Parametric Effects of Radiation on a Plasma*

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The parametric excitation of the modes of an infinite plasma by intense incident radiation is studied on the basis of the Vlasov equation. It is found that the modes can be driven into unstable oscillations for incident frequencies in the three regions $\omega_0 \simeq \omega_{pe}$, $\omega_{pe}+\omega_i$, and $2\omega_{pe}$, where ω_{pe} is the electron plasma frequency, and ω_i is the ion acoustic frequency. In the limit of weak intensities, the features of the two resonances $\omega_0 \simeq \omega_{pe}+\omega_i$ and $2\omega_{pe}$ are found to be in substantial agreement with the results of DuBois and Goldman. For larger intensities it is found that the resonance $\omega_0 \simeq \omega_{pe}+\omega_i$ is restricted to frequencies ω_0 which are not more than $4\omega_{pi}$ above or ω_i below this value, and has a maximum growth rate of $0.05\omega_{pe}$. The resonance near $\omega_0 \simeq \omega_{pe}$ is found to be dominated by collisional damping if $\gamma/\omega_{pe} > 10^{-4}$, and limited to a range of frequencies ω_0 of only $\omega_{pi}/100$. The present results do not generally agree with the results obtained by Silin. These results indicate that the usual harmonic approximation for the plasma is justified except in the above-mentioned frequency regions.

I. INTRODUCTION

In recent years there has been considerable interest in the effects which arise from the interaction of intense radiation with a plasma. To investigate the nonlinear interactions between the radiation and the plasma, at least two simplifying approaches are possible. The most common approach has been to treat the plasma as a linear system which can be described by the usual linear modes of oscillation [given by the zeros of the linear dielectric function $\epsilon(\mathbf{k},\omega)$]. These modes are then assumed to be excited by, and subsequently scatter, the incoming radiation. Studies based on this approach have been used to examine such effects as optical mixing,[1] "light-by-light" scattering,[2] and stimulated Raman scattering.[3] A unified treatment of these various effects can be found in a recent paper by Baym and Hellwarth.[4] All of these phenomena require radiation of extreme intensity (10^9-10^{10} W/cm^2) to produce even marginally observable results. The essential point, for our purposes, is simply to note that all of these effects have been studied under the assumption that the plasma can be treated in a linear fashion (what Baym and Hellwarth termed the "harmonic approximation"), whereas the radiation is treated nonlinearly. In particular it has been assumed that the radiation does not produce any significant modifications of the linear modes of the plasma.

A second possible approach is to ignore the scattering of the radiation as it passes through the plasma, and instead to concentrate on the modification which it induces in the linear modes of the plasma. This approach can then be used to examine the harmonic approximation on which the above studies are based. Obviously, if the harmonic approximation fails, and the radiation also becomes strongly modified, then it becomes necessary to treat both the radiation and the plasma in a nonlinear manner. However, in order to determine the validity of the harmonic approximation, it suffices to treat the radiation as given, and the amplitudes of the plasma oscillations as small (i.e., neglecting mode-mode coupling). It should be noted that the latter approximation does not generally hold for a finite plasma where large amplitude oscillations may be excited by intense radiation and mode-mode coupling becomes important.

The most interesting effect which arises from this approach is the possibility of generating instabilities in the modes by a parameteric action of the radiation field. This possibility has also been noted recently by Silin[5] and by DuBois and Goldman.[6] The analysis of DuBois and Goldman is based on a Green's-function perturbative analysis, which is restricted to the case when the radiation-induced energy of the particles is small compared to their thermal energy. They showed that, even with this restriction, the plasma can be unstable to certain applied frequencies. However, in order to justify the harmonic approximation for greater intensities, it is important to estimate the range of frequencies which produce instabilities when the intensity is very large. In the present analysis their restriction on the intensity is removed, and the parametric effects are examined from the point of view of the Vlasov equation. The analysis of Silin is largely based on the hydrodynamic equations for a cold plasma. Therefore, he considers the case when the radiation-induced energy of the particles is large compared to their thermal energy, which should complement the work of DuBois and Goldman. However, since he neglects the spatial variation of the applied electric field, he failed to obtain one unstable region and his

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other regions of instability do not appear to bear any similarity with what is found in the present study. Moreover, it will be shown that there can be a "fine structure" in the unstable regions near ω_{pe}, which neither of the investigators appear to have noticed.

The physical origin of the parametric effects of radiation on the modes in an infinite plasma is fairly easy to understand on a qualitative basis. An applied electric field $\mathbf{E}_0 \cos(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$ generates drift velocities in the components of the plasma $(q\mathbf{E}_0/m\omega_0) \times \sin(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$, where q and m is the charge and mass of that component. The density perturbations of wave number $\mathbf{k}$ then have their charged components shifted by a relative distance $(q/m - q'/m')(\mathbf{k} \cdot \mathbf{E}_0/\omega_0^2 k) \times \cos(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$. Provided that the charge-to-mass ratio of the mobile components is not the same, the field then causes the modes of frequency ω to acquire frequency components $n\omega_0 \pm \omega$. In this way, the relatively undamped high- and low-frequency modes of the plasma (with frequency ω_H and ω_L, respectively) acquire frequency components $n\omega_0 \pm \omega_H$ and $n\omega_0 \pm \omega_L$. Several possibilities then arise. As noted by DuBois and Goldman, if $\omega_0 \simeq \omega_H + \omega_L$ then $\omega_0 - \omega_H \simeq \omega_L$ so that the radiation, together with the high-frequency mode, tends to excite the low-frequency mode. Moreover, since $\omega_0 - \omega_L \simeq \omega_H$, the low-frequency mode and the radiation act to also excite the high-frequency mode. This interplay between the high-frequency and low-frequency modes and the radiation can then lead to instabilities of both modes. This may occur even if the low-frequency mode is strongly Landau damped, as in the case when the electron and ion temperatures are comparable. Another possibility which was apparently not considered by DuBois and Goldman, is when $\omega_0 \simeq \omega_H$. In this case the high-frequency mode and the radiation produce a density modulation with a frequency near $2\omega_H$, and this in turn interacts again with the high-frequency mode, producing a frequency component ω_H, which can again lead to instabilities. This, together with still another resonance near $\omega_0 \simeq \omega_H - \omega_L$, produces a fine structure in the unstable region near the electron plasma frequency. Since the separation between these resonances is only of the order of the ion plasma (or acoustic) frequency, this fine structure may only be of academic interest. However, since the physical mechanism which produces these resonances is quite distinct, and since it is difficult to judge *a priori* the width of these resonances, we will consider both of them in this study. The work of Silin does not distinguish between these resonances, nor is it clear to us what physical mechanism is responsible for his instabilities. Finally, the spatial variation of the applied electric field can produce an instability at the higher frequency $\omega_0 \simeq 2\omega_H$, provided that $\mathbf{k} \cdot \mathbf{k}_0$ does not vanish. This instability, which was pointed out by DuBois and Goldman, does not depend on the interaction between two types of modes in the plasma. It is the highest frequency parametric instability in a plasma. The infinite number of subharmonic instabilities (e.g., $\omega_0 \simeq \omega_H/n$), which were considered by Silin, will not be studied here, since these cases do not appear to correspond to a physically realizable situation (the frequency ω_0 must be larger than the electron plasma frequency in order for the radiation to enter the plasma).

In the following section we will first obtain the basic equations on which our analysis will be based. In Sec. III, we will examine instabilities which are present in the dipole approximation ($k_0 = 0$), and make some comparisons of our results with those obtained by Silin and by DuBois and Goldman. In Sec. IV, we will investigate the instabilities which arise from the spatial variation of the applied electric field.

II. BASIC EQUATIONS

We begin by considering an infinite plasma described by the Vlasov equation. We assume that the plasma is subjected to a transverse electric field

$$\mathbf{E}_0(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t), \quad (\mathbf{k}_0 \cdot \mathbf{E}_0 = 0). \tag{1}$$

The modification of this radiation field due to the plasma can be approximated by an index of refraction, $n(\omega_0) = (1 - \omega_{pe}^2/\omega_0^2)^{1/2}$, and the usual relationship $ck_0 = n(\omega_0)\omega_0$. In order for this field to penetrate the plasma, we must take ω_0 to be somewhat larger than the electron plasma frequency $\omega_{pe} = (4\pi n e^2/m)^{1/2}$. Finally, the effects due to the magnetic field will be ignored, since we will assume that the thermal velocities of the particles is much less than the velocity of light.

If there are no density variations in the plasma, then the distribution function for each component satisfies

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} \mathbf{E}_0(\mathbf{r},t) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0.$$

A solution of this equation, which satisfies the condition of no density variations, is $f(\mathbf{r},\mathbf{v},t) = f_0(\mathbf{u})$, where

$$\mathbf{u} = \mathbf{v} - (q\mathbf{E}_0/m) \int_\tau^t dt' \cos(\mathbf{k}_0 \cdot [\mathbf{r} - \mathbf{v}(t-t')] - \omega_0 t')$$

are constants of the motion [τ may also be a function of the constants $\mathbf{k}_0 \cdot (\mathbf{r} - \mathbf{v}t)$ and $\mathbf{E}_0 \times \mathbf{v}$]. The functions $f_0(\mathbf{u})$ are arbitrary, and can be selected on the basis of considerations which are actually not included explicitly in the collisionless approximation used here. Thus, for example, collisions within each component would tend to make the functions Maxwellian (possibly with a constant drift velocity). While we will take the functions $f_0(\mathbf{u})$ to be Maxwellian in order to obtain quantitative results, their functional form can be left arbitrary for the present. We will refer to any group of particles which is described by a different function $f_0(\mathbf{u})$ as a different "component" of the plasma, even if the charge and mass is the same (e.g., two groups of electrons moving relative to each other).

The constants of the motion $\mathbf{u}$ can be simplified if it is assumed (as we have) that the thermal velocities are all much less than the velocity of light. In this case, the expression for $\mathbf{u}$ may be approximated by

$$\mathbf{u}=\mathbf{v}-(q\mathbf{E}_0/m\omega_0)\sin(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t). \qquad (2)$$

We must consider perturbations of the plasma which are superimposed on this stationary state. The linearized equations for the perturbed distribution functions are

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}}+\frac{q}{m}\mathbf{E}_0(\mathbf{r},t)\cdot\frac{\partial f_1}{\partial \mathbf{v}}-\frac{q}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial \mathbf{v}}=0. \qquad (3)$$

This is a linear homogeneous equation which contains a given function $\mathbf{E}_0(\mathbf{r},t)$ as a parameter (or coefficient). For this reason, it is common to call excitations due to such a term parametric excitations. The electrostatic potential $\phi(\mathbf{r},t)$ satisfied Poisson's equation

$$\nabla^2\phi=-4\pi\sum_\sigma\int d^3v \quad q_\sigma f_{1\sigma}(\mathbf{r},\mathbf{v},t), \qquad (4)$$

where the sum is over the various components (as described above). We will now follow a method which is very similar to one used by Aliev and Silin,[7] except that we will retain the spatial variation of the applied electric field. We take the spatial variation of the perturbed quantities in the direction perpendicular to $\mathbf{k}_0$ to go as $e^{i\mathbf{k}_\perp\cdot r}$ (where $\mathbf{k}_0\cdot\mathbf{k}_\perp=0$), and set

$$f_1(\mathbf{r},\mathbf{v},t)=F(r_{\parallel},\mathbf{k}_\perp,\mathbf{u},t)e^{i\mathbf{k}_\perp\cdot\mathbf{r}} \times\exp\{-i(q\mathbf{k}\cdot\mathbf{E}_0/m\omega_0^2)\cos(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)\}, \qquad (5)$$

where $r_{\parallel}=\mathbf{k}_0\cdot\mathbf{r}/k_0$. The new distribution functions, $F(r_{\parallel},\mathbf{k}_\perp,\mathbf{u},t)$, describe the behavior of the component in a frame of reference in which there is no induced drift velocity. These functions satisfy the equations

$$\left[\frac{\partial F}{\partial t}+i\mathbf{k}_\perp\cdot\mathbf{u}F+u_{\parallel}\frac{\partial F}{\partial r_{\parallel}}\right] \times\exp\{\ \}-i\frac{q}{m}\phi\mathbf{k}_\perp\cdot\frac{\partial f_0}{\partial \mathbf{u}}-\frac{q}{m}\frac{\partial\phi}{\partial r_{\parallel}}\frac{\partial f_0}{\partial u_{\parallel}}=0, \qquad (6)$$

where $\{\ \}$ is the same as in Eq. (5). Dividing Eq. (6) by this term, and using the Bessel function identity[8]

$$\exp[iA\cos(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)]=\sum_{n=-\infty}^{\infty}(i)^n J_n(A)e^{in(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}, \qquad 7)$$

and then taking the Fourier transform with respect to $r_{\parallel}$ and time yields

$$(\mathbf{k}\cdot\mathbf{u}-\omega)F(\mathbf{k},\mathbf{u},\omega)-(q/m)\sum_{n=-\infty}^{\infty}(i)^n J_n(\mu)(\mathbf{k}+n\mathbf{k}_0)\cdot\frac{\partial f_0}{\partial \mathbf{u}} \times\phi(\mathbf{k}+n\mathbf{k}_0,\ \omega+n\omega_0)=0, \qquad (8)$$

where $\mu_\sigma=q_\sigma\mathbf{k}\cdot\mathbf{E}_0/m_\sigma\omega_0^2$. In the present notation, the perturbed quantities f_1 go as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ times the exponential factor in Eq. (5). Introducing the function

$$\rho(\mathbf{k},\omega)=q\int d^3uF(\mathbf{k},\mathbf{u},\omega),$$

then, from (8) we obtain

$$\rho(\mathbf{k},\omega)=-\frac{q^2}{m}\sum_{n=-\infty}^{\infty}(i)^n J_n(\mu)\int\frac{(\mathbf{k}+n\mathbf{k}_0)\cdot(\partial f_0/\partial\mathbf{u})}{\omega-\mathbf{k}\cdot\mathbf{u}+i\delta}d^3u \times\phi(\mathbf{k}+n\mathbf{k}_0,\ \omega+n\omega_0), \qquad (9)$$

where δ is a positive infinitesimal which yields the usual Landau contour (this will be suppressed in what follows). Also, if (5) is substituted into Poisson's equation, the Fourier transformed equation becomes [again with the use of (7)],

$$\phi(\mathbf{k},\omega)=\frac{4\pi}{k^2}\sum_\sigma\sum_{n=-\infty}^{\infty}(i)^n J_n(-\mu_\sigma) \times\rho_\sigma(\mathbf{k}+n\mathbf{k}_0,\ \omega+n\omega_0). \qquad (10)$$

There are now several possible ways to proceed. The most obvious thing to do at this point is to eliminate the quantities $\rho_\sigma(\mathbf{k},\omega)$ between Eqs. (9) and (10). The resulting expression for the Fourier components of ϕ is

$$\phi(\mathbf{k},\omega)=-\sum_{m=-\infty}^{\infty}S_m(\mathbf{k},\omega,\mathbf{k}_0,\omega_0)\phi(\mathbf{k}+m\mathbf{k}_0,\ \omega+m\omega_0), \qquad (11)$$

where

$$S_m(\mathbf{k},\omega,\mathbf{k}_0,\omega_0)=(i)^m\sum_\sigma\sum_{n=-\infty}^{\infty}J_n(\mu_\sigma)J_{n-m}(\mu_\sigma)\frac{4\pi q_\sigma^2}{k^2 m_\sigma} \times\int\frac{\mathbf{k}+m\mathbf{k}_0\cdot(\partial f_{0\sigma}/\partial\mathbf{u})}{\omega+n\omega_0-(\mathbf{k}+n\mathbf{k}_0)\cdot\mathbf{u}}d^3u \qquad (12)$$

acts as an electric susceptibility matrix. It will be noted that the various components appear in an additive fashion in this matrix. Now, as discussed in the Introduction, we know that this system is stable provided that we can set $k_0=0$, and if all of the μ_σ are equal. In fact, under these conditions, the dispersion relation reduces (as we will show later) to the usual form, namely

$$1+\sum_\sigma\chi_\sigma(k,\omega)=0, \qquad (13)$$

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[7] Yu M. Aliev and V. P. Silin, Zh. Eksperim. i Teor. Fiz. **48**, 901 (1965) [English transl.: Soviet Phys.—JETP **21**, 601 (1965)].

[8] A rather trivial point should be mentioned. The choice of $\cos(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)$ instead of $\sin(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)$ has been made because it introduces some simplicity into the analysis at a later stage (Sec. III, Case B). It is essentially for the same reason that the Mathieu and Hill equations are written with cosine parametric terms. This choice introduces the factors $(i)^n$ in (7), which always drop out in the final dispersion relations, so we do not use the notation $I_n(iA)$.

where

$$\chi_\sigma(\mathbf{k},\omega)=\frac{4\pi q_\sigma^2}{k^2 m_\sigma}\int\frac{\mathbf{k}\cdot(\partial f_{0\sigma}/\partial\mathbf{u})}{\omega-\mathbf{k}\cdot\mathbf{u}}d^3u \tag{14}$$

is the linear electric susceptibility of the component σ. Thus, the total effect of the electric field in this case is contained in the exponential factor in Eq. (5). While these facts can be proved starting with Eq. (11), they are by no means obvious just from the form of the equations. In order to make this feature explicit, it appears to be necessary to use a somewhat more awkward approach, and to eliminate $\phi(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0)$ between Eqs. (9) and (10). In this case, we obtain the system of equations

$$\rho_\sigma(\mathbf{k},\omega)=-\frac{4\pi q_\sigma^2}{m_\sigma}\sum_{\sigma'}\sum_{n,m}(i)^{n+m}J_n(\mu_\sigma)\times J_m(-\mu_{\sigma'})(\mathbf{k}+n\mathbf{k}_0)^{-2}\int d^3u\frac{(\mathbf{k}+n\mathbf{k}_0)}{\omega-\mathbf{k}\cdot\mathbf{u}}\cdot\frac{\partial f_{0\sigma}}{\partial\mathbf{u}}\times\rho_{\sigma'}[\mathbf{k}+(m+n)\mathbf{k}_0,\,\omega+(m+n)\omega_0], \tag{15}$$

where the fact that $\mathbf{k}_0\cdot\mathbf{E}_0=0$ has been used in the arguments of the Bessel functions. To simplify this expression further, we will now make use of the fact that in most cases the wave number k is much larger than k_0. Moreover, as we will show below, only those components of $\rho_\sigma(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0)$ for which n is a small integer have an appreciable magnitude. Thus, as a first approximation, we may set k_0 equal to zero in (15)—the so-called dipole approximation. In this case, by making the substitution $m=n'-n$, the sum on n can be performed with the aid of the relationship

$$\sum_{n=-\infty}^{\infty}J_n(a)J_{n'-n}(b)=J_{n'}(a+b), \tag{16}$$

and one obtains (dropping the prime on n')

$$\rho_\sigma(\mathbf{k},\omega)=-\chi_\sigma(\mathbf{k},\omega)\sum_{\sigma'}\sum_{n=-\infty}^{\infty}(i)^nJ_n(\mu_{\sigma\sigma'})\times\rho_{\sigma'}(\mathbf{k},\,\omega+n\omega_0), \tag{17}$$

where $\chi_\sigma(\mathbf{k},\omega)$ is given by (14), and

$$\mu_{\sigma\sigma'}=\mu_\sigma-\mu_{\sigma'}=\left(\frac{q_\sigma}{m_\sigma}-\frac{q_{\sigma'}}{m_{\sigma'}}\right)\frac{\mathbf{k}\cdot\mathbf{E}_0}{\omega_0^2}=-\mu_{\sigma'\sigma}. \tag{18}$$

Equation (17) now shows that if $\mu_{\sigma\sigma'}=0$, then only the term $n=0$ is nonzero in the sum, and thus one readily recovers the dispersion relationship (13). Equation (17) was used by Aliev and Silin to consider the stabilizing effect of radiation on the drift instabilities in a plasma, when $\omega_0\gg\omega_{pe}$. Because they considered such high frequencies, the drift instability could only be modified if the electric field is strong enough to reverse the direction of the drift motion for an appreciable fraction of the period of the applied field. More recently, Silin has used the cold hydrodynamic approximation to Eq. (17) to study the parametric effects for lower values of ω_0.

To obtain the first-order corrections to (17) due to finite values of k_0, we approximate $(\mathbf{k}+n\mathbf{k}_0)^{-2}(\mathbf{k}+n\mathbf{k}_0)$ in Eq. (15) by $k^{-2}\mathbf{k}-nk^{-4}(2\mathbf{k}\cdot\mathbf{k}_0\mathbf{k}-k^2\mathbf{k}_0)$. Noting that $nJ_n(A)=1/2A\,(J_{n+1}+J_{n-1})$, one can again perform one of the sums in (15) with the help of (16). One then obtains

$$\rho_\sigma(\mathbf{k},\omega)=-\chi_\sigma(\mathbf{k},\omega)\sum_{\sigma'}\sum_n(i)^nJ_n(\mu_{\sigma\sigma'})\rho_{\sigma'}(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0)-\frac{4\pi q_\sigma^2}{m_\sigma k^4}\int d^3u\frac{(2\mathbf{k}_0\cdot\mathbf{kk}-\mathbf{k}^2\mathbf{k}_0)\cdot(\partial f_{0\sigma}/\partial\mathbf{u})}{\omega-\mathbf{k}\cdot\mathbf{u}}\times\tfrac{1}{2}\mu_\sigma\sum_{\sigma'}\sum_n(i)^n[J_{n+1}(\mu_{\sigma\sigma'})+J_{n-1}(\mu_{\sigma\sigma'})]\times\rho_{\sigma'}(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0). \tag{19}$$

Now the second group of terms in (19) does not vanish even if all of the $\mu_{\sigma\sigma'}$ vanish (i.e., all charge to mass ratios are equal), and since $\mathbf{k}\cdot\mathbf{E}_0$ appears as a coefficient of this term, the electric field can still have an influence on the stability of the system. If one assumes that the functions $f_0(\mathbf{u})$ are isotropic in $\mathbf{u}$ (in particular, that there are no drift velocities to the components), then (19) can be written in the more compact form

$$\rho_\sigma(\mathbf{k},\omega)=-\chi_\sigma(\mathbf{k},\omega)\sum_{\sigma'}\sum_n(i)^nJ_n(\mu_{\sigma\sigma'})\rho_{\sigma'}(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0)-\tfrac{1}{2}(\mathbf{k}_0\cdot\mathbf{k}/k^2)\mu_\sigma\chi_\sigma(\mathbf{k},\omega)\sum_{\sigma'}\sum_n(i)^n\times[J_{n+1}(\mu_{\sigma\sigma'})+J_{n-1}(\mu_{\sigma\sigma'})]\times\rho_{\sigma'}(\mathbf{k}+n\mathbf{k}_0,\,\omega+n\omega_0). \tag{19a}$$

It is a simple matter to obtain higher order corrections to (19) following the above procedure, but we shall not consider them here.

We shall now consider the dipole approximation ($k_0=0$), and then return to the more general case (19) in Sec. IV.

III. THE DIPOLE APPROXIMATION ($k_0=0$)

In this section we shall neglect the spatial variation of the applied electric field over distances of the order of the perturbation wavelength. If $k\simeq0.1k_D$, where k_D is the Debye wave number, then for $\omega_0\simeq1.1\omega_{pe}$, $k_0\simeq0.4\omega_0/c$ [see discussion following Eq. (10)], so

$$k_0\simeq0.44\omega_{pe}/c=0.44k_D(v_T/c)=4.4k(v_T/c),$$

where $v_T=(\kappa T/m)^{1/2}$ is the thermal velocity. Thus, if $4v_T/c\ll1$, then $k\gg k_0$ and the dipole approximation is justified—at least for those phenomena which do not vanish when k_0 is set equal to zero. As we shall see, there are instabilities which only occur if $k_0\neq0$, and these will be taken up in the following section.

In the present approximation, we can use Eq. (17) and, if we restrict our considerations to two components (electrons and ions), we can write this equation in the form

$$\epsilon_\sigma(\omega+m\omega_0)\rho_\sigma(\omega+m\omega_0)=-\chi_\sigma(\omega+m\omega_0)\times\sum_{n=-\infty}^{\infty}(i)^{n-m}J_{n-m}(\mu_{\sigma\sigma'})\rho_{\sigma'}(\omega+n\omega_0)\,,\quad(20)$$

where $\epsilon_\sigma(k,\omega)=1+\chi_\sigma(k,\omega)$ is the usual linear dielectric function, and we have suppressed the wave number k. In Eq. (20), σ refers to one component and σ' to its conjugate component [and not a summation index, as in (17)]. Thus, (20) represents two (infinite) sets of equations. With the recurrence relationship written in the form (20), where m is an arbitrary integer, one can require that $\mathrm{Re}(\omega)\leq\omega_0/2$ without any loss in generality.

Our primary interest is to determine whether there are unstable roots ω to Eq. (20) for any (real) values of ω_0. Following the discussion in the Introduction, we expect that there might be instabilities if ω_0 is near a high-frequency mode ω_H and possibly in the region of $2\omega_H$ (for this is frequently the case in parametric excitations). To illustrate the nature of the high-frequency mode, we will first obtain an approximate expression for ω_H, and also for the low-frequency mode ω_L in the limit of very high ω_0. Modifications of these results for lower ω_0 will be taken up in the following sections.

If ω_0 is very large (compared to all plasma frequencies), the only terms in (20) which are of any importance are the ones for which $m=n=0$, for all other $\chi_\sigma(\omega+m\omega_0)$ are very small. Hence, in this limit, the frequency ω is determined by the roots of

$$\epsilon_e(\omega)\epsilon_i(\omega)-J_0^2(\mu_{ei})\chi_e(\omega)\chi_i(\omega)=0\,.\quad(21)$$

When $E_0=0$, this reverts to (13), but if $E_0\neq0$ the factor $J_0^2(\mu_{ei})$ gives the approximate effect of the applied field on the frequency of the modes. Obviously, if J_0 vanishes, then the neglected terms become dominant, but nonetheless they always represent only small corrections. The frequencies ω_H and ω_L are then given by the high- and low-frequency roots of (21) which have the smallest damping rate. To obtain approximate expressions for these roots, it is sometimes useful to use the hydrodynamic approximation, for which

$$\chi_\sigma(\mathbf{k},\omega)=\omega_{p\sigma}^2/(3k^2v_{T\sigma}^2-\omega^2)\,,\quad(22)$$

where $\omega_{p\sigma}^2=4\pi n_\sigma^0q_\sigma^2/m_\sigma$, and $v_{T\sigma}=(\kappa T_\sigma/m_\sigma)^{1/2}$. This approximation neglects Landau damping and is only justified if $\omega/k\gg v_T$. Therefore, this approximation does not adequately describe the low-frequency mode, unless the ion temperature is much less than the electron temperature. In this approximation the high- and low-frequency modes (ω_H^2,ω_L^2) are given by

$$\tfrac{1}{2}(\omega_{ke}^2+\omega_{ki}^2)\pm\tfrac{1}{2}\{(\omega_{ke}^2-\omega_{ki}^2)^2+4J_0^2(\mu_{ei})\omega_{pe}^2\omega_{pi}^2\}^{1/2},\quad(23)$$

where $\omega_{k\sigma}^2=\omega_{p\sigma}^2+3k^2v_{T\sigma}^2$. As E_0 is increased, the high-frequency mode decreases toward its minimum value ω_{ke}, whereas the low-frequency mode increases towards its maximum value ω_{ki}. For more accurate estimates of these frequencies, one can use the usual asymptotic expressions [assuming that $f_0(\mathbf{u})$ is Maxwellian]

$$\chi(\mathbf{k},\omega)\simeq(kD)^{-2}\left[-\frac{1}{2x^2}\left(1+\frac{3}{2x^2}\right)+i\pi^{1/2}xe^{-x^2}+\frac{iy}{x^3}\right]\begin{pmatrix}x^2\gg1\\y^2\ll1\end{pmatrix},\quad(24)$$

$$\chi(k,\omega)\simeq(kD)^{-2}[1+ix\pi^{1/2}](x^2,y^2\ll1)\,,$$

where $D=(\kappa T/4\pi n_0q^2)^{1/2}$ is the Debye length, and $(\omega/k)(m/2\kappa T)^{1/2}=x+iy$. Even these expressions can only be used to determine the low-frequency mode, provided that $T_e\gg T_i$. If $T_e\simeq T_i$, then this mode must be determined numerically, and is rather heavily damped.[9] The effect of the term J_0^2 on these modes has been described in more detail by Aliev and Silin.

To investigate the stability of the system, we note first that since ω_0 cannot be much smaller than the electron-plasma frequency, then $\chi(\omega+m\omega_0)$ is quite small unless $m=0$, ±1 [note again that we are using the convention $\omega_0/2\geq\mathrm{Re}(\omega)$]. Thus, in Eq. (20), we will restrict our considerations to the components $\rho_\sigma(\omega)$ and $\rho_\sigma(\omega\pm\omega_0)$ and obtain the six equations

$$\epsilon_\sigma(\omega)\rho_\sigma(\omega)=-\chi_\sigma(\omega)\{J_0\rho_{\sigma'}(\omega)+iJ_1(\mu_{\sigma\sigma'})\times[\rho_{\sigma'}(\omega+\omega_0)+\rho_{\sigma'}(\omega-\omega_0)]\}\,,\quad(25)$$
$$\epsilon_\sigma(\omega\pm\omega_0)\rho_\sigma(\omega\pm\omega_0)=-\chi_\sigma(\omega\pm\omega_0)\{J_0\rho_{\sigma'}(\omega\pm\omega_0)+J_1(\mu_{\sigma\sigma'})\rho_{\sigma'}(\omega)-J_2\rho_{\sigma'}(\omega\mp\omega_0)\}\,.$$

These equations are sufficiently general to describe the entire frequency range above the electron-plasma frequency quite accurately. However, the resulting six-by-six determinant is not particularly transparent, so we will consider various unstable regions in which some simplifications can be made. These regions can be identified, in the limit of zero intensity, by the following classification:

Case A: $\omega_0\simeq\omega_H+\omega_L$ $(\omega\simeq\omega_L)$,
Case B: $\omega_0\simeq\omega_H$ $(\omega\simeq0)$,
Case C: $\omega_0\simeq2\omega_H$ $(\omega\simeq\omega_H)$.

We shall now consider these cases individually.

Case A ($\omega_0\simeq\omega_H\pm\omega_L$)

To illustrate these two unstable regions, we will consider the case $\omega_0\simeq\omega_H+\omega_L$ and comment below on the analogous calculation for $\omega_0\simeq\omega_H-\omega_L$. If $\omega_0\simeq\omega_H+\omega_L$ and $\omega\simeq\omega_L$ then, as E_0 goes to zero ($\mu\to0$), the most important frequency components are $\rho_\sigma(\omega-\omega_0)$ and $\rho_\sigma(\omega)$. The remaining terms in (25), $\rho_\sigma(\omega+\omega_0)$, tends to

[9] B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961).

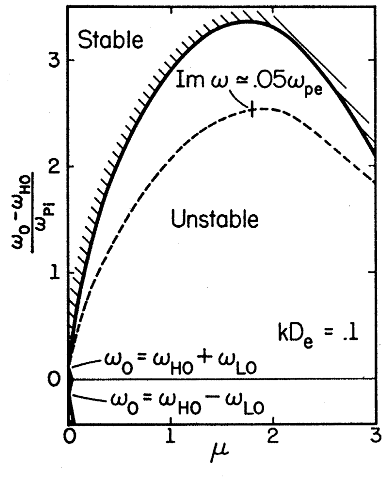

FIG. 1. The unstable regions emanating from $\omega_0=\omega_{H0}\pm\omega_{L0}$, where $\omega_{H0}^2=\omega_{ke}^2+\omega_{pi}^2$ and $\omega_{L0}=\omega_{pi}kD_e$ and $kD_e=0.1$. The dashed curve gives the frequency for maximum growth for a given value of μ. The line $\omega_0=\omega_{H0}$ separates the two (mirror-image) unstable regions.

be less important because the frequency $\omega+\omega_0\simeq\omega_H+2\omega_L$ is further off resonance from any of the linear modes of the system. However, for larger values of E_0, the present unstable region extends down to values of ω_0 near ω_H, in which case $\rho_\sigma(\omega+\omega_0)$ is no longer negligible. Therefore, rather than ignore the terms $\rho_\sigma(\omega+\omega_0)$ entirely, we first eliminate them from the remaining four equations in (25) and then neglect all terms proportional to J_2. This approximation takes into account the dominant effects arising from $\rho_\sigma(\omega+\omega_0)$. The resulting four equations then yield the dispersion relation

$$\begin{aligned}&[1-(J_0^2+J_1^2)\Gamma_e(\omega-\omega_0)\Gamma_i(\omega-\omega_0)]\\&\times[1-(J_0^2+J_1^2)\Gamma_e(\omega)\Gamma_i(\omega)]\\&+J_1^2[\Gamma_e(\omega-\omega_0)-\Gamma_e(\omega)][\Gamma_i(\omega-\omega_0)-\Gamma_i(\omega)]\\&-J_1^2\Gamma_i(\omega)\Gamma_e(\omega+\omega_0)J_1^2[1-\Gamma_i(\omega-\omega_0)\Gamma_e(\omega-\omega_0)J_0^2]/\\&[1-\Gamma_i(\omega+\omega_0)\Gamma_e(\omega+\omega_0)J_0^2]=0,\quad(26)\end{aligned}$$

where $\Gamma_\sigma(\omega)\equiv\chi_\sigma(\omega)/\epsilon_\sigma(\omega)$. The last term of (26) is the principal contribution due to the components $\rho_\sigma(\omega+\omega_0)$, and contains a near-resonant denominator as $\omega_0+\omega$ decreases toward ω_H. The condition for the onset of instability is determined by the behavior of $\Gamma_\sigma(\omega)$ for real values of ω (as well as ω_0). In the case of Maxwellian distributions $f_0(\mathbf{u})$, this function can be evaluated from known functions.[10] In general, Eq. (26) can only be solved by numerical methods. However, aside from the question of the onset of instability for weak intensities (when the Landau damping becomes important), the hydrodynamic approximation for $\Gamma_\sigma(\omega)$ may be used to simplify Eq. (26). We shall therefore first consider this approximation, and return to considerations of damping afterwards.

If we use the approximation given in Eq. (22), so that $\Gamma_\sigma(\omega)\simeq\omega_{p\sigma}^2/\omega_{k\sigma}^2-\omega^2$, then Eq. (26) can be put in the form

$$\begin{aligned}&[(\omega-\omega_0)^2-\omega_H^2][(\omega-\omega_0)^2-\omega_L^2][\omega^2-\omega_H^2][\omega^2-\omega_L^2]\\&+J_1^2\omega_{pe}^2\omega_{pi}^2\{[\omega^2-(\omega-\omega_0)^2]^2+\omega_H^4[\omega_H^2-(\omega-\omega_0)^2]/\\&[\omega_H^2-(\omega+\omega_0)^2]\}=0,\quad(27)\end{aligned}$$

where some simplifying approximations have been used in the last term. Here ω_H and ω_L are the same as in Eq. (23), except that J_0^2 is replaced by $(J_0^2+J_1^2)$. This modification tends to make ω_H and ω_L somewhat less dependent on μ—at least for small values of μ. To determine the region of instability, we replace ω^2 by ω_L^2 and $(\omega-\omega_0)^2$ by ω_H^2 in all terms in (27) which do not thereby vanish, except for the denominator of the last term. If we now set[11]

$$\omega=\omega_L+\delta\omega,\quad\omega_0=\omega_H+\omega_L+\delta\omega_0$$

and assume that $\omega_H\gg|\delta\omega_0-\delta\omega|$, then (27) yields the quartic equation for $\delta\omega$

$$\begin{aligned}&[(\delta\omega)^2+2\omega_L\delta\omega]^2-[2\omega_L\delta\omega_0+(\delta\omega_0)^2][(\delta\omega)^2+2\omega_L\delta\omega]\\&+\omega_{pe}^2\omega_{pi}^2J_1^2[\omega_L+\delta\omega_0]/\omega_H=0.\quad(28)\end{aligned}$$

It follows from this that the boundary between the stable and unstable region occurs for values of $\delta\omega_0$ which satisfy

$$[(\delta\omega_0)^2+2\omega_L\delta\omega_0]^2=4(\omega_{pe}^2\omega_{pi}^2/\omega_H)J_1^2[\omega_L+\delta\omega_0].$$

This unstable region is shown in Fig. 1, where the expressions (31) have been used for ω_H and ω_L, and we have taken $kD_e=0.1$. The most significant feature of the unstable region is that it is limited to a range of frequencies $\omega_H+4\omega_{pi}\gtrsim\omega_0>\omega_H$. For even larger values of μ, the unstable region may have a wavy structure, but the above bounds on ω_0 should still be valid.

To obtain the maximum growth rate for a fixed value of μ, we solve Eq. (28) for $(\delta\omega)^2+2\omega_L\delta\omega$ and determine the value of $\delta\omega_0$ which produces the largest imaginary term in this equation. This yields the dashed curve shown in Fig. 1. A rough approximation of this curve can be obtained by noting that $\omega_{pe}^2\omega_{pi}^2J_1^2/\omega_H\omega_L^3$ is much larger than unity when $2\gtrsim\mu\gtrsim0.1$, in which case one finds that $\delta\omega_0/\omega_L\simeq(\omega_{pe}^2\omega_{pi}^2J_1^2/\omega_H\omega_L^3)^{1/3}$. Using this value of $\delta\omega_0$ in Eq. (28) yields the maximum growth rate as a function of μ,

$$\mathrm{Im}\,\omega\simeq\tfrac{1}{2}\omega_{pe}(m_eJ_1^2/m_i)^{1/3}\quad(2\gtrsim\mu\gtrsim0.1).\quad(29)$$

This indicates that the maximum growth rate is $0.028\omega_{pe}$ (for $\mu\simeq1.8$), compared to a more accurate value of $0.046\omega_{pe}$. To the degree that $J_1^2(\mu)$ can be approximated by $(\mu/2)^2$, Eq. (29) agrees with the result of Silin who found that the growth rate goes as $\omega_{pe}(m_e\mu^2/m_i)^{1/3}$. However, Silin gave no upper bound to this growth rate. Moreover, the unstable regions which we find (Figs. 1, 2, and 3) bear no similarity to the ones described by Silin. He appears to find two unstable regions which both emanate from $\omega_0^2=\omega_{pe}^2+\omega_{pi}^2$ as μ is increased from zero. One region consists of *all* frequencies for which $\omega_{pe}^2+\omega_{pi}^2-\omega_0^2>0$, with no cutoff either at low frequencies or low powers, while the second region lies

[10] B. D. Fried and S. C. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

[11] In the case where $\omega_0\simeq\omega_H-\omega_L$, if one makes analogous approximations and sets $\omega_0=\omega_H-\omega_L-\delta\omega_0$, and $\omega=\omega_L+\delta_\omega$, then one again recovers Eq. (29). Thus this unstable region is the mirror image of the present region reflected about the frequency ω_H. (See Figs. 1 and 2.)

somewhere above $\omega_0^2=\omega_{pe}^2+\omega_{pi}^2$. He also finds a zero growth rate along $\omega_0^2=\omega_{pe}^2+\omega_{pi}^2$, whereas we find this to be an unstable region (Case B). Considering these differences, the above-mentioned agreement may be fortuitous.

In order to analyze the effect of Landau damping, we first neglect the last term of Eq. (26), since we will consider only the region $\omega_0\simeq\omega_H+\omega_L$. Furthermore, we will assume that $T_e>T_i$ and that

$$\omega_L/kv_{Ti},\quad \omega_H/kV_{Te}\gg 1,\quad \text{and}\quad 1\gg\omega_L/kv_{Te},$$

so that the asymptotic expressions (24) may be used for $\chi_\sigma(\mathbf{k},\omega)$. In this case, one finds that the real part of Eq. (26) can be approximated by

$$\begin{aligned}&[(\omega-\omega_0)^2-\omega_H^2](\omega^2-\omega_L^2)\\&\quad-2\omega_{pe}^2[\gamma_{Le}(\omega-\omega_0)/\omega-\omega_0](\pi/2)^{1/2}\\&\quad\times(\omega/kv_{Te})[\omega^2-(1-J_0^2-J_1^2)\omega_{pi}^2]\\&\quad-J_1^2\omega_{pi}^2[\omega_{pe}^2+(\omega-\omega_0)^2]=0,\end{aligned}\qquad(30)$$

if ω is real and $1\gg kD_e$. Here γ_L is the linear Landau damping factor

$$\gamma_{L\sigma}(\omega)/\omega=(\pi/8)^{1/2}(\omega/kv_{T\sigma})^3\exp(-\omega^2/2k^2v_{T\sigma}^2),$$

and

$$\begin{aligned}\omega_L^2&=\omega_{pi}^2[(kD_e)^2-(1-J_0^2-J_1^2)],\\ \omega_H^2&=\omega_{ke}^2+\omega_{pi}^2-(1-J_0^2-J_1^2)\omega_{pi}^2.\end{aligned}\qquad(31)$$

Equation (30) is analogous to Eq. (27) without the last term. It is easy to show that (30) has real solutions for ω (when $\delta\omega_0=0$) only if the sum of the last two terms is positive. Hence, the condition for stability is found to be

$$(\pi/2)^{1/2}(\omega_L\gamma_{Le}(\omega_H)/\omega_H kv_{Te})>J_1^2/(kD_e)^2.\qquad(32)$$

The most important effect of damping is when the power of the incident radiation is weak, in which case $(1-J_0^2-J_1^2)(kD_e)^{-2}\ll1$, even though $1\gg kD_e$. In this

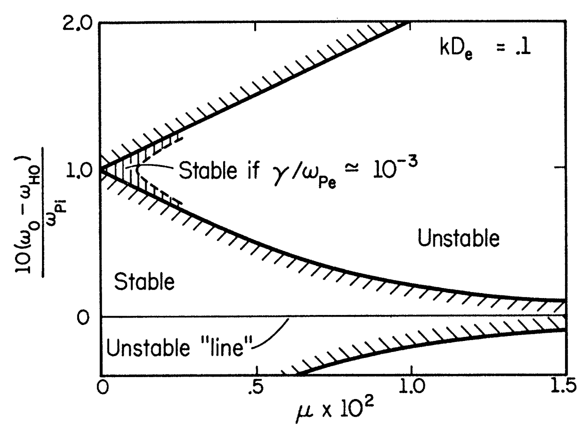

FIG. 2. Details of the unstable region (and its mirror image) near $\omega_0=\omega_{H0}$ in Fig. 1. The region of stabilization due to a damping rate of $\gamma/\omega_{pe}\simeq10^{-3}$ is shown. The unstable "line" along $\omega_0=\omega_{H0}$ is a narrow unstable region.

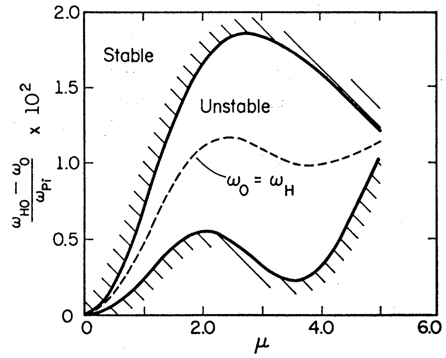

FIG. 3. The narrow unstable region emanating from $\omega_0=\omega_{H0}$. The growth rate along $\omega_0=\omega_H$ is roughly $\omega_{pi}J_2(\mu)$ (neglecting damping).

case, the condition for stability (32) reduces to

$$\left(\frac{\pi m_e}{2m_i}\right)^{1/2}\frac{\gamma_{Le}(\omega_H)}{\omega_H}>\frac{J_1^2(\mu)}{(kD_e)^2}\simeq\left(\frac{\mu}{2kD_e}\right)\simeq\frac{1}{4}\frac{(\mathbf{E}_0\cdot k)^2}{k^2 4\pi n\kappa T_e},\qquad(33)$$

where we have set $\omega_0\simeq\omega_{pe}$ in μ. The quantity $(E_0^2/4\pi n\kappa T_e)$ is essentially the ratio of the energy density in the radiation to the kinetic energy density of the electrons. It is this parameter which DuBois and Goldman used for their perturbative analysis, and the present limit $(1\gg\mu/kD_e)$ corresponds to the case which they considered. The present result differs from their result by a factor-of-4 increase in the power required for instability. Despite this increased power requirement, (33) still predicts instability for modest intensities. To illustrate this fact, assume that $10^{-3}\gtrsim\gamma(\omega_H)/\omega_H$. Then the system is unstable if $\mu/kD_e>1.1\times10^{-2}$. Now, if we set $\omega_0=\omega_p$ in the expression for μ, we obtain

$$\mu/kD_e\simeq1.55\times10^6(I/nT)^{1/2},\qquad(34)$$

where n is in cm^{-3}, T in °K, and the intensity $I=E_0^2c/4\pi$ is in W/cm². Thus, if $n=10^{12}$ and $T=10^4$, the system is unstable for an intensity of 1 W/cm². The details of the unstable region (together with a portion of the mirror region below ω_H) for small values of μ is shown in Fig. 2. The region which is stabilized by a damping rate $\gamma/\omega_{pe}\simeq10^{-3}$ is also shown for $\omega_0\simeq\omega_H+\omega_L$. The region in the immediate vicinity of $\omega_0=\omega_H$, labeled as an unstable "line" in Fig. 2, corresponds to another instability which we will now consider.

Case B ($\omega_0\simeq\omega_H$)

In this case, the frequency of the incident radiation is near the lower limit of the frequencies which will be transmitted by the plasma. Presumably, still lower frequencies (where other subharmonic resonances can occur) are therefore largely of only academic interest.

The present case is interesting, not only because it shows that there are other resonances besides Case A which are observable, but also because the mode interaction which produces the present resonance is quite different from the Case A. We note first that if we set $\omega=0$ in Eq. (25), and we ignore the imaginary part of $\chi_\sigma(\pm\omega_0)$, then there are two classes of solutions, namely,

$$\text{(a)}\quad \rho_\sigma(\omega_0)=-\rho_\sigma(-\omega_0)\,,\quad \rho_\sigma(0)=0$$
$$\text{(b)}\quad \rho_\sigma(\omega_0)=\rho_\sigma(-\omega_0)\,,\quad \rho_\sigma(0)\neq 0.$$

These solutions are analogous to the usual $\sin(n\omega_0 t)$ and $\cos(n\omega_0 t)$ solutions which separate the stable and unstable regions of simpler parametric equations (e.g., the Mathieu or Hill equations). The purpose of expressing the applied electric field in terms of a cosine function (see Ref. 8 in Sec. II) was to get this simple separation [into (a) and (b)] at this point. The question now arises as to whether these periodic solutions form boundaries between stable and unstable solutions in the present system of equations. We will show that this is the case for the solutions (a), but it is not the case for the solutions (b). In the present case, as contrasted with Case A, the instability does not depend upon the existence of a low-frequency mode (although it may be strongly affected by it), but instead it is caused by the nonlinear interaction of the mode on itself through the action of the electric field. (This is in fact the more common type of parametric instability.) It should be noted that while the instability does not depend on the second mode, it will only occur if there are two components with different charge-to-mass ratios.

We will first examine the solutions which for $\omega=0$ go over into solutions of type (a). To do this we ignore[12] $\rho_\sigma(\omega)$ for small ω, and then obtain from (25) the condition

$$\begin{aligned}&[1-\Gamma_e(\omega+\omega_0)\Gamma_i(\omega+\omega_0)J_0^2][1-\Gamma_e(\omega-\omega_0)\Gamma_i(\omega-\omega_0)J_0^2]\\&\quad-[\Gamma_e(\omega+\omega_0)\Gamma_i(\omega-\omega_0)+\Gamma_e(\omega-\omega_0)\Gamma_i(\omega+\omega_0)]J_2^2\\&\quad-\Gamma_e(\omega-\omega_0)\Gamma_e(\omega+\omega_0)\Gamma_i(\omega-\omega_0)\Gamma_i(\omega+\omega_0)\\&\qquad\qquad\times[2J_0^2J_2^2-J_2^4]=0.\qquad(35)\end{aligned}$$

If we first examine this using the hydrodynamic approximation (22), then we obtain

$$\begin{aligned}&[(\omega+\omega_0)^2-\omega_H^2][(\omega-\omega_0)^2-\omega_H^2][\omega_H^2-\omega_L^2]^2\\&\qquad-(\omega_{pe}\omega_{pi})^4(4J_0^2J_2^2-J_2^4)=0,\qquad(36)\end{aligned}$$

where ω_H and ω_L are given by (23). This is readily solved to yield

$$\begin{aligned}\omega^2=\omega_0^2+\omega_H^2-\{&(\omega_0^2+\omega_H^2)^2-(\omega_H^2-\omega_0^2)^2\\&+(\omega_{pe}\omega_{pi})^4(4J_0^2J_2^2-J_2^4)/(\omega_H^2-\omega_L^2)^2\}^{1/2},\end{aligned}$$

which predicts that these solutions are unstable if

$$(\omega_{pe}\omega_{pi})^4J_2^2(4J_0^2-J_2^2)/(\omega_H^2-\omega_L^2)^2>(\omega_H^2-\omega_0^2)^2.\qquad(37)$$

As will be shown shortly, the expression (37) is not accurate when $J_2\gtrsim 2J_0$, corresponding to a value of μ between two and three. The maximum growth rate in the present case is predicted to be of the order of $\omega_{pi}J_2$, as compared with the growth rate $\omega_{pe}(m_eJ_1^2/m_i)^{1/3}$ in the Case A [Eq. (29)]. Therefore, the present instability is usually weaker than the instability in Case A.

To examine the effects due to damping, Eq. (35) can be analyzed in the same way as used in Case A [except now the condition $T_e/T_i>1$ is not required, but only that $(\omega_0\pm\omega/k)\gg v_{Te},\ v_{Ti}$]. The real part of (35) can then be put in the form

$$\begin{aligned}&[(\omega+\omega_0)^2-\omega_H^2][(\omega-\omega_0)^2-\omega_H^2][\omega_H^2-\omega_L^2]^2\\&\qquad+4\omega_{pe}^4\omega_H^4[\gamma_L(\omega_H)/\omega_H]^2\\&\qquad\qquad-\omega_{pe}^4\omega_{pi}^4J_2^2[2+2J_0^2-J_2^4]=0.\end{aligned}$$

This equation differs from (36) not only in the addition of the damping term, but also in a slight change in the last term. If we set $\omega_H\simeq\omega_{pe}$, this result shows that the system is unstable only if[13]

$$J_2^2[2+2J_0^2-J_2^2]>4(m_i/m_e)^2(\gamma/\omega_{pe})^2.\qquad(38)$$

Because of the large mass ratio factor, the present instability will only occur if γ/ω_{pe} is less than about 10^{-4}—which may not be satisfied by the collisional damping. The unstable region, as given by Eq. (37)—with $4J_0^2-J_2^2$ replaced by $2+2J_0^2-J_2^2$, is shown in Fig. 3. The outstanding feature of this region is that is has an extremely narrow frequency range—of the order of $\omega_{pi}/100$. This fact, coupled with the condition (38), shows that this instability is probably only of academic interest, for it would be extremely difficult to observe.

We now turn to the stability of the solutions of type (b), and assume that, for small ω, $\rho_\sigma(\omega+\omega_0)\simeq\rho_\sigma(\omega-\omega_0)$, but now retain $\rho_\sigma(\omega)$. In this case, Eq. (25) yields (in the hydrodynamic approximation)

$$\begin{aligned}&[(\omega+\omega_0)^2-\omega_H^2][\omega_L^2-\omega^2][\omega_H^2-\omega_L^2]\\&\qquad\qquad-2\omega_{pe}^2\omega_{pi}^2\omega_H^2J_1^2=0,\end{aligned}$$

from which it readily follows that ω is real. Hence, the periodic solutions of type (b) do not separate a stable and unstable region.

Case C ($\omega_0\simeq 2\omega_H$)

In many of the standard parametric equations, if the system has a natural frequency ω' and it is parametrically excited at a frequency near $2\omega'$, then it becomes unstable. The purpose of the present section is to show that this is not the case with a plasma, provided that we neglect the spatial variation of the electric field (i.e., within the dipole approximation). It is easy to see that if $\omega_0\simeq 2\omega_H$ and $\omega\simeq\omega_H$, then we may again use the same equations as in Case A, for reasons discussed there. In fact, since here $\omega_0+\omega\simeq 3\omega_H$ is quite far from any resonance, the approximation is better in the present case

[12] This depends on the fact that $\omega_L\gg\omega$.

[13] This condition, together with the assumption $\omega_L\gg\omega$ and the above value for ω, is equivalent to the requirement that $\omega_L\simeq\omega_i\gg\gamma$. DuBois and Goldman considered the case $\gamma\gg\omega_i$, in which case the present resonance does not occur. I am indebted to Dr. DuBois for bringing this point to my attention.

than in Case A. If we use Eq. (27), and now make use of the fact that $\omega_0 \simeq 2\omega_H$ and $\omega \simeq \omega_H$, then we obtain

$$[\omega_H^2-(\omega-\omega_0)^2][\omega_H^2-\omega^2][\omega_H^2-\omega_L^2]^2 + J_1^2\omega_{pe}^2\omega_{pi}^2[\omega^2-(\omega-\omega_0)^2]=0.$$

Substituting

$$\omega=\omega_H+\delta\omega, \quad \omega_0=2\omega_H+\delta\omega_0$$

into this equation, and assuming that $\omega_H \gg \delta\omega$ and $2\omega_H \gg \delta\omega_0$, it is found that $\delta\omega$ is always real so that the system is stable in this region. It will be shown in the next section that this region is not stable if one takes into account spatial variation of the electric field.

IV. SPATIAL VARIATION OF THE APPLIED FIELD

In this section, we will consider the effects which arise from the finite values of k_0. At the same time, we can fortunately simplify the equations by considering only one (mobile) component. That is, the instabilities we shall now consider do not arise from the relative motion of two charged components, but are due to the relative motion of a component and the electromagnetic wave. In the dipole approximation ($k_0=0$) there is no wave front with which the motion of the charge components can be compared. The effect of the radiation on each component could then be replaced by a Galilean transformation. In the present case this is not possible, and new effects thereby arise.

Returning to Eq. (19a), and setting $\sigma'=\sigma$ (one dynamic component), we obtain

$$\rho(\mathbf{k}+n\mathbf{k}_0, \omega+n\omega_0)=-\alpha(\mathbf{k},\mathbf{k}_0,\omega_0)\Gamma(\mathbf{k}+n\mathbf{k}_0, \omega+n\omega_0)i \times\{\rho[\mathbf{k}+(n+1)\mathbf{k}_0, \omega+(n+1)\omega_0] -\rho[\mathbf{k}+(n-1)\mathbf{k}_0, \omega+(n-1)\omega_0]\}, \quad (39)$$

where $\alpha(\mathbf{k},\mathbf{k}_0,\omega_0)=\frac{1}{2}(\mathbf{k}\cdot\mathbf{k}_0/k^2)(q\mathbf{k}\cdot\mathbf{E}_0/m\omega_0^2)$, and we consistantly neglect terms of order $(k_0/k)^2$. Here n is an arbitrary integer, and we can therefore require that $\omega_0/2 \geq \mathrm{Re}(\omega)$. We will now reinvestigate Case C of the last section, namely, when $\omega_0 \simeq 2\omega_H$ and $\omega \simeq \omega_H$. Again keeping only the components $\rho(\mathbf{k}-\mathbf{k}_0, \omega-\omega_0)$ and $\rho(k,\omega)$, we obtain the dispersion relation

$$1-\alpha^2(\mathbf{k},\mathbf{k}_0,\omega_0)\Gamma(\mathbf{k},\omega)\Gamma(\mathbf{k}-\mathbf{k}_0, \omega-\omega_0). \quad (40)$$

The dependency of $\Gamma(\mathbf{k}-\mathbf{k}_0, \omega-\omega_0)$ on $\mathbf{k}_0$ can be neglected in this equation. In the present frequency range the hydrodynamic approximation gives an adequate first approximation, and we then obtain

$$(\omega^2-\omega_k^2)[(\omega-\omega_0)^2-\omega_k^2]-\alpha^2(\mathbf{k},\mathbf{k}_0,\omega_0)\omega_p^4=0,$$

where $\omega_k^2=\omega_p^2+3k^2v_T^2$. If we set

$$\omega=\omega_k+\delta\omega, \quad \omega_0=2\omega_k+\delta\omega_0,$$

where $\omega_k \gg \delta\omega$, and $2\omega_k \gg \delta\omega_0$, then this yields

$$\delta\omega=\tfrac{1}{2}\delta\omega_0 \pm [(\delta\omega_0)^2-\alpha^2(\mathbf{k},\mathbf{k}_0,2\omega_k)(\omega_p^4/\omega_k^2)]^{1/2}. \quad (41)$$

Therefore, if we neglect damping, the system is unstable to perturbations of wave number $\mathbf{k}$, provided that

$$\tfrac{1}{4}(q\mathbf{k}\cdot\mathbf{E}_0/4m\omega_k^2)^2(\mathbf{k}_0\cdot\mathbf{k}/k^2)^2(\omega_p^4/\omega_k^2) > (\delta\omega_0)^2. \quad (42)$$

Thus, the most unstable perturbations are those for which $\mathbf{k}$ lies in the plane of $\mathbf{k}_0$ and $\mathbf{E}_0$ and bisects the right angle between them [so that $(\mathbf{k}\cdot\mathbf{E}_0)(\mathbf{k}\cdot\mathbf{k}_0)/k^2 = k_0E_0/2$]. It is clear from (40) that, if $\omega_k/k \gg v_T$, the Landau damping effects are negligible, and the collisional damping largely dominates. To estimate the power required for the onset of instability, we can balance the maximum growth rate, predicted by (41), against the damping rate γ. Doing this, we conclude that the system is unstable if

$$(qk_0E_0/16m\omega_p^2) > (\gamma/\omega_p).$$

If we set $k_0=\omega_0/c \simeq 2\omega_p/c$, then this yields the following numerical condition for instability

$$I > 0.15(\gamma/\omega_p)^2 n, \quad (43)$$

where $I=E_0^2c/4\pi$ is the intensity in $\mathrm{W/cm^2}$ and n is in $\mathrm{cm^{-3}}$. These results agree in most respects with the results obtained by DuBois and Goldman, except that the condition for instability (43) appears to require an intensity twenty times higher than they found. While the present instability is considerably more difficult to excite than in Case A of the last section, the condition (44) is generally well below the intensity obtainable from lasers (10^9 $\mathrm{W/cm^2}$). Thus for laser beams in this frequency range, the harmonic approximation cannot be used for a plasma.

V. CONCLUSION

It has been shown that the modes of an infinite plasma can be made unstable by intense radiation. The most important unstable regions are found to be those which, in the limit of weak intensities, correspond to the instabilities described by DuBois and Goldman. The unstable region which emanates from $\omega_0 \simeq \omega_{pe}+\omega_{pi}(kD_e)$ is shown to be confined to a range of frequencies $\omega_{pe}+4\omega_{pi} \gtrsim \omega_0 \gtrsim \omega_{pe}$, regardless of the intensity of the radiation. In the most unstable region, the growth rate is approximately given by $\frac{1}{2}\omega_{pe}[m_eJ_1^2(\mu)/m_i]^{1/3}$, where $\mu=e\mathbf{k}\cdot\mathbf{E}_0/m\omega_0^2$ and $2 \gtrsim \mu \geq 0.1$. The largest growth rate is found to be slightly less than $0.05\omega_{pe}$. The onset of instability for weak intensities ($1 \gg \mu/kD_e$) is given by $I/nT > 4\times10^{-14}(\gamma/\omega_{pe})$, where I is in $\mathrm{W/cm^2}$, and n is in $\mathrm{cm^{-3}}$. A mirror-image unstable region, reflected about $\omega_0 \simeq \omega_{pe}$, presumably cannot be directly excited, since we must have $\omega_0 \gtrsim \omega_{pe}$ for the radiation to enter the plasma. A second instability, which occurs for $\omega_0 \simeq \omega_{pe}$, is relatively weak and confined to a very narrow range of frequencies (of the order of $\omega_{pi}/100$). The condition for instability in this case is $J_2^2(\mu) > (m_i/m_e)^2(\gamma/\omega_{pe})$, and hence the instability does not occur if $\gamma/\omega_{pe} > 10^{-4}$. It is worth noting, however, that for a finite plasma, where the modes can be excited to large amplitudes, it may be possible to excite all three instabilities through mode coupling (which can be ignored in the infinite plasma). The final unstable region occurs near $\omega_0=2\omega_{pe}$ and, while more difficult to excite, should be significant for laser beams in this frequency range. The condition for instability in this case is found to be $I>0.15(\gamma/\omega_{pe})^2n$, and the range of unstable frequencies is given by Eq. (42). It is not clear whether or not instability will be bounded in frequencies (for large intensities) as are the above-mentioned instabilities. Finally, from these results, we conclude that the harmonic approximation (discussed in the Introduction) is justified except in the region of ω_{pe} and $2\omega_{pe}$.

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It is a pleasure to express my appreciation to Dr. D. F. DuBois for giving me a prepublication copy of Dr. M. V. Goldman's thesis on this subject.[14]

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Comments on the Solution to the Boltzmann Equation for a Weakly Ionized Plasma*

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Hydrodynamic equations describing the motion of electrons in a weakly ionized plasma are derived formally from the Boltzmann equation by means of the Chapman-Enskog procedure. Two cases are considered, each with a stationary Maxwellian distribution ascribed to the atoms. In the first, electron-electron collisions are ignored, and the electron distribution function is determined by a balance between the electron-atom collisions and the electric field. There is only one conservation law, a hydrodynamic equation for the density. In the second case, electron-electron collisions are dominant, the distribution function is a local Maxwellian, and there are five conservation laws—equations for the density, drift velocity, and temperature. The equations used previously by the author to describe low-frequency oscillations are obtained in either case if the electron-atom collision frequency is independent of velocity. Otherwise, the zeroth-order equations are still exact, but first-order corrections are required, as illustrated by the example of the velocity-independent mean free path. Our results are somewhat different from those of Davidov, who made different assumptions about the dominant collision mechanisms.

I. INTRODUCTION

In this paper we discuss the derivation of hydrodynamic equations for electrons in a weakly ionized gas in an external electric field. We employ a spherical harmonic expansion of the electron-atom collision integral[1–3] (which leads to simple equations for small values of the electron-atom mass ratio, m/M) and a modification of the Chapman-Enskog procedure[4] (which is valid for slow variations of the distribution function in space and time). Our purpose is twofold. First, we wish to establish somewhat more rigorously the equations previously assumed to describe certain low-frequency oscillations.[5] We find, in general, that corrections to these equations are required. Second, we wish to distinguish mathematically the forms taken by the Chapman-Enskog heirarchy for this problem when there are different numbers of conservation laws. The classic work on the motion of electrons in gases is that of Davidov,[6] whose results are still quoted today.[7] At the end of the paper we will discuss the similarities and differences between our equations and Davidov's.

The basic mathematical problem is the solution of the equation[8]

$$\left(\frac{\partial}{\partial t}+\mathbf{c}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{e}{m}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{c}}\right)f(\mathbf{r},\mathbf{c},t)=-J(f), \qquad (1)$$

where $f(\mathbf{r},\mathbf{c},t)$ is the electron distribution function, $J(f)$ is a sum of collision integrals, and

$$\mathbf{E}=\mathbf{E}_0+\mathbf{E}_s(\mathbf{r},t), \qquad (2)$$

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[8] We use the notation of Ref. 3. For example, **c** always denotes a velocity, and $\partial/\partial\mathbf{r}$ and $\partial/\partial\mathbf{c}$ denote spatial and velocity gradients, respectively.