in an annulus. The results depend on the presence or absence of circulation Γ_1 about the inner cylinder. If Γ_1 can change with the external angular velocity, then the critical angular velocity Ω_0 for the appearance of vortices in a narrow annulus ($R_2 - R_1 \ll R_2 + R_1$) is given by $\Omega_0 = (\kappa/\pi d^2)\ln(2d/\pi a)$, where $d = R_2 - R_1$ is the width of the annulus. At this angular velocity, the equilibrium value of the circulation about

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[1] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

[2] R. P. Feynman, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

[3] Although multiply quantized physical vortices are possible in principle, they require appreciably higher energy and have never been observed.

[4] Detailed experiments have been carried out for a wide annulus consisting of a cylinder containing a fine wire along its axis. Only the quantum state $n=1$ was detected in the original experiment of W. F. Vinen, Proc. Roy. Soc. (London) **A260**, 218 (1961); in more recent studies, S. C. Whitmore and W. Zimmerman, Jr., Phys. Rev. Letters **15**, 389 (1965), have also observed the values $n=2$ and 3.

[5] P. J. Bendt, Phys. Rev. **127**, 1441 (1962), has reported irrotational flow in a narrow annulus with circulation much larger than h/m.

[6] See, for example, G. B. Hess and W. M. Fairbank, in *Low Temperature Physics LT9*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, New York, 1965), Part A, p. 188.

[7] D. S. Tsakadze, Zh. Eksperim. i Teor. Fiz. **46**, 505 (1964) [English transl.: Soviet Phys.—JETP **19**, 343 (1964)], has reported irrotational (vortex-free) flow in an annulus $R_1 \approx 0.5$ cm, $R_2 \approx 0.8$ cm up to angular velocities the order of 1 rad/sec. This experiment is cited as evidence that the He II in the annulus forms an inner irrotational region surrounded by a dense array of vortices. Since the outer region was not observed directly, however, it appears possible to interpret the above experiment as a measurement of the critical angular velocity for creation of vortices in an annulus. Preliminary findings of D. J. Tanner, B. E. Springett, and R. J. Donnelly, in *Low Temperature Physics LT9*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, New York, 1965), Part A, p. 346, seem to confirm the absence of vortices in a narrow annulus at angular velocities the order of 1 rad/sec.

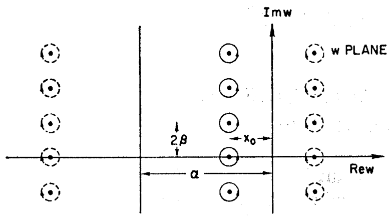

FIG. 1. Distribution of images for an infinite row of vortices in a channel.

the inner cylinder is $2\pi R_1{}^2\Omega_0 = 2\kappa(R_1/d)^2 \ln(2d/\pi a)$, which is much larger than κ. On the other hand, if Γ_1 is constrained to vanish, then the critical angular velocity Ω_c for the appearance of vortices in a narrow annulus is given by $\Omega_c = O(\kappa/2\pi R_1 d)$, which is equivalent to Feynman's critical velocity[2] $v_c = O(\hbar/md)$ with $\kappa = h/m$. For a wide annulus ($R_1 \ll R_2$), our expressions reproduce Vinen's calculations[4] for the same geometry. In the limit $R_1 \to 0$, we recover the standard results for a simply connected cylinder of radius R_2.[6]

The complex potential for a system of rectilinear vortices in an annulus is derived in Sec. II with the method of images. Section III contains the calculation of various physical quantities: the fluid velocity **v**, the circulation $\Gamma(r)$ about a circle of radius r, the angular momentum **L**, and the energy E of the total system. The low-lying states are then computed by minimizing the free energy $F = E - \Omega L$. The circulation Γ_1 may be treated as continuous for a narrow annulus, and it is necessary to consider both unrestricted variations with respect to Γ_1 (Sec. IV) and restricted variations, where Γ_1 vanishes (Sec. V); for a wide annulus, Γ_1 is discrete and each quantum state must be treated separately (Sec. VI).

II. METHOD OF IMAGES APPLIED TO AN ANNULUS

A system of rectilinear vortices in an incompressible fluid confined to a finite region forms one of the standard boundary-value problems of classical mathematical physics. The essential difficulty lies in satisfying the condition that the normal component of fluid velocity vanish at the boundary[8]; a convenient approach in such problems is the method of images. For a vortex inside a cylinder, only a single image is required. In an annulus, however, an infinite series of images in needed, since there are two boundaries. It is simplest to solve this problem in two stages: we first consider a line of equally spaced vortices between parallel boundaries, shown in Fig. 1. The solution for the annulus can then be obtained with an elementary conformal transformation.[9] The vortices in Fig. 1 are a distance 2β apart, while the separation between the boundaries is α. The complex plane illustrated in Fig. 1 will be called the w plane, so that the boundaries are specified by $\mathrm{Re}w = 0$ and $\mathrm{Re}w = -\alpha$. If x_0 is the distance from the row of vortices to the right boundary, the doubly infinite set of images is specified by the following set of points:

$$\begin{aligned} &\text{positive vortices at} \quad -x_0 + 2\alpha m + 2i\beta n\,, \\ &\text{negative vortices at} \quad x_0 + 2\alpha m + 2i\beta n\,, \end{aligned} \tag{1}$$

where m and n take all integral values.

The complex potential in the w plane may be constructed with the theta function $\vartheta_1(w\,|\,\tau)$.[10,11] This is an integral function of w with a doubly periodic array of simple zeros at the points

$$w = \pi m + \pi\tau n\,, \tag{2}$$

where τ is a complex number with positive imaginary part. Thus the function

$$\frac{\vartheta_1[(\pi/2\alpha)(w+x_0)\,|\,i\beta/\alpha]}{\vartheta_1[(\pi/2\alpha)(w-x_0)\,|\,i\beta/\alpha]} \tag{3}$$

has a simple zero at the position of each positive vortex and a simple pole at the position of each negative vortex. The complex potential $f_0(w)$ is proportional to the logarithm of this function

$$f_0(w) = \frac{i\kappa}{2\pi} \ln\left\{\frac{\vartheta_1[(\pi/2\alpha)(w+x_0)]}{\vartheta_1[(\pi/2\alpha)(w-x_0)]}\right\}; \tag{4}$$

here and subsequently, the dependence on the parameter τ will be omitted whenever $\tau = i\beta/\alpha$.

The transformation to an annular region in the z plane may be performed with the substitution

$$w = \ln(z/R_2)\,, \tag{5}$$

where $z = re^{i\varphi}$; the lines $\mathrm{Re}w = \text{const}$ correspond to the circles $\ln(r/R_2) = \text{const}$. The constant α must be taken as $\alpha = \ln(R_2/R_1)$, while β is determined from the periodicity in the angle φ. If we consider a ring of l vortices symmetrically spaced at a radius r_l (Fig. 2), then $\beta = \pi/l$, and $x_0 = \ln(R_2/r_l)$. Substitution of Eq. (5) into Eq. (4) yields

$$f_1(z) = \frac{i\kappa}{2\pi} \ln\left\{\frac{\vartheta_1[\gamma \ln(z/r_l)\,|\,\tau/l]}{\vartheta_1[\gamma \ln(zr_l/R_2{}^2)\,|\,\tau/l]}\right\}, \tag{6}$$

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[8] C. C. Lin [*On the Motion of Vortices in Two Dimensions* (University of Toronto Press, Toronto, Canada, 1943)] has devised an elegant solution for this problem, based on Green's functions. Lin's formalism is most useful for general proofs and has been used to study the equilibrium configuration of many rectilinear vortices in an arbitrary container [A. L. Fetter, Phys. Rev. **152**, 183 (1966)]. For simplicity, the present paper starts from first principles, but it can also be considered as a special application of Lin's theory; the relevant Green's function for an annular region is easily derived from our Eq. (12).

[9] The basic method used here is that of P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, pp. 1238–1243. Their complex potential differs from that derived here and appears to reduce to an incorrect limit as $\alpha \to \infty$, β and x_0 remaining finite.

[10] L. Rosenhead, Phil. Trans. Roy. Soc. (London) **A228**, 275 (1929), has treated the similar problem of the Kármán vortex street in a finite channel.

[11] All the relevant properties of the theta functions may be found in E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1962), 4th ed., Chap. XXI.

where

$$\gamma=\tfrac{1}{2}\pi[\ln(R_2/R_1)]^{-1}, \qquad (7)$$
$$\tau=i\pi[\ln(R_2/R_1)]^{-1}=2i\gamma.$$

Equation (6) cannot be the complete complex potential since, as shown in Sec. III, it implies a finite circulation about the inner cylinder. It is obvious physically that such a circulation cannot be induced merely by the presence of vortices in the annulus, so that a further image vortex must be placed at the center of the cylinder. In the simple case of a vortex outside of a cylinder ($R_2\to\infty$), this effect is well known.[12] The additional complex potential will be written as

$$f_2(z)=\frac{i\ln(z/R_2)}{2\pi}\left[\Gamma_1+l\kappa\frac{\ln(R_2/r_l)}{\ln(R_2/R_1)}\right]. \qquad (8)$$

Here, the first term in square brackets represents the physical circulation Γ_1 about the inner cylinder, and the second term represents the image vortex at the center. The total complex potential $f(z)$ for a ring of l vortices is the sum of the contributions from all the vortices including images,

$$f(z)=f_1(z)+f_2(z). \qquad (9)$$

A slightly different system is an array of identical vortices at arbitrary positions specified by the set $\{\mathbf{r}_j\}=\{r_j,\varphi_j\}$; a straightforward generalization of Eq. (9) shows that the complex potential in this case is

$$f(z)=\frac{i\Gamma_1}{2\pi}\ln(z/R_2)+\frac{i\kappa}{2\pi}\ln(z/R_2)\sum_j\frac{\ln(R_2/r_j)}{\ln(R_2/R_1)}+\frac{i\kappa}{2\pi}\sum_j\ln\left\{\frac{\vartheta_1[\gamma\ln(z/r_je^{i\varphi_j})\,|\,\tau]}{\vartheta_1[\gamma\ln(zr_j/R_2{}^2e^{i\varphi_j})\,|\,\tau]}\right\}, \qquad (10)$$

where τ is given in Eq. (7), and the sums run over all vortices. It can be shown by direct manipulation of the theta functions that Eq. (10) reduces to Eq. (9) when the array is a symmetric ring of l vortices.[13] These two equations represent exact solutions of the hydrodynamic boundary-value problem of vortices in an annulus and will now be used to compute the physical properties of the system.

III. PROPERTIES OF VORTICES IN AN ANNULUS

The following calculations will be based on the stream function $\psi(r,\varphi)$ defined as

$$\psi(r,\varphi)=\mathrm{Im}f(re^{i\varphi}). \qquad (11)$$

For most physical quantities, the ring of l vortices is merely a particular case of the system of many vortices and need not be treated separately. The one exception is the energy of a symmetric ring, which assumes an especially elegant form; this result is most simply derived directly from Eq. (9).

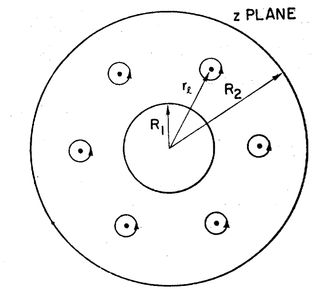

FIG. 2. Geometry of a ring of l vortices in an annulus, where l is taken as 6 for definiteness.

The stream function for a system of vortices in an annulus is obtained from Eqs. (10) and (11) as

$$\psi(r,\varphi)=\frac{\Gamma_1}{2\pi}\ln(r/R_2)+\frac{\kappa}{2\pi}\ln(r/R_2)\sum_j\frac{\ln(R_2/r_j)}{\ln(R_2/R_1)}+\frac{\kappa}{2\pi}\sum_j\mathrm{Re}\ln\left\{\frac{\vartheta_1[\gamma\ln(r/r_j)+i\gamma(\varphi-\varphi_j)]}{\vartheta_1[\gamma\ln(rr_j/R_2{}^2)+i\gamma(\varphi-\varphi_j)]}\right\}. \qquad (12)$$

Given ψ, the fluid velocity at the point (r,φ) may be calculated with the relations

$$v_r=-r^{-1}\partial\psi/\partial\varphi,\quad v_\varphi=\partial\psi/\partial r. \qquad (13)$$

We shall first verify that Eq. (10) represents the correct solution of the boundary-value problem by showing that the radial component of velocity vanishes at the boundary. When $r=R_2$, the stream function vanishes identically for all φ since ϑ_1 is an odd function of its function of its argument; it follows immediately that

$$v_r(R_2,\varphi)=-R_2{}^{-1}\partial\psi(R_2,\varphi)/\partial\varphi=0. \qquad (14)$$

When $r=R_1$, the first two terms of Eq. (12) are constant, while the last term vanishes.[14] Hence $\psi(R_1,\varphi)$ is independent of φ and $v_r(R_1,\varphi)=0$.

The tangential component of the velocity in the bulk of the fluid is important in calculating the circulation and the angular momentum. Differentiation of Eq. (12) yields

$$v_\varphi(r,\varphi)=\frac{\Gamma_1}{2\pi r}+\frac{\kappa}{2\pi r}\sum_j\frac{\ln(R_2/r_j)}{\ln(R_2/R_1)}+\frac{\kappa\gamma}{2\pi r}\sum_j\mathrm{Re}\left\{\frac{\vartheta_1'[\gamma\ln(r/r_j)+i\gamma(\varphi-\varphi_j)]}{\vartheta_1[\gamma\ln(r/r_j)+i\gamma(\varphi-\varphi_j)]}-\frac{\vartheta_1'[\gamma\ln(rr_j/R_2{}^2)+i\gamma(\varphi-\varphi_j)]}{\vartheta_1[\gamma\ln(rr_j/R_2{}^2)+i\gamma(\varphi-\varphi_j)]}\right\}, \qquad (15)$$

where the prime on ϑ_1 denotes differentiation with respect to its argument. The circulation $\Gamma(r)$ about a circle of radius r is defined as

$$\Gamma(r)=r\int_{-\pi}^{\pi}d\varphi\, v_\varphi(r,\varphi), \qquad (16)$$

[12] L. M. Milne-Thomson, *Theoretical Hydrodynamics* (Macmillan and Company, London, 1960), 4th ed., p. 362.

[13] It is necessary to use the infinite product representation of the theta function, Ref. 11, p. 469.

[14] This may be proved by repeated use of the relation $\vartheta_1(z+\frac{1}{2}\pi)=\vartheta_2(z)$, Ref. 11, p. 464.

which depends explicitly on the position of all the vortices. The relevant integral is evaluated in the Appendix; we find

$$\Gamma(r)=\Gamma_1+\kappa\sum_j\eta(r-r_j)\,, \tag{17}$$

where $\eta(x)$ is the step function $\eta(x)=\frac{1}{2}[1+(x/|x|)]$. Equation (17) shows that the circulation about the inner cylinder is Γ_1, which for the moment may be chosen arbitrarily. As expected from physical considerations, the circulation increases by κ each time the contour includes a new vortex.

Both the energy E and angular momentum $\mathbf{L}$ of the system must be calculated by integrating over the whole volume of the fluid:

$$E=\tfrac{1}{2}\rho\int d^2r\ v^2, \tag{18a}$$

$$\mathbf{L}=\rho\int d^2r\ \mathbf{r}\times\mathbf{v}. \tag{18b}$$

The final results depend parametrically on the position of each vortex and may be interpreted as the energy and angular momentum of a system of vortices. It is, however, incorrect to combine the contributions from the fluid and from the vortices. An analogous situation occurs in electrostatics, where the energy may be expressed either as the integral of the energy in the electric field or as the sum of Coulomb interaction energy between the charged particles. In the present geometry, $\mathbf{L}$ is parallel to the axis of rotation, and its magnitude is given by

$$L=\rho\int d^2r\ rv_\varphi. \tag{19}$$

The angular integral in Eq. (19) is proportional to the circulation, so that L may be rewritten as

$$L=\rho\int_{R_1}^{R_2} rdr\ \Gamma(r)=\tfrac{1}{2}\rho\Gamma_1(R_2^2-R_1^2)+\tfrac{1}{2}\rho\kappa\sum_j(R_2^2-r_j^2). \tag{20}$$

Equation (20) shows that the contribution of a given vortex to the angular momentum decreases as it moves away from the center of the system.

The calculation of the energy is more difficult since E is quadratic in the velocity field. Equation (18a) may be transformed by means of the stream function

$$\begin{aligned}E&=\tfrac{1}{2}\rho\int d^2r[v_y(\partial\psi/\partial x)-v_x(\partial\psi/\partial y)]\\&=\tfrac{1}{2}\rho\int d^2r\left\{\left[\frac{\partial}{\partial x}(\psi v_y)-\frac{\partial}{\partial y}(\psi v_x)\right]-\psi\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right\}\\&=\tfrac{1}{2}\rho\oint_{C_2}d\mathbf{l}\cdot\mathbf{v}\psi-\tfrac{1}{2}\rho\oint_{C_1}d\mathbf{l}\cdot\mathbf{v}\psi\\&\qquad-\tfrac{1}{2}\rho\int d^2r\ \psi\,|\mathrm{curl}\mathbf{v}|\,,\end{aligned} \tag{21}$$

where the contours C_1 and C_2 are circles of radii R_1 and R_2, taken in the positive sense. The second line of Eq. (21) is obtained by partial integration, while the last line is an application of Green's theorem.[15] The integral around C_2 vanishes because $\psi(R_2,\varphi)=0$; the integral about C_1 is

$$\begin{aligned}-\tfrac{1}{2}\rho\oint_{C_1}d\mathbf{l}\cdot\mathbf{v}\psi&=-\tfrac{1}{2}\rho\psi(R_1)\Gamma(R_1)\\&=(4\pi)^{-1}\rho\Gamma_1^2\ln(R_2/R_1)+(4\pi)^{-1}\rho\Gamma_1\kappa\sum_j\ln(R_2/r_j).\end{aligned} \tag{22}$$

The last term of Eq. (21) depends on the specific model adopted for the vortex core. We shall assume that the vorticity $\boldsymbol{\zeta}(\equiv\mathrm{curl}\ \mathbf{v})$ is constant over a circular core of radius a and vanishes outside of that region. Since the vortices do not overlap, the integral reduces to a sum of terms evaluated at the position of each vortex,

$$\int d^2r\ \zeta\psi=\sum_k\int_k d^2r\ \zeta\psi\,, \tag{23}$$

where the subscript means that the integral is taken over the core of the kth vortex. The stream function diverges logarithmically near the position of each vortex, and it is convenient to isolate the singular contribution of the kth vortex, which will be written as ψ_{0k}. Equation (23) thus becomes

$$\sum_k\int_k d^2r\ \zeta\psi=\sum_k\int_k d^2r\ \zeta[\psi-\psi_{0k}]+\sum_k\int_k d^2r\ \zeta\psi_{0k}. \tag{24}$$

In the limit of vanishing core radius, the first term of Eq. (24) may be evaluated by setting $\zeta=\kappa\delta(\mathbf{r}-\mathbf{r}_k)$; this yields

$$\kappa\sum_k\psi_{(k)}\,, \tag{25a}$$

where

$$\psi_{(k)}=\lim_{\mathbf{r}\to\mathbf{r}_k}[\psi(\mathbf{r})-(2\pi)^{-1}\kappa\ln(|\mathbf{r}-\mathbf{r}_k|/a)]. \tag{25b}$$

Substitution of Eq. (12) into Eq. (25) shows that

$$\begin{aligned}\psi_{(k)}=&\frac{\Gamma_1}{2\pi}\ln(r_k/R_2)+\frac{\kappa}{2\pi}\ln(r_k/R_2)\sum_j\frac{\ln(R_2/r_j)}{\ln(R_2/R_1)}\\&+\frac{\kappa}{2\pi}\sum_{j(\neq k)}\mathrm{Re}\ln\left\{\frac{\vartheta_1[\gamma\ln(r_k/r_j)+i\gamma(\varphi_k-\varphi_j)]}{\vartheta_1[\gamma\ln(r_kr_j/R_2^2)+i\gamma(\varphi_k-\varphi_j)]}\right\}\\&-\frac{\kappa}{2\pi}\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_k)]}{\vartheta_1'(0)}\frac{r_k}{\gamma a}\right\},\end{aligned} \tag{26}$$

where $\vartheta_1'(0)$ is the derivative of $\vartheta_1(z)$ evaluated at $z=0$. The second term of Eq. (24) represents the small

[15] See, for example, T. M. Apostol, *Mathematical Analysis* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1957), pp. 283–292. The derivation given here is a straightforward generalization of that of H. Lamb, *Hydrodynamics* (Dover Publication, Inc., New York, 1945), 6th ed., p. 229.

contribution to the energy from the core of each vortex. With the models of uniform vorticity, the integral is easily computed as[16]

$$\int_k d^2r\, \zeta\psi_{0k}=\tfrac{1}{4}(2\pi)^{-1}\kappa. \tag{27}$$

A combination of Eqs. (21), (22), (26), and (27) leads to the final form for the energy of the system

$$E=\frac{\rho\ln(R_2/R_1)}{4\pi}\left\{\Gamma_1+\kappa\sum_j\frac{\ln(R_2/r_j)}{\ln(R_2/R_1)}\right\}^2+\frac{\rho\kappa^2}{4\pi}\sum_{jk}{}' \times\operatorname{Re}\ln\left\{\frac{\vartheta_1[\gamma\ln(r_jr_k/R_2^2)+i\gamma(\varphi_k-\varphi_j)]}{\vartheta_1[\gamma\ln(r_k/r_j)+i\gamma(\varphi_k-\varphi_j)]}\right\}+\frac{\rho\kappa^2}{4\pi}\sum_j \times\left[\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_j)]}{\vartheta_1'(0)}\frac{r_j}{\gamma a}\right\}+\tfrac{1}{4}\right], \tag{28}$$

where the primed sum means that we omit the terms $j=k$. In the limiting case of a wide annulus ($R_1\rightarrow 0$), the fluid becomes simply connected, and the circulation Γ_1 must vanish. Equation (28) then reduces to[17]

$$E=(4\pi)^{-1}\rho\kappa^2\sum_{jk}\ln\{a^{-1}[R_2^2-2r_jr_k\cos(\varphi_j-\varphi_k)+(r_jr_k/R_2)^2]^{1/2}\}-(4\pi)^{-1}\rho\kappa^2\sum_{jk}{}'\times\ln(|\mathbf{r}_j-\mathbf{r}_k|/a)+(4\pi)^{-1}\rho\kappa^2\sum_j\tfrac{1}{4}, \tag{29}$$

which is the correct expression for the energy of a system of vortices in a cylinder of radius R_2.[18] Equation (29) has a simple interpretation as the interaction energy between pairs of real vortices in the cylinder plus the interaction energy between each real vortex and all of the images including its own.

A special configuration of particular interest is l vortices symmetrically placed in a ring of radius r_l. The stream function is the imaginary part of Eq. (9), and the calculation of the energy E_l is essentially the same as for a general array of vortices. The final expression is

$$E_l=\frac{\rho}{4\pi}\ln(R_2/R_1)\left[\Gamma_1+l\kappa\frac{\ln(R_2/r_l)}{\ln(R_2/R_1)}\right]^2+\frac{\rho\kappa^2 l}{4\pi}\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_l)|\tau/l]}{\vartheta_1'(0|\tau/l)}\frac{r_l}{\gamma a}\right\}+\frac{\rho\kappa^2 l}{16\pi}, \tag{30}$$

where the parameter τ/l appearing in the theta function has been made explicit. Equation (30) becomes simple in two important limiting cases. When $R_1/R_2\rightarrow 0$, the circulation Γ_1 must vanish, and Eq. (30) reduces to

$$E_l=\frac{\rho\kappa^2 l}{4\pi}\left\{\ln\left[\frac{R_2^l-r_l^{2l}R_2^{-l}}{lar_l^{l-1}}\right]+\tfrac{1}{4}\right\}, \tag{31}$$

which is the correct expression for the energy of a symmetric ring of l vortices in a cylinder of radius R_2.[19] In the opposite limit when $l(R_2-R_1)\ll(R_2+R_1)$, Eq. (30) becomes

$$E_l=\frac{\rho}{4\pi}\left(\frac{R_2-R_1}{R_2}\right)\left[\Gamma_1+l\kappa\left(\frac{R_2-r_l}{R_2-R_1}\right)\right]^2+\frac{\rho\kappa^2 l}{4\pi}\left\{\ln\left[\frac{2(R_2-R_1)}{\pi a}\sin\left(\frac{\pi(R_2-r_l)}{R_2-R_1}\right)\right]+\tfrac{1}{4}\right\}. \tag{32}$$

The first term, which represents the residual effect of the multiply-connected annulus, may become important if Γ_1 or l is large, while the second term is l times the energy of a single vortex in a channel of width R_2-R_1.[20] Equations (30) and (31) may also be obtained directly from Eqs. (28) and (29) by a sequence of algebraic manipulations.[13]

IV. NARROW ANNULUS: UNRESTRICTED FREE-ENERGY VARIATION

The equilibrium state of an arbitrary system rotating with angular velocity Ω is obtained by minimizing the free energy $F=E-\Omega L$. In the special case of an inviscid fluid containing vortices, it can be shown that each vortex in an equilibrium configuration must remain stationary in the rotating coordinate system.[6] An equivalent form of this condition is obtained by transforming to the laboratory coordinate system, where the only possible equilibrium configurations are those in which the vortex array rotates rigidly with an angular velocity Ω. Such behavior occurs only for symmetric vortex arrangements, and the present work will be restricted to a ring of l vortices equally spaced on the circumference of a ring of radius r_l. The free energy F_l for this system is easily derived from Eqs. (20) and (30), and it is convenient to introduce a dimensionless free energy $\mathfrak{F}_l=4\pi F_l/\rho\kappa^2=(4\pi/\rho\kappa^2)(E_l-\Omega L_l)$, which is given explicitly as

$$\mathfrak{F}_l=\ln(R_2/R_1)\left\{\frac{\Gamma_1}{\kappa}+l\frac{\ln(R_2/r_l)}{\ln(R_2/R_1)}\right\}^2+l\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_l)|\tau/l]}{\vartheta_1'(0|\tau/l)}\frac{r_l}{\gamma a}\right\}-\frac{2\pi\Omega}{\kappa}\frac{\Gamma_1}{\kappa}(R_2^2-R_1^2)-\frac{2\pi\Omega}{\kappa}l(R_2^2-r_l^2). \tag{33}$$

In Eq. (33), the small corrections associated with the core structure have been neglected; equivalently, these terms can be absorbed in a redefinition of the core radius a.

[16] The details of this standard calulation may be found in E. S. Raja Gopal, Ann. Phys. (N. Y.) 25, 196 (1963).

[17] Since $\tau\rightarrow 0$ when $R_1\rightarrow 0$, it is necessary to use Jacobi's imaginary transformation for the theta functions, Ref. 11, p. 474.

[18] See, for example, A. L. Fetter, Phys. Rev. 138, A429 (1965).

[19] See, for example, A. L. Fetter and R. J. Donnelly, Phys. Fluids 9, 619 (1966).

[20] See, for example, Ref. 16, Eq. (3.7) for the energy of a vortex pair symmetrically placed in a channel. The energy of a single vortex in a channel of width α is just $\frac{1}{2}$ that of the pair in a channel of width 2α, since the total system of images is the same.

We shall assume temporarily that the quantum number of circulation Γ_1/κ about the inner cylinder is large; this assumption will be verified below in the case of a narrow annulus, but it is not permissible for a wide annulus (Sec. VI). Thus Γ_1/κ in a narrow annulus may be treated as a continuous variable, and $\mathfrak{F}_l$ must be minimized with respect to the two independent parameters r_l and Γ_1/κ. In contrast, the number of vortices need not be large, so that the equilibrium value of $\mathfrak{F}_l$ must be calculated separately for each l. The actual number of vortices present for a fixed angular velocity Ω is given by the integer l corresponding to the lowest free energy.

The minimum value of $\mathfrak{F}_l$ is determined by the following equations:

$$\partial\mathfrak{F}_l/\partial(\Gamma_1/\kappa)=0\,, \tag{34}$$

$$\partial\mathfrak{F}_l/\partial r_l=0\,, \tag{35}$$

which must be solved simultaneously for the equilibrium values of r_l and Γ_1. These conditions may be evaluated explicitly with Eq. (33),

$$\frac{\Gamma_1}{\kappa}+l\frac{\ln(R_2/r_l)}{\ln(R_2/R_1)}=\frac{\pi\Omega(R_2^2-R_1^2)}{\kappa\ln(R_2/R_1)}\,, \tag{36}$$

$$\tfrac{1}{2}+\frac{2\pi\Omega r_l^2}{\kappa}=\frac{\Gamma_1}{\kappa}+l\frac{\ln(R_2/r_l)}{\ln(R_2/R_1)}+\gamma\frac{\vartheta_1'[2\gamma\ln(R_2/r_l)\,|\,\tau/l]}{\vartheta_1[2\gamma\ln(R_2/r_2)\,|\,\tau/l]}\,. \tag{37}$$

Substitution of Eq. (36) into Eq. (37) provides a somewhat simpler equation

$$\tfrac{1}{2}+\frac{2\pi\Omega r_l^2}{\kappa}=\frac{\pi\Omega(R_2^2-R_1^2)}{\kappa\ln(R_2/R_1)}+\gamma\frac{\vartheta_1'[2\gamma\ln(R_2/r_l)\,|\,\tau/l]}{\vartheta_1[2\gamma\ln(R_2/r_l)\,|\,\tau/l]}\,, \tag{38}$$

which is independent of Γ_1. The solution to Eq. (38) yields the equilibrium radius r_l for a fixed Ω and l; the corresponding equilibrium circulation Γ_1 is then obtained directly from Eq. (36). It can be verified with Eq. (37) that each vortex in the ring rotates about the center of the annulus with angular velocity Ω.

The equilibrium value of the free energy is given by a combination of Eqs. (33) and (36):

$$[\mathfrak{F}_l]_{\text{eq}}=-\frac{\pi^2\Omega^2(R_2^2-R_1^2)^2}{\kappa^2\ln(R_2/R_1)}+l\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_l)\,|\,\tau/l]}{\vartheta_1'(0\,|\,\tau/l)}\frac{r_l}{\gamma a}\right\}-\frac{2\pi\Omega l}{\kappa}\left[R_2^2-r_l^2-\frac{(R_2^2-R_1^2)\ln(R_2/r_l)}{\ln(R_2/R_1)}\right], \tag{39}$$

where r_l is the solution of Eq. (38). The first term in Eq. (39) is the free energy $[\mathfrak{F}_0]_{\text{eq}}$ associated with pure equilibrium circulation

$$[\Gamma_1]_{\text{eq}}=\frac{\pi\Omega(R_2^2-R_1^2)}{\ln(R_2/R_1)} \tag{40}$$

about the inner cylinder in the absence of vortices ($l=0$), while the second and third terms represent additional contributions from the vortices. It is useful to isolate the free energy of the vortices by considering the quantity

$$[\mathfrak{F}_l-\mathfrak{F}_0]_{\text{eq}}=l\ln\left\{\frac{\vartheta_1[2\gamma\ln(R_2/r_l)\,|\,\tau/l]}{\vartheta_1'(0\,|\,\tau/l)}\frac{r_l}{\gamma a}\right\}-\frac{2\pi\Omega l}{\kappa}\left[R_2^2-r_l^2-\frac{(R_2^2-R_1^2)\ln(R_2/r_l)}{\ln(R_2/R_1)}\right], \tag{41}$$

which vanishes if $l=0$. The square bracket in Eq. (41) is positive[21] for $R_1<r_l<R_2$, and the free energy associated with the formation of vortices necessarily becomes negative for sufficiently large angular velocities. The critical angular velocity for the formation of vortices is the smallest value of Ω for which $[\mathfrak{F}_l-\mathfrak{F}_0]_{\text{eq}}$ vanishes.

Equations (36), (38), and (39) provide an exact description of the equilibrium configuration of l vortices symmetrically placed in an annulus. If the ratio R_2/R_1 is unrestricted, the equilibrium state can be determined only with numerical computation. Explicit results may be obtained in limiting cases, however, and we shall now examine one of them: a narrow annulus. More precisely, the quantity

$$q_l^2=\exp(2i\pi\tau/l)=\exp[-2\pi^2/l\,\ln(R_2/R_1)] \tag{42}$$

is assumed to be small. This condition is satisfied if

$$2\pi>l\ln(R_2/R_1)\,, \tag{43}$$

since q_l^2 is then bounded by $e^{-\pi}\approx 0.043$. In the limit of a narrow annulus, the width $d=R_2-R_1$ becomes much smaller than the mean radius $R=\frac{1}{2}(R_2+R_1)$ and Eq. (43) reduces to

$$2\pi R>ld\,. \tag{44}$$

Equation (44) requires that the distance between vortices $(2\pi R/l)$ be larger than the width of the annulus.

The analysis of the equilibrium configuration for small q_l^2 is straightforward. We assume that the radius of the ring r_l may be written as

$$r_l=r_0[1+O(q_l^2)]\,, \tag{45}$$

where r_0 is independent of l. The theta function has an expansion[22] in ascending powers of q_l^2 and substitution

[21] It is easy to prove that the function $f(x)=(1-x)[\ln(1/x)]^{-1}$ has a positive first derivative in the range $0<x<1$ [see, for example, F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill Book Company, Inc., New York, 1965), p. 618]; hence $f(x)$ is an increasing function and $f(r_l^2/R_2^2)-f(R_1^2/R_2^2)>0$.

[22] Reference 11, p. 464.

of Eq. (45) into Eq. (38) yields a zero-order equation for r_0

$$\tfrac{1}{2}+\frac{2\pi r_0{}^2\Omega}{\kappa}=\frac{\pi\Omega(R_2{}^2-R_1{}^2)}{\kappa\ln(R_2/R_1)}+\gamma\cot[2\gamma\ln(R_2/r_0)]. \quad (46)$$

Equation (46) is independent of l, which justifies the form assumed in Eq. (45). A similar expansion of Eq. (41) leads to

$$[\mathfrak{F}_l-\mathfrak{F}_0]_{\text{eq}}=l\ln\{(r_0/\gamma a)\sin[2\gamma\ln(R_2/r_0)]\}-2\pi\Omega l\kappa^{-1}\times\{R_2{}^2-r_0{}^2-(R_2{}^2-R_1{}^2)\ln(R_2/r_0)[\ln(R_2/R_1)]^{-1}\}. \quad (47)$$

This equation represents the zero-order approximation to the free energy associated with the creation of a ring of l vortices in the annulus. Apart from the linear dependence on the number of vortices, Eq. (47) is independent of l. Thus the mathematical assumption of small $q_l{}^2$ is equivalent to the physical assumption of negligible interaction between different vortices in the ring. If the condition (44) is satisfied, the energy of each vortex arises solely from its interaction with the infinite sequence of self-images in the two walls. The zero-order approximation to the critical angular velocity Ω_0 for the appearance of vortices is easily found from Eq. (47) to be

$$\Omega_0=\frac{\ln\{(r_0/\gamma a)\sin[2\gamma\ln(R_2/r_0)]\}}{2\pi\kappa^{-1}\{R_2{}^2-r_0{}^2-(R_2{}^2-R_1{}^2)\ln(R_2/r_0)[\ln(R_2/R_1)]^{-1}\}}. \quad (48)$$

Equation (48) is independent of l; hence the number of vortices in the ring is indeterminate in this zero-order approximation, which arises from neglecting the interaction energy between vortices. Inclusion of the small correction terms of order $q_l{}^2$ removes this unphysical feature, as is shown below.

If $\Omega<\Omega_0$, the equilibrium state of the fluid is pure irrotational flow with circulation Γ_1 about the inner cylinder. Γ_1 would be continuous in the absence of a quantization condition, and the equilibrium circulation at an angular velocity Ω would then be given by Eq. (40). In fact, Γ_1 is restricted to integral multiples of $\kappa=h/m$, and we must examine the free energy $\mathfrak{F}_0(n)$ describing irrotational (vortex-free) flow with n units of quantized circulation

$$\mathfrak{F}_0(n)=n^2\ln(R_2/R_1)-2\pi\Omega n\kappa^{-1}(R_2{}^2-R_1{}^2). \quad (49)$$

The transition from a state with circulation $n\kappa$ to one with circulation $(n+1)\kappa$ occurs at an angular velocity $\Omega(n+\frac{1}{2})$, obtained as the solution of

$$\mathfrak{F}_0(n+1)=\mathfrak{F}_0(n). \quad (50)$$

An elementary calculation shows that

$$\Omega(n+\tfrac{1}{2})=\frac{(n+\frac{1}{2})\kappa\ln(R_2/R_1)}{\pi(R_2{}^2-R_1{}^2)}. \quad (51)$$

An irrotational state with circulation $n\kappa$ is energetically favorable only in the range $\Omega(n-\frac{1}{2})<\Omega<\Omega(n+\frac{1}{2})$. As Ω increases continuously from 0 to Ω_0, the equilibrium state passes through a discrete sequence, in which the circulation changes successively by one unit.[23] In the limit of large quantum numbers, the additive term $\frac{1}{2}$ becomes negligible, and Eq. (51) reduces to Eq. (40). The maximum quantum number obtainable in purely irrotational flow is

$$(\Gamma_1/\kappa)_{\max}=\pi\Omega_0\kappa^{-1}(R_2{}^2-R_1{}^2)[\ln(R_2/R_1)]^{-1}, \quad (52)$$

which is much larger than one in most cases of interest.

The above expressions may be simplified in the limit of a narrow annulus ($d\ll R$). If $\zeta(=R_2-r_0)$ is the distance from the ring to the outer wall, then the explicit solution of Eq. (46) is

$$\zeta=\tfrac{1}{2}d[1+O(d/R)]. \quad (53)$$

Hence the equilibrium position for the ring of vortices is midway between the walls. The critical angular velocity for the appearance of vortices in a narrow annulus is

$$\Omega_0=(\kappa/\pi d^2)\ln(2d/\pi a), \quad (54)$$

apart from corrections which vanish as $d/R\to 0$. Typical values for liquid He II ($\kappa=h/m\approx 10^{-3}$ cm^2 sec^{-1}, $R\approx 1$ cm, $d\approx 0.1$ cm, $a\approx 10^{-8}$ cm) lead to $\Omega_0\approx\frac{1}{2}$ rad sec^{-1}, which agrees in order of magnitude with the values reported in Ref. 7. This high value of Ω_0 can be understood by considering the critical angular velocity $(\kappa/2\pi R^2)\ln(R/a)$ for vortex creation in a cylinder of radius R; Eq. (54) is essentially the critical angular velocity for a cylinder of radius $d/\sqrt{2}$, which is the approximate width of a vortex in the annulus. If $\Omega<\Omega_0$, the equilibrium state is an irrotational flow, and the transition from a circulation $n\kappa$ to $(n+1)\kappa$ occurs at

$$\Omega(n+\tfrac{1}{2})=(n+\tfrac{1}{2})\kappa/2\pi R^2. \quad (55)$$

In the limit of large quantum numbers, the equilibrium circulation about the inner cylinder is $2\pi R^2\Omega$, which is also the value that would occur if the inner cylinder were filled with a uniform density of vortices $2\Omega/\kappa$. The maximum circulation associated with the irrotational flow occurs at $\Omega=\Omega_0$ and is given by

$$(\Gamma_1/\kappa)_{\max}=2\pi\Omega_0R^2/\kappa =2(R/d)^2\ln(2d/\pi a). \quad (56)$$

With the same numerical values as above, the lowest circulation state ($n=1$) in liquid He II appears at $\Omega(\frac{1}{2})\approx 8\times 10^{-5}$ rad sec^{-1}, while the maximum quantum number of irrotational circulation is $(\Gamma_1/\kappa)_{\max}\approx 3.1\times 10^3$. This large circulation agrees qualitatively with that reported in Ref. 5.

The above results for the low-lying states in a narrow annulus can also be obtained with the following ele-

[23] Quantized flux in a thin superconducting ring exhibits similar behavior, which has been examined by N. Byers and C. N. Yang, Phys. Rev. Letters 7, 46 (1961).

mentary arguments: The lowest state at a given angular velocity is that most closely approximating solid-body rotation.[2,6] This condition means that the fluid is essentially stationary in the rotating frame of reference, with deviations arising only from the quantization of circulation. Equivalently, the equilibrium configuration minimizes the relative velocity between the fluid and the walls. Thus Γ_1 changes by one unit whenever

$$|(\Gamma_1/2\pi R)-\Omega R|\geqslant\tfrac{1}{2}(\kappa/2\pi R),\qquad(57)$$

which immediately reproduces Eq. (55). Purely irrotational flow, in which $v(r)=\Gamma_1/2\pi r$, cannot persist at arbitrarily large angular velocities, since the fluid moves faster at the inner wall than at the outer wall. Such a flow pattern becomes unstable when a singly quantized vortex in the middle of the annulus can just compensate for the velocity difference:

$$\Gamma_1/2\pi(R+\tfrac{1}{2}d)+(\kappa/\pi d)\approx\Gamma_1/2\pi(R-\tfrac{1}{2}d)-(\kappa/\pi d).\qquad(58)$$

Equation (58) suggests that the maximum quantum number of irrotational (vortex-free) circulation is of order $(R/d)^2$, which agrees with Eq. (56), apart from the logarithmic factor. Furthermore, the critical angular for the appearance of vortices is of order κ/d^2, as in Eq. (54). The equilibrium condition that the vortices remain stationary in the rotating frame of reference[6] is here satisfied by placing the vortices at the midpoint of the channel, equidistant from the set of images in either wall. In the laboratory coordinate system, each vortex moves in the large circulating velocity field associated with Γ_1; the equilibrium condition (40) ensures that the vortices rotate with angular velocity Ω.

The precise equilibrium configuration for $\Omega>\Omega_0$ can be determined only by including the interaction energy between vortices. For this purpose, it is necessary to expand Eqs. (38) and (41) through first order in q_l^2. The calculation is not difficult in the limit of a narrow annulus ($d\ll R$), and we find a discrete sequence of states in which the number of vortices in the ring increases monotonically from unity. The transition from a ring of $l-1$ vortices to a ring of l vortices occurs at an angular velocity

$$\begin{aligned}\Omega_l&=\Omega_0+4\kappa q_l/\pi d^2\\&=(\kappa/\pi d^2)[\ln(2d/\pi a)+4\exp(-2\pi^2R/ld)].\end{aligned}\qquad(59)$$

The first vortex appears in the fluid at Ω_1, which provides a more exact expression for the critical angular velocity. Each time Ω passes through one of the values Ω_l, the number of vortices in the ring increases by one. For small l, the change in angular velocity associated with the addition of one vortex is extremely small.

Equation (59) is valid only for $2\pi R\geqslant ld$, so that our analysis breaks down when the distance between vortices becomes equal to d. Presumably a second ring then starts to appear. The total increase in the angular velocity $\Delta\Omega$ corresponding to the change from a state with no vortices to one with $2\pi R/d$ vortices is given by

$$\Delta\Omega\approx(4\kappa/\pi d^2)e^{-\pi};\qquad(60)$$

this value represents only a small fractional change in Ω:

$$\Delta\Omega/\Omega_0\approx4e^{-\pi}[\ln(2d/\pi a)]^{-1},\qquad(61)$$

which is the order of 1% for the numerical values used below Eq. (54). In this narrow range of angular velocity, the equilibrium circulation about the inner cylinder and the total angular momentum remain approximately constant at $[\Gamma_1]_{\mathrm{eq}}\approx2\pi\Omega_0R^2$ and $[L]_{\mathrm{eq}}=\rho\Gamma_1Rd\approx2\pi\rho\Omega_0R^3d$; this last value is just that associated with classical solid-body rotation. Hence a direct measurement of the angular momentum[6] cannot detect the appearance of vortices in a narrow annulus, and other experimental methods, such as attentuation of second sound[24] or ion trapping,[25] must be used to measure Ω_0.

V. NARROW ANNULUS: RESTRICTED FREE-ENERGY VARIATION

It has been assumed in Sec. IV that the fluid always attains the state of lowest free energy. In certain circumstances, however, this assumption fails to provide a realistic description of liquid He II, and we shall now consider the effect of a restricted variation of the free energy, in which the circulation Γ_1 is constrained to vanish. Such a situation occurs when annulus containing superfluid He II is accelerated from rest at a temperature $T\ll T_\lambda$. In this case, experiments[26] indicate that the fluid remains stationary until the relative velocity between the fluid and the walls exceeds the critical velocity v_c. The same restriction of vanishing circulation also describes superfluid flow in a straight pipe, where states of quantized circulation are meaningless. Various calculations involving creation of vortex pairs or vortex rings[2,27] have suggested that $v_c=O(\hbar/md)$ for flow in a channel of width d. A similar result is obtained in this section with the restricted variation of the free energy in a large annulus.[28]

The free energy for a ring of l vortices in the absence of circulation is easily obtained from Eq. (33) by setting $\Gamma_1=0$. In addition, if Eqs. (43) and (45) are satisfied, then the theta functions may be simplified and $\mathfrak{F}_l$ is

[24] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) **A238**, 204 (1956).

[25] G. Careri, W. D. McCormick, and F. Scaramuzzi, Phys. Letters **1**, 61 (1962); R. L. Douglass, Phys. Rev. Letters **13**, 791 (1964); B. E. Springett, D. J. Tanner, and R. J. Donnelly, *ibid.* **14**, 585 (1965).

[26] J. B. Mehl and W. Zimmerman, Jr., Bull. Am. Phys. Soc. **11**, 479 (1966); and private communication.

[27] W. F. Vinen, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. III, p. 1; J. C. Fineman and C. E. Chase, Phys. Rev. **129**, 1 (1963); A. L. Fetter, Phys. Rev. Letters **10**, 507 (1963).

[28] The experimental relevance of these calculations is discussed in greater detail by R. J. Donnelly and A. L. Fetter, Phys. Rev. Letters **17**, 747 (1966).

given explicitly as

$$\mathfrak{F}_l = l[\ln(R_2/r_0)]^2[\ln(R_2/R_1)]^{-1} + l\ln\{(r_0/\gamma a)\sin[2\gamma\ln(R_2/r_0)]\} - 2\pi\Omega\kappa^{-1}l(R_2{}^2-r_0{}^2), \quad (62)$$

which is to be compared with Eq. (47). In the limit of a narrow annulus, Eq. (62) becomes

$$\mathfrak{F}_l = l\ln[(2d/\pi a)\sin(\pi\zeta/d)] - 4\pi l\Omega R\zeta/\kappa, \quad (63)$$

where terms that vanish as $R\to\infty$ have been neglected. The lowest value of $\mathfrak{F}_l$ is obtained by placing the vortices near the inner wall, so that

$$\zeta_{\text{eq}} \approx d - Ca, \quad (64)$$

where C is a constant of order unity. A combination of Eqs. (63) and (64) yields

$$\mathfrak{F}_l = l\ln(2C) - 4\pi l\Omega Rd/\kappa. \quad (65)$$

The stationary fluid represents the zero of free energy, and a ring of vortices is energetically favorable only at a critical angular velocity

$$\Omega_c = (\kappa/4\pi Rd)\ln(2C), \quad (66)$$

when $\mathfrak{F}_l$ becomes negative. As in Sec. IV, this zero-order calculation cannot determine the number of vortices.

A slightly different configuration of interest is a vortex pair, consisting of two oppositely directed vortices (Fig. 3). If the vortices (with circulation $\kappa_1=\kappa$, $\kappa_2=-\kappa$) are placed at $\mathbf{r}_1$ and $\mathbf{r}_2$ ($r_1<r_2$), the exact free energy may be obtained with a straightforward generalization of Eqs. (20) and (28). For a narrow annulus, the reduction of the free energy is very similar to the analysis of Sec. IV, and only the final formulas will be given here. In the limit $d\ll R$, the pair must be symmetrically placed in the channel, with coordinates

$$r_1 = R - \tfrac{1}{2}\epsilon, \qquad r_2 = R + \tfrac{1}{2}\epsilon. \quad (67)$$

Neglecting terms that vanish as R becomes infinite, we find

$$\mathfrak{F} = (d/R)(\Gamma_1/\kappa)^2 + 2(\epsilon/R)(\Gamma_1/\kappa) + 2\ln\sin(\pi\epsilon/d) + 2\ln(d/\pi a) - 4\pi\Omega\Gamma_1\kappa^{-2}Rd - 4\pi\Omega\kappa^{-1}R\epsilon. \quad (68)$$

The equilibrium state predicted by Eq. (68) depends on the assumed value of the circulation Γ_1. If Γ_1 is allowed to vary freely, then the equilibrium conditions are

$$\partial\mathfrak{F}/\partial(\Gamma_1/\kappa) = (2d\Gamma_1/R\kappa) + (2\epsilon/R) - (4\pi\Omega Rd/\kappa) = 0, \quad (69)$$

$$\partial\mathfrak{F}/\partial\epsilon = 2\pi d^{-1}\cot(\pi\epsilon/d) + (2\Gamma_1/\kappa R) - (4\pi\Omega R/\kappa) = 0. \quad (70)$$

Equation (69) yields the equilibrium circulation

$$[\Gamma_1]_{\text{eq}} = 2\pi\Omega R^2 - \kappa\epsilon/d; \quad (71)$$

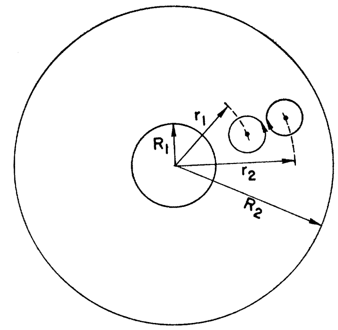

FIG. 3. Geometry of a vortex pair in an annulus.

substitution of Eq. (71) into Eq. (70) leads to

$$\cot(\pi\epsilon/d) = (\epsilon/\pi R), \quad (72)$$

with the solution

$$\epsilon = \tfrac{1}{2}d[1+O(d/R)]. \quad (73)$$

Thus each vortex is equidistant from the wall and the center of the channel, which is physically obvious because the vortices must be stationary in the rotating coordinate system.[6] The free energy associated with vortex-free equilibrium circulation is

$$[\mathfrak{F}_\Gamma]_{\text{eq}} = -4\pi^2\Omega^2R^3d\kappa^{-2}, \quad (74)$$

while the free energy for equilibrium circulation combined with a vortex pair is

$$[\mathfrak{F}_{\Gamma p}]_{\text{eq}} = -4\pi^2\Omega^2R^3d\kappa^{-2} + 2\ln(d/\pi a). \quad (75)$$

Comparison of Eq. (75) with Eq. (74) shows that a vortex pair never represents the equilibrium state in the limit $R\to\infty$. This conclusion is confirmed by a detailed calculation, which shows that the critical angular velocity for the creation of a vortex pair is of order $\kappa R/d^3$ if the circulation Γ_1 is treated as a variational parameter.

The other possibility considered here is that of vanishing circulation ($\Gamma_1=0$); the corresponding free energy for a vortex pair is obtained from Eq. (68):

$$\mathfrak{F} = 2\ln\sin(\pi\epsilon/d) + 2\ln(d/\pi a) - 4\pi\Omega\kappa^{-1}R\epsilon. \quad (76)$$

Except for the unphysical limit of $\epsilon\approx 0$, Eq. (76) is a decreasing function of ϵ and attains its lowest value when the vortices are near the outer walls,

$$\epsilon_{\text{eq}} \approx d - C'a, \quad (77)$$

where C' is a constant of order unity. A combination of Eqs. (76) and (77) yields

$$\mathfrak{F} = 2\ln(C') - 4\pi\Omega\kappa^{-1}Rd. \quad (78)$$

The vortex pair becomes energetically favorable when its free energy [Eq. (78)] is lower than that of the stationary fluid ($\mathfrak{F}=0$). This condition leads to the critical angular velocity Ω_{cp} for creation of a vortex pair under the restriction of vanishing circulation

$$\Omega_{cp} = (\kappa/2\pi Rd)\ln(C'). \quad (79)$$

Equation (79) is approximately twice that found in Eq. (66) for creation of a ring of identical vortices.

The critical linear velocities associated with Eqs. (66) and (79) are

$$v_c = (\hbar/2md)\ln(2C),$$
$$v_{cp} = (\hbar/md)\ln(C'), \tag{80}$$

where the quantum of circulation ($\kappa = h/m$) has been made explicit. These values are only approximate because of the unknown logarithmic constants. Nevertheless, they agree in order of magnitude with the Feynman calculation of v_c,[2] and with estimates based on the Landau criterion[29] applied to vortex creation.[27] In a strict sense, Eq. (80) only describes a finite annulus, but it is possible to consider a sequence of systems in which R becomes large while d remains finite. Since the expression for v_c is independent of R, the calculation applies equally well to superfluid flow in a long straight channel of width d.

The present formulation has one important new feature: the total volume of fluid is finite, so that the energy and angular momentum are well defined. Thus we avoid the ambiguity associated with the definition of linear momentum in an infinite system. Previous calculations[27] have assumed that the correct procedure is to calculate v_c using the impulse, which remains finite even for an infinite system. It is easy to see that the approach used here partially justifies this assumption. Equation (20) shows that the angular momentum of a vortex pair in an annulus is given exactly as $\frac{1}{2}\rho\kappa(r_2^2 - r_1^2)$. In the limit $R\rightarrow\infty$, this value reduces to $R\rho\kappa\epsilon$, which is R times the impulse P of a vortex pair a distance ϵ apart.[30] Hence the free energy for a vortex pair in a large annulus (with $\Gamma_1 = 0$) is approximately $E - vP$, where $v = R\Omega$ is the linear velocity of the walls. The critical velocity is obtained by comparison with the stationary fluid ($F = 0$), which immediately reproduces the Landau criterion $v_c = (E/P)_{\min}$ for creation of a vortex pair.

Equation (80) provides an interesting comparison with the linear velocity v_0 corresponding to vortex creation with the unrestricted variation of the free energy. If the circulation about the inner cylinder is allowed to change with the external angular velocity, then vortices cannot appear until the free energy of the vortices plus circulation is lower than that of solid-body rotation. This criterion leads to the critical velocity

$$v_0 = R\Omega_0 \approx (\kappa R/\pi d^2)\ln(2d/\pi a), \tag{81}$$

which is larger than Eq. (80) by a factor R/d. The critical velocity v_0 should be relevant to experiments in which the superfluid is created in a state of rotation. In this case, the unrestricted variation of the free energy $E - \Omega L$ is equivalent to minimizing the energy subject to the constraint of fixed angular momentum; the parameter Ω is a Lagrange multiplier which is eventually identified with the angular velocity of the container. The superfluid automatically adjusts the circulation about the inner cylinder to achieve the most favorable value.

It must be emphasized that irrotational vortex-free circulation always has a lower free energy than *any* configuration of vortices as long as $\Omega < \Omega_0$. Hence restricted variation of the free energy can never lead to absolute equilibrium. The absence of circulation ($\Gamma_1 = 0$) also means that the vortices no longer move with the external angular velocity Ω; such vortices in liquid He II would therefore be subject to viscous normal-fluid forces. This lack of self-consistency also affects the Landau criterion, which merely answers the kinematical question: How fast must the superfluid flow between stationary walls before it becomes energetically favorable to create a quasiparticle?

VI. WIDE ANNULUS

The limiting case of a wide annulus ($R_1 \ll R_2$) presents a problem of experimental interest, since the first evidence for quantized circulation[4] in liquid He II was obtained with a rotating cylinder ($R_2 \approx 0.2$ cm) containing a fine wire ($R_1 \approx 10^{-3}$ cm) along its axis. A finite circulation Γ_1 about the wire removes the degeneracy of its transverse vibrational modes, allowing a direct measurement of Γ_1. Only the first quantum state ($n = 1$) was observed in the original experiment, but higher states ($n = 2,3$) have been found in more recent work.[4] It is clearly not permissible to treat Γ_1 as a continuous variable, and the free energy must be calculated separately for each integral value of Γ_1/κ.

We therefore consider the problem of fluid in a wide annulus containing l vortices placed in a ring of radius r_l and a circulation $n\kappa$ about the inner cylinder. The dimensionless free energy $\mathfrak{F}_{nl}$ for this system may be obtained directly from Eq. (33). Unfortunately, this form is inconvenient in the limit $R_1 \ll R_2$, and it is necessary to use the transformation properties of the theta functions[17] to rewrite the free energy as

$$\mathfrak{F}_{nl} = n^2 \ln(R_2/R_1) + 2nl \ln(R_2/r_l) + l \ln\left\{\frac{2\vartheta_1[il\ln(R_2/r_l)\,|\,\tau' l]}{il\vartheta_1'(0\,|\,\tau' l)}\frac{r_l}{a}\right\} - G\{n[1-(R_1^2/R_2^2)] + l[1-(r_l^2/R_2^2)]\}, \tag{82}$$

where

$$\tau' = -\tau^{-1} = i\pi^{-1}\ln(R_2/R_1), \tag{83}$$

and

$$G = 2\pi\Omega R_2^2 \kappa^{-1} \tag{84}$$

is a dimensionless angular velocity. Equation (82) must be minimized with respect to r_l for each value of l and

[29] L. Landau, J. Phys. (USSR) **5**, 71 (1941).

[30] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., p. 229.

n; comparison of $\mathfrak{F}_{nl}$ for different l and n then yields the physical state of lowest free energy.

The analysis of the equilibrium configuration is similar to that of Sec. IV, and most of the details will be omitted. The theta function in Eq. (82) has a rapidly converging expansion[22] in ascending powers of the parameter

$$q_l' = \exp(i\pi\tau' l) = (R_1/R_2)^l, \tag{85}$$

which is very small for the experiments of interest ($R_1/R_2 \approx 5\times10^{-3}$). Detailed examination of the free energy shows that the minimum occurs when the ring is far from either wall, so that the inequality

$$R_1 \ll r_l \ll R_2 \tag{86}$$

is satisfied. The form of $\mathfrak{F}_{nl}$ can be greatly simplified in this limit,[31] and we find

$$\mathfrak{F}_{nl} = \tfrac{1}{2}n^2 \ln(1/y) + \tfrac{1}{2}l(2n+l-1)\ln(1/x) + l\ln(R_2/al) + l\ln[1-x^l-(y/x)^l] - G[n(1-y)+l(1-x)]. \tag{87}$$

Here, the abbreviations

$$y = (R_1/R_2)^2, \quad x = (r_l/R_2)^2 \tag{88}$$

have been introduced, and corrections of order y^l have been neglected. The equilibrium condition for fixed l and n is

$$\partial\mathfrak{F}_{nl}/\partial x = 0, \tag{89}$$

which leads to an equation for x:

$$2Gx+1 = 2n+l+2l(x^{2l}-y^l)(x^l-x^{2l}-y^l)^{-1}. \tag{90}$$

In the limit $n=0$, $y\rightarrow 0$, it can be verified that Eqs. (87) and (90) reduce to the known equations for the low-energy states of a simply connected cylinder.[32]

Equations (87) and (90) are particularly suitable for numerical evaluation, and it is straightforward to compute the equilibrium free energy $\mathfrak{F}_{nl}$ as a function of the angular velocity G. The detailed results are essentially identical with Vinen's calculations (Fig. 8 of Ref. 4): The fluid remains stationary for small angular velocities. At a critical value $G=\frac{1}{2}\ln(1/y)[\Omega=\kappa(2\pi R_2^2)^{-1} \times\ln(R_2/R_1)]$, irrotational flow with a single quantum of circulation about the inner cylinder becomes the equilibrium state. Two quanta of circulation appear at the higher value $G=\frac{3}{2}\ln(1/y)[\Omega=3\kappa(2\pi R_2^2)^{-1}\ln(R_2/R_1)]$. Further increase in Ω leads to vortex formation in the bulk of the fluid, and the precise sequence of states depends on the core radius a; Vinen[4] has studied this this question in detail.

VII. DISCUSSION

The present paper has analyzed the low-lying states of an inviscid fluid in a rotating annulus. Equilibrium arrays of vortices necessarily have a high degree of symmetry, and only a single ring of equally spaced vortices is considered here. In the limit of a narrow annulus, the lowest levels represent irrotational flow with quantized circulation about the inner cylinder. These states form an equally spaced sequence, and each level is stable in a small interval of angular velocity with a width $\kappa/2\pi R^2$. Vortices appear in the fluid only at a much larger angular velocity $\Omega_0 = (\kappa/\pi d^2)\ln(2d/\pi a)$, at which point the quantum number of circulation is given by $2(R/d)^2\ln(2d/\pi a)\gg 1$. The vortices are formed midway between the walls, and the number of vortices grows rapidly as Ω increases beyond Ω_0. In certain situations, however, the circulation about the inner cylinder is not free to change and remains equal to zero; the corresponding critical angular velocity for vortex creation is the order of $\kappa/2\pi Rd$. This calculation leads to a critical linear velocity v_c for the destruction of superfluidity of order $\hbar/md$, in agreement with previous estimates.[2,27]

The equilibrium states are quite different in a wide annulus. Vortices appear at much lower angular velocities, and the maximum quantum number for irrotational flow is a small integer. These relations cannot be expressed by simple formulas like those for a narrow annulus, but the numerical values are easily computed in any specific situation. The theoretical predictions agree qualitatively with the rather meager experimental data, both for a narrow annulus[5,7] and for a wide annulus.[4]

Superfluid flow in an annulus reveals several striking analogies to superconductivity. The irrotational fluid motion in a narrow annulus is quantized in precisely the same way as the persistent current in a superconducting ring.[23] At an angular velocity Ω_0, it becomes favorable for vortices to form in the bulk of the fluid, and the number of vortices increases rapidly with Ω. An identical effect occurs in a type-II superconductor[33] at a magnetic field H_{c1} when the magnetic flux enters the sample in the form of quantized flux lines or vortices. In both systems, the phenomena depend nonanalytically on the number of vortices [$\propto$ exp(-const/l)]. The only essentially new feature in the charged system is the existence of a second characteristic length λ, the London penetration depth. Supercurrents are confined to a thickness λ, which acts as the natural large-distance cutoff in a bulk type-II superconductor. Thus the properties of a type-II superconductor are generally insensitive to the size or shape of the sample. In contrast, a neutral superfluid has no corresponding shielding length, so that the behavior of liquid He II depends explicitly on the size of the container. In particular, the critical magnetic field H_{c1} is of order $(\varphi_0/\lambda^2) \times\ln(\lambda/\xi)$, where $\varphi_0 = hc/2e$ is the quantum of magnetic flux and ξ is the coherence length (or core radius). This expression is obviously similar to that for Ω_0 [Eq. (54)] or Ω_c [Eq. (66)]. It would be interesting to study a thin

[31] It is convenient to use the transformation $\vartheta_1(z+\frac{1}{2}\pi\tau|\tau) = iM^{-1}\vartheta_4(z|\tau)$ where $M=\exp(\frac{1}{4}i\pi\tau)\exp(iz)$ [Ref. 11, p. 464].

[32] G. B. Hess (private communication).

[33] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

cylinder made of type-II superconductor, or a thin superconducting film in the shape of a narrow annulus. These superconducting systems should exhibit the essential features of superfluid He II in a narrow annulus, while the difficult experimental problem of observing the flow states would be greatly simplified by the electromagnetic effects associated with the changed supercurrents.

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APPENDIX

This Appendix contains the evaluation of the following integral

$$I(r,r')=\gamma \operatorname{Re}\int_{-\pi}^{\pi} d\varphi \left\{\frac{\vartheta_1'[\gamma \ln(r/r')+i\gamma(\varphi-\varphi')]}{\vartheta_1[\gamma \ln(r/r')+i\gamma(\varphi-\varphi')]} - \frac{\vartheta_1'[\gamma \ln(rr'/R_2^2)+i\gamma(\varphi-\varphi')]}{\vartheta_1[\gamma \ln(rr'/R_2^2)+i\gamma(\varphi-\varphi')]}\right\}. \quad \text{(A1)}$$

Since the physical properties of the system are unchanged by a change of coordinate axis, it is clear that Eq. (A1) is independent of φ', which may be set equal to zero for convenience. The integral is most simply evaluated by exploiting the periodicity of the theta functions in the complex plane. We shall therefore consider the following contour integral

$$\int_C dz[\vartheta_1'(z)/\vartheta_1(z)] \quad \text{(A2)}$$

taken over a rectangular path shown in Fig. 4, where the corners are given by the points $z=\gamma \ln(r/r')\pm i\gamma\pi$ and $z=\gamma\ln(rr'/R_2^2)\pm i\gamma\pi$. The value of the contour integral is $2\pi i$ times the number of zeros of $\vartheta_1(z)$ enclosed, because $\vartheta_1(z)$ is an integral function.[34] The only relevant zero of $\vartheta_1(z)$ is at the origin, which lies inside the contour if $r>r'$ and otherwise lies outside.

[34] Reference 11, pp. 119 and 465.

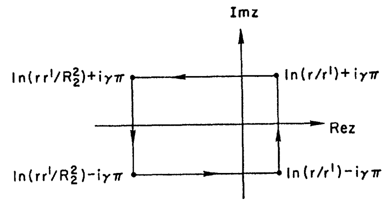

FIG. 4. Integration contour for Eq. (A2).

Hence Eq. (A2) may be written as

$$\int_C dz[\vartheta_1'(z)/\vartheta_1(z)]=2\pi i\eta(r-r'), \quad \text{(A3)}$$

where η is the step function defined below Eq. (17).

The integral along the horizontal portions of the contour is

$$\int_{x_0}^{x_1} dx\left\{\frac{\vartheta_1'(x-i\gamma\pi)}{\vartheta_1(x-i\gamma\pi)}-\frac{\vartheta_1'(x+i\gamma\pi)}{\vartheta_1(x+i\gamma\pi)}\right\}, \quad \text{(A4)}$$

where

$$x_0=\gamma \ln(rr'/R_2^2),$$
$$x_1=\gamma \ln(r/r'). \quad \text{(A5)}$$

Equation (7) shows that $i\gamma\pi=\frac{1}{2}\pi\tau$, and the integrand of Eq. (A4) may be rewritten as

$$\frac{\vartheta_1'(x-i\gamma\pi)}{\vartheta_1(x-i\gamma\pi)}-\frac{\vartheta_1'(x-i\gamma\pi+\pi\tau)}{\vartheta_1(x-i\gamma\pi+\pi\tau)}=2i, \quad \text{(A6)}$$

where the right side follows from the periodicity of the theta function.[35] Substitution of Eq. (A6) into Eq. (A4) shows that the contribution from the horizontal sides of the rectangle is $2i(x_1-x_0)=4i\gamma \ln(R_2/r')$. The integral along the vertical portion of the contour is just $iI(r,r')$, and Eq. (A3) thus becomes

$$iI(r,r')+4i\gamma \ln(R_2/r')=2\pi i\eta(r-r'). \quad \text{(A7)}$$

Substitution of Eq. (7) yields

$$I(r,r')=2\pi\eta(r-r')-2\pi \ln(R_2/r')[\ln(R_2/R_1)]^{-1}. \quad \text{(A8)}$$

The evaluation of the circulation [Eqs. (16) and (17)] is straightforward using Eq. (A8).

[35] Reference 11, p. 465.