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Determination of S-Wave Pion-Pion Scattering Lengths*

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The S-wave pion-pion scattering lengths a_0 and a_2 in the channels of total isospin 0 and 2, respectively, are determined by requiring that the high-energy limit of the pion-pion total cross section be the same in all isospin channels. The determination consists of using the once-subtracted dispersion relation and the phase representation which are satisfied by the crossing-symmetric forward pion-pion amplitudes and also the unsubtracted dispersion relation valid for the crossing-antisymmetric amplitude. The specific approximations to be made are that the scattering becomes asymptotic fairly rapidly above the ρ and f resonances in respective channels, that these are the only $\pi\pi$ resonances in the energy region up to the f resonance, that the S wave dominates below the resonances, and that the conventional effective-range expansion is valid for the S wave with the effective range between zero and $2\mu^{-1}$ (where μ^{-1} is the pion Compton wavelength and the pion-pion force range is expected to be $0.5\mu^{-1}$ because of 2-pion exchange). The scattering lengths are determined as $\mu a_0 = 0.25 \pm 0.08$ and $\mu a_2 = 0.0 \pm 0.03$. The uncertainties are based upon the variations in a_0 and a_2 due to changes in the parametrization of the $\pi\pi$ scattering used in the present determination. It is found that the unknown details of high-energy scattering are relatively unimportant in this determination of a_0 and a_2. It is shown that the above values of a_0 and a_2 are consistent with the partially conserved axial-vector current sum rule due to Adler. This is contrary to the conclusion of previous authors; we attribute the difference to a different use of the sum rule. When one of the conjectured resonances (σ and ϵ) is added as a true resonance, no solution is found to make the high-energy limit of the total cross section the same with the parametrization of the phase and the cross section considered in the present work.

I. INTRODUCTION AND SUMMARY

THE high-energy limit of the $\pi\pi$ total cross section, $\sigma(\infty)$, can be expressed in terms of integrals over the $\pi\pi$ total cross section and the phase of the $\pi\pi$ forward scattering amplitude. If one assumes that the forward amplitude becomes pure imaginary sufficiently rapidly in the high-energy region, the above expression for $\sigma(\infty)$ allows one to estimate $\sigma(\infty)$ in terms of the lower energy information. A study was made earlier of the $\pi^+\pi^0$ channel.[1] It was found that enough information appears to be available to carry out a reasonably accurate estimate of $\sigma(\infty)$ except for the contribution coming from the low-energy region. An estimate of $\sigma(\infty)$ was made,[1] therefore, by treating the low-energy scattering as unknown but parametrized by the scattering length appropriate to this channel (isospin 2).

In principle, $\sigma(\infty)$ can be calculated for all three isospin channels by this procedure. However they are not all independent; in fact, the Pomeranchuk theorem implies one relation between them so at most two are independent. These are, for example, $\sigma(\infty)$ for the channels $\pi^+\pi^0$ and $\pi^0\pi^0$. It is generally granted that the cross section for all three isospin channels, or equivalently for all physical $\pi\pi$ channels, approach the same limit.

It is the purpose of the present work to determine the S-wave $\pi\pi$ scattering lengths a_0 and a_2, in the channels of total isospin $I=0$ and 2, respectively, by requiring that the high-energy limit of the $\pi\pi$ total cross section is the same in all channels. The main reason why such a determination is possible is that the ratio of the high-energy cross sections in the channels $\pi^+\pi^0$ and $\pi^0\pi^0$ is determined essentially by these scattering lengths as long as the low-energy scattering is parametrized by the conventional S-wave effective-range expansion with the effective range not exceeding twice the pion Compton wavelength. (The force range which corresponds to the exchange of two pions is half the pion Compton wavelength.)

For convenience, we deal in the present work with the $\pi^+\pi^0$ and $\pi^0\pi^0$ channels, which are crossing-symmetric and involve both a_0 and a_2. The requirement that the cross sections for these two channels approach the same limit provides a relation between a_0 and a_2. Another relation is provided by the unsubtracted dispersion relation for the crossing antisymmetric amplitude. This is sufficient to determine both a_0 and a_2 from the existing experimental information on the $\pi\pi$ system, namely the mass and width of the ρ resonance ($I=1$) and the f resonance ($I=0$).

We assume, on the basis of recent experiments,[2] that there are no other low-energy resonances which contribute. However, the effects of the conjectured σ and ϵ resonances,[3] treated as true resonances, are also considered in Sec. IV. According to a preliminary analysis, neither of these resonances seems to be consistent with the requirement that the cross sections approach the same high-energy limit. The possible existence of high energy resonances affects the analysis very little.

* Supported by the National Science Foundation.
[1] F. T. Meiere and M. Sugawara, Phys. Rev. 144, 1308 (1966), hereafter referred to as I.
[2] H. O. Cohn *et al.*, Phys. Rev. Letters 15, 906 (1965); I. Corbett *et al.*, Nuovo Cimento 39, 979 (1965).
[3] L. Durand and Y. Chiu, Phys. Rev. Letters 14, 329 (1965); L. Brown and P. Singer, *ibid.* 8, 460 (1962).

In Sec. II we review the basic equations for $\sigma(\infty)$, which follow essentially from the phase representation[4] and also the dispersion relations available for the forward $\pi\pi$ scattering amplitudes. The explicit evaluation of the integrals involved and the determination of the scattering lengths are given in Sec. III. The accuracy of the method is discussed in Sec. IV. It is shown in Sec. V that our results are consistent with Adler's sum rule[5] relating $\pi\pi$ scattering to axial-vector renormalization constant in β decay. In the Appendix, a rigorous proof is given that the effective-range expansion is valid at least in some vicinity of threshold in the partial-wave dispersion theory. Arguments are also given in the Appendix that the effective range can not be too large compared with the actual force range.

The results of the present work can be summarized as follows: For zero scattering lengths, $\sigma_{+0}(\infty)\gg\sigma_{00}(\infty)$, while for large scattering lengths, $\sigma_{+0}(\infty)\ll\sigma_{00}(\infty)$. Thus, the requirement that the total cross sections approach the same limit determines the scattering lengths fairly unambiguously as $\mu a_0=0.25\pm0.08$ and $\mu a_2=0.00\pm0.03$ (μ^{-1} is the pion Compton wavelength). For these values of a_0 and a_2, $\sigma_{\pi\pi}(\infty)=30\pm10$ mb. The uncertainties in a_0 and a_2 are based upon the actual variations in a_0 and a_2 when changes are made in the parametrization of the phase and the cross sections. The effective range is allowed to vary between zero and $2\mu^{-1}$. Also varied are the energies which define the energy regions in which S wave dominates and the high-energy region in which scattering is asymptotic. Some modifications of the effective-range expansion is also considered. It is found that the unknown details of the high-energy behavior of the scattering are unimportant in this determination of a_0 and a_2 since, although they can affect the value of the individual $\sigma(\infty)$ greatly, they hardly change the ratio $\sigma_{+0}(\infty)/\sigma_{00}(\infty)$. Presumably the most essential assumption to the present determination is that the low-energy scattering is well parametrized by the effective-range expansion with an effective range not greater than $2\mu^{-1}$ in the energy region roughly one pion mass above threshold (in the total c.m. energy). The details of the S-wave scattering above this energy region are quite unimportant in the present determination as long as the S wave does not resonate.

Our result, $\sigma(\infty)=30\pm10$ mb, is in general agreement with the conclusion reached earlier[1] that $\sigma_{+0}(\infty)$ can hardly be made smaller than 20 mb but unknown details of the high-energy behavior make it difficult to set a precise upper limit for $\sigma_{+0}(\infty)$.

II. FORMALISM

The derivation of a convenient expression for $\sigma(\infty)$ has been given in an earlier paper,[1] but the essential steps are reviewed here. The once-subtracted dispersion relation and the phase representation[4] are valid for the forward scattering amplitude $A(\omega)$. For a crossing symmetric amplitude, these take the forms

$$A(\omega)=A(\mu)+\frac{2(\omega^2-\mu^2)}{\pi}\int_0^\infty \frac{dq'}{\omega'^2-\omega^2}\sigma(\omega'), \quad (1a)$$

$$A(\omega)=P(\omega^2)\exp\left(\frac{2\omega^2}{\pi}\int_\mu^\infty \frac{d\omega'\delta(\omega')}{\omega'(\omega'^2-\omega^2)}\right), \quad (1b)$$

where μ is the pion mass, (ω,q) is the lab pion energy-momentum, and the normalization is such that the total cross section is given by $\mathrm{Im}A(\omega)=q\sigma(\omega)$. The relation to the c.m. pion energy-momentum (E,p) is summarized by $\mu q=2pE$. The total phase is defined by $A(\omega)=\pm|A(\omega)|e^{i\delta(\omega)}$ on the real axis, where the $\pm$ sign is determined to be the sign of $A(\mu)$ by the requirement that $\delta(\omega)$ be zero on the gap, $-\mu\leq\omega<\mu$, where $A(\omega)$ is real. $P(\omega^2)$ is a polynomial in ω^2. Note that as long as $\sigma(\omega)$ is non-negative, $\mathrm{Im}A(\omega)=q\sigma(\omega)\geq0$ and hence $0\leq\delta(\omega)\leq\pi$ for $A(\mu)>0$ or $-\pi\leq\delta(\omega)\leq0$ for $A(\mu)<0$.

We assume that the amplitude at high energies is dominated by inelastic processes and hence becomes imaginary ($\delta(\infty)=\pm\frac{1}{2}\pi$) fast enough so that the total cross section approaches a nonzero constant. It is then possible to express $\sigma(\infty)$ in terms of the phase, $\delta(\omega)$, and the over-all constant c which appears in the polynomial $P(\omega^2)$. The highest power of ω^2 in $P(\omega^2)$ is found to be either one or zero since $\mathrm{Im}A(\omega)\propto P(\omega^2)\omega^{-2\delta(\infty)/\pi}$ as $\omega\to+\infty$. The over-all constant c of $P(\omega^2)$ can be expressed in terms of $\delta(\omega)$ and $\sigma(\omega)$ by equating Eqs. (1a) and (1b) at $\omega=0$ and $\omega=\mu$. The resulting expressions are

$$\sigma(\infty)=(c/\mu)\exp\left(-\frac{2}{\pi}\int_\mu^\infty \frac{d\omega}{\omega}[\delta(\omega)-\delta(\infty)]\right), \quad (2a)$$

where, for $A(\mu)>0(\delta(\infty)=\frac{1}{2}\pi)$,

$$c=\frac{2\mu^2}{\pi}\int_0^\infty dq\frac{\sigma(\omega)}{\omega^2} -A(\mu)\left[1-\exp\left(-\frac{2\mu^2}{\pi}\int_\mu^\infty\frac{d\omega}{\omega}\frac{\delta(\omega)}{\omega^2-\mu^2}\right)\right], \quad (2b)$$

and, for $A(\mu)<0(\delta(\infty)=-\frac{1}{2}\pi)$,

$$c=\frac{2\mu^2}{\pi}\int_0^\infty dq\frac{\sigma(\omega)}{\omega^2}-A(\mu). \quad (2c)$$

There are two independent amplitudes which are crossing symmetric and, therefore, satisfy all the above relations. These are, for example, those for $\pi^+\pi^0$

[4] M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962); Phys. Rev. **130**, 2127 (1963).
[5] S. Adler, Phys. Rev. **140**, B736 (1965).

and $\pi^0\pi^0$ given by

$$A_{+0}(\omega)=\tfrac{1}{2}A_1(\omega)+\tfrac{1}{2}A_2(\omega),$$
$$A_{00}(\omega)=\tfrac{1}{3}A_0(\omega)+\tfrac{2}{3}A_2(\omega),\tag{3}$$

where $A_I(\omega)$ are the amplitudes in the channels of total isospin $I=0$, 1, and 2. The third amplitude can be made crossing antisymmetric, which is

$$f(\omega)=6[A_{+-}(\omega)-A_{++}(\omega)]$$
$$=2A_0(\omega)+3A_1(\omega)-5A_2(\omega).\tag{4}$$

As was mentioned earlier, we assume here that all $A_I(\omega)$ become pure imaginary at high energies fast enough so that all $\sigma_I(\infty)$ are finite, nonzero cross sections. It then follows that $f(\omega)/\omega$ satisfies an unsubtracted dispersion relation

$$f(\omega)=\frac{2\omega}{\pi}\int_\mu^\infty \frac{d\omega'\,\mathrm{Im}f(\omega')}{\omega'^2-\omega^2}.\tag{5}$$

The threshold value of $f(\omega)$ is related to the S-wave scattering lengths by $f(\mu)=16\pi(2a_0-5a_2)$, so that

$$2a_0-5a_2=\frac{\mu}{8\pi^2}\int_0^\infty \frac{dq}{\omega}[2\sigma_0(\omega)+3\sigma_1(\omega)-5\sigma_2(\omega)].\tag{6}$$

Equations (2) and (6) constitute the basic expressions for determining a_0 and a_2.

A study of $\sigma_{+0}(\infty)$ based upon the expressions (2) was made in a previous work.[1] It was found, in particular, that the S waves play an essential role in determining $\sigma_{+0}(\infty)$, though the final estimate of $\sigma_{+0}(\infty)$ turns out to be rather insensitive to the S-wave parameters. This was simply because a cancelation took place among the S-wave terms in the expressions (2). However, such a cancelation does not happen in the case of $\sigma_{00}(\infty)$, but $\sigma_{00}(\infty)$ depends sensitively on a_0 and a_2. One of the main reasons for this difference is that there are two S-waves contributing to $\sigma_{00}(\infty)$ while $\sigma_{+0}(\infty)$ depends only on one of them.

Because of this remarkable difference in the dependence of $\sigma_{+0}(\infty)$ and $\sigma_{00}(\infty)$ on the S-wave scattering lengths, it is proposed in the present work to determine a_0 and a_2 by requiring that

$$\sigma_{+0}(\infty)=\sigma_{00}(\infty)\tag{7}$$

which, combined with the Pomeranchuk theorem, implies that all $\sigma_I(\infty)$ are the same.

The isospin amplitudes $A_I(\omega)$ can be expressed in terms of the phase shifts by

$$A_I(\omega)=\frac{16\pi E}{p\mu}\sum_l(2l+1)\frac{e^{2i\delta_{l,I}}-1}{2i}.\tag{8}$$

The ρ resonance appears in A_1 and hence $\pi^+\pi^0$ scattering, while the f resonance appears in A_0 and hence $\pi^0\pi^0$ scattering.

Several remarks are pertinent here. First, the basic equations (2) and (6) are all exact, as long as $A_I(\omega)$ become pure imaginary at high energies sufficiently rapidly. Second, since the expression for $\sigma(\infty)$ involves both the phase and the cross section, it permits one to make better use of the available knowledge of the $\pi\pi$ system than the usual dispersion relation does. In particular, the high-energy region can be dealt with knowing that the over-all phase approaches that of an imaginary amplitude. Third, the constant c and hence $\sigma(\infty)$ can be expressed in a variety of different but equivalent ways. The expressions (2) are chosen so as to put emphasis on the low-energy region. Finally, although essentially the same analysis can be carried through for any other amplitude, there is a definite technical advantage to choosing the amplitudes A_{+0} and A_{00} which possess crossing symmetry and whose over-all phase is conveniently bounded.

III. DETERMINATION OF SCATTERING LENGTHS

The various phase and cross section integrals in Eqs. (2) and (6) can be broken into three parts according to the range of integration; the low-energy region, $\mu\leq\omega\leq\bar\omega$, where $\bar\omega$ is some point close to but below the resonance, the resonance region, $\bar\omega\leq\omega\leq\omega_{\mathrm{inel}}$, where ω_{inel} is somewhere above the resonance, and high-energy region, $\omega_{\mathrm{inel}}\leq\omega$. In the high-energy region, we set $\delta(\omega)=\delta(\infty)$ and $\sigma(\omega)=\sigma(\infty)$. In the resonance region, we assume that the phase and cross section are dominated by the resonant partial wave. In the low-energy region, we assume that the S waves dominate. Moreover, we assume that an effective-range expansion is valid for the S waves,

$$p\cot\delta_{0,I}=(1/a_I)+\tfrac{1}{2}r_Ip^2.\tag{9}$$

Since the S waves are always very important in the integrals in (2) and (6), the assumption that the expression (9) is valid in the low-energy region is essential to our determination of a_0 and a_2. In partial-wave dispersion theory, one can prove the effective-range expansion (9) rigorously at least in some vicinity of threshold, as is shown in the Appendix. The effective range r_I is treated in the present work as a parameter in the range between 0 and $2\mu^{-1}$, in spite of the fact that the force range is about $0.5\mu^{-1}$ for $\pi\pi$ scattering. Arguments are presented in the Appendix that the effective range is not likely to be too large compared with the force range, so that 0 to $2\mu^{-1}$ should be sufficient.

Arguments are presented in I that a meaningful procedure to define the resonance region, $\bar\omega<\omega<\omega_{\mathrm{inel}}$ would be to choose it from one full width below the resonance to 1.5 full widths above the resonance, in units of the c.m. energy. This would mean that

$$\begin{array}{lll}\bar\omega(\rho)=9.48 & \omega_{\mathrm{inel}}(\rho)=22.1 & \text{for}\quad \pi^+\pi^0\\ \bar\omega(f)=31.9 & \omega_{\mathrm{inel}}(f)=52.1 & \text{for}\quad \pi^0\pi^0\end{array}\tag{10}$$

in units of μ.

For the case $A(\mu)>0(\delta(\infty)=\pi/2)$, the contributions from threshold to $\bar{\omega}$ may be considered as coming purely from S waves. However, as long as the S waves do not resonate, the S-wave contributions come mainly from the very low-energy region. Therefore, we simply integrate the S-wave contributions up to $\bar{\omega}$ as a reference and consider the effect of changing the S-wave cutoff in Sec. IV. For the case $A(\mu)<0(\delta(\infty)=-\frac{1}{2}\pi)$, the S-wave cutoff needs to be carefully examined. This is discussed in detail in I and is not repeated here since the threshold values are positive according to the present work.[6]

One simplification occurs for the phase integrals in (2a) and (2b). The over-all phase rises rapidly in the resonance region passing through $\pi/2$. Contributions from either side of the resonance tend to average and hence we may replace $\delta(\omega)$ by its average, $\frac{1}{2}\pi$, in the resonance region. In fact, requiring this to be exact was used in the previous work[1] to determine the values in (10) for $\bar{\omega}$ and ω_{inel}. Thus,

$$\frac{2}{\pi}\int_\mu^\infty \frac{d\omega}{\omega}[\delta(\omega)-\delta(\infty)]=\frac{2}{\pi}\int_\mu^{\bar\omega} \frac{d\omega}{\omega}\delta(\omega)-\ln\frac{\bar{\omega}}{\mu},$$

$$\frac{2\mu^2}{\pi}\int_\mu^\infty \frac{d\omega}{\omega}\frac{\delta(\omega)}{(\omega^2-\mu^2)}=\frac{2\mu^2}{\pi}\int_\mu^{\bar\omega} \frac{d\omega}{\omega}\frac{\delta(\omega)}{(\omega^2-\mu^2)}-\tfrac{1}{2}\ln\left(\frac{\bar{\omega}^2-\mu^2}{\bar{\omega}^2}\right). \quad (11)$$

Both of these integrals can now be evaluated in terms of (9) and (10).

The cross-section integral in (2b) and (2c) is written as

$$\frac{2\mu^2}{\pi}\int_0^\infty \frac{dq}{\omega^2}\sigma(\omega)=\frac{2\mu^2}{\pi}\int_0^{\bar q} \frac{dq}{\omega^2}\sigma(\omega)+\frac{2\mu^2}{\pi}\int_{\bar q}^{q_{\text{inel}}} \frac{dq}{\omega^2}\sigma(\omega)+\frac{2\mu^2}{\pi}\int_{q_{\text{inel}}}^{\infty} \frac{dq}{\omega^2}\sigma(\infty), \quad (12)$$

where $\bar{q}$ and q_{inel} are the momenta which correspond to $\bar{\omega}$ and ω_{inel}. The resonant contribution in (12) is evaluated using

$$\sigma_I(\omega)=4\pi^2(2l+1)\Gamma\delta(M-2E)/p^2 \quad (13)$$

where M and Γ are the mass and the full width of the resonance, and I and l are its isospin and angular momentum, respectively. (This δ-function approximation was checked numerically for the ρ and found equivalent to using a Breit-Wigner form for $\sigma_I(\omega)$ over a range of 1.5 full widths, and hence sufficiently accurate for our purpose.) The low-energy contribution in (12) is evaluated using

$$\sigma_I(\omega)=8\pi/(p^2+p^2\cot^2\delta_{0,I}) \quad (14)$$

with the effective-range expansion (9). The high-energy contribution in (12) is

$$(2\mu/\pi)\sigma(\infty)\tan^{-1}(\mu/q_{\text{inel}}),$$

which contains, however, $\sigma(\infty)$ yet to be determined. This difficulty can be circumvented by solving (2b) for c, in terms of (2a), as

$$c=c'/(1-\gamma),$$

where

$$c'=\frac{2\mu^2}{\pi}\int_{\bar q}^{q_{\text{inel}}}\frac{dq}{\omega^2}\sigma(\omega)+\frac{2\mu^2}{\pi}\int_0^{\bar q}\frac{dq}{\omega^2}\sigma(\omega)-A(\mu)\left[1-\left(\frac{\bar{\omega}^2-\mu^2}{\bar{\omega}^2}\right)^{1/2}\times\exp\left(-\frac{2\mu^2}{\pi}\int_\mu^{\bar\omega}\frac{d\omega}{\omega}\frac{\delta(\omega)}{(\omega^2-\mu^2)}\right)\right], \quad (15)$$

and

$$\gamma=\frac{2}{\pi}\frac{\bar{\omega}(\rho)}{\mu}\exp\left(-\frac{2}{\pi}\int_\mu^{\bar\omega(\rho)}\frac{d\omega}{\omega}\delta(\omega)\right)\times\left[\tfrac{1}{2}\tan^{-1}\frac{\mu}{q_{\text{inel}}(\rho)}+\tfrac{1}{2}\tan^{-1}\frac{\mu}{q_{\text{inel}}(f)}\right],\quad \text{for } \pi^+\pi^0,$$

$$=\frac{2}{\pi}\frac{\bar{\omega}(f)}{\mu}\exp\left(-\frac{2}{\pi}\int_\mu^{\bar\omega(f)}\frac{d\omega}{\omega}\delta(\omega)\right)\left\{\tan^{-1}\frac{\mu}{q_{\text{inel}}(f)}\right\},\quad \text{for } \pi^0\pi^0.$$

Then

$$\sigma(\infty)=\frac{c\bar{\omega}}{\mu^2}\exp\left(-\frac{2}{\pi}\int_\mu^{\bar\omega}\frac{d\omega}{\omega}\delta(\omega)\right). \quad (16)$$

The high-energy correction $1/(1-\gamma)$ is one of the most ambiguous parts in (15), but fortunately affects the scattering-length determination very little.

A similar analysis is assumed for the cross-section integrals in the unsubtracted dispersion relation (6). The $I=0$ cross section is given by its S-wave component in the low-energy region, by the f in the resonance region and by $\sigma(\infty)$ in the high-energy region which is above $\omega_{\text{inel}}(f)$. The $I=1$ cross section has a resonance contribution from the ρ and is set equal to $\sigma(\infty)$ above $\omega_{\text{inel}}(\rho)$. The $I=2$ cross section gives an S-wave contribution, no resonance contribution, and is set equal to $\sigma(\infty)$ above $\omega_{\text{inel}}(f)$. This behavior is summarized in Fig. 1. In this procedure, the integral in (6) cuts off at $\omega_{\text{inel}}(f)$. The Pomeranchuk theorem ensures that the integrand in (6) goes to zero at high energy but there is an ambiguity in the actual high-energy contribution. However, assuming reasonable high-energy behavior, one can estimate the high-energy contribution in the integral in (6) to demonstrate that this integral is actually dominated by the low-energy contribution. Therefore, we assume the above procedure, and also

[6] L. D. Jacobs and W. Selove, Phys. Rev. Letters **16**, 669 (1966).

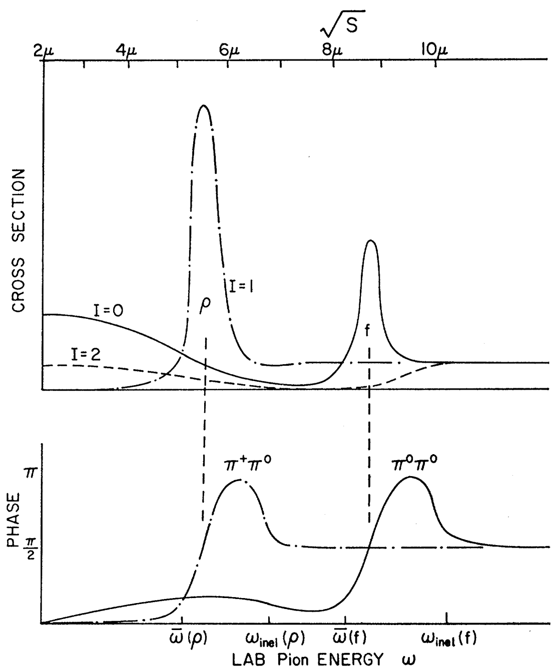


FIG. 1. The qualitative behavior of the $\pi\pi$ cross section and phase assumed in this analysis. Fig. 1(b) is related to Fig. 1(a) by usual isospin conservation.

assume $\sigma_1=\sigma(\infty)=25$ mb between $\omega_{\text{inel}}(\rho)$ and $\omega_{\text{inel}}(f)$ to evaluate the integral in (6).

All these parametrizations are based on an intuitively appealing picture of the $\pi\pi$ scattering which certainly matches the qualitative behavior of the phase and the total cross section. In the present work, we assume the values in (10) as a reference, but also vary $\bar{\omega}$ and ω_{inel} in order to discuss how sensitive our results are to these quantities.

In order to determine a_0 and a_2, we first solve numerically the unsubtracted dispersion relation (6) for a_0 as a function of a_2; the results are given in Fig. 2. Using these values, we then calculate $\sigma_{+0}(\infty)$ and $\sigma_{00}(\infty)$ from (15) and (16). The results are shown in Fig. 3 for several values of $r_0=r_2$. Representative values, based on an effective range $r_2=r_0=0.5\mu^{-1}$, an S-wave cutoff at the ρ resonance, and the choice of parameters given by (10), are $\mu a_0=0.23$, $\mu a_2=0.0$, and $\sigma_{\pi\pi}(\infty)=35$ mb. If the only uncertainties arise from choice of the parameters such as the effective range, $\bar{\omega}$, and ω_{inel} which are discussed in Sec. IV, then the results of the present work can be summarized as

$$\mu a_0=0.25\pm0.08\,,\quad \mu a_2=0.0\pm0.03\,,$$
$$\sigma_{\pi\pi}(\infty)=30\pm10\text{ mb}. \qquad (17)$$

IV. DISCUSSION

We now discuss the effect of changing our parametrization of the $\pi\pi$ cross section and phase by varying the effective range, the S-wave cutoff, the form of the S-wave effective-range expansion, and the parameters $\bar{\omega}$ and ω_{inel}.

As discussed in the Appendix, use of the effective-range expansion is justified by partial-wave dispersion theory in some region above threshold. It is also discussed in the Appendix that the effective range should be in the vicinity of, or less than, $0.5\mu^{-1}$, which is the range of the force arising from the exchange of the lightest system possible, namely, 2π. If, however, the effective ranges r_0 and r_2 are varied independently in the interval from 0 to $2\mu^{-1}$, taking the S-wave region for both channels from threshold to the ρ ($\sqrt{s}=5.46\mu$), the results vary from $\mu a_0=0.20$, $\mu a_2=-0.01$, $\sigma(\infty)=36$ mb to $\mu a_0=0.30$, $\mu a_2=\pm0.02$, $\sigma(\infty)=33$ mb. Keeping the effective range fixed at $r_2=r_0=1\mu^{-1}$ but reducing the range of integration for the S-wave contributions to about one pion mass above threshold ($\sqrt{s}=2.9\mu$) changes the results from $\mu a_0=0.24$, $\mu a_2=0.0$, $\sigma(\infty)=34$ mb to $\mu a_0=0.22$, $\mu a_2=0.0$, $\sigma(\infty)=35$ mb. The effective-range expansion was modified to read $p\cot\delta=1/a+\frac{1}{2}rp^2+Pp^4$, but the coefficient P restricted so that the correction does not change $p\cot\delta$ by more than a factor of two up to the ρ resonance. For $r=1\mu^{-1}$, the results could be made to vary from $\mu a_0=0.21$, $\mu a_2=-0.01$, $\sigma(\infty)=36$ mb to $\mu a_0=0.25$, $\mu a_2=+0.01$, $\sigma(\infty)=33$ mb. In short, the results are rather insensitive to the details of the S-wave scattering other than the scattering lengths themselves if the effective range is between 0 and $2\mu^{-1}$. For larger values of the effective range, the results become too sensitive to the details to be useful.

The point $\bar{\omega}$ is a measure of where the phase begins to rise to the resonance value of $\frac{1}{2}\pi$. If this point is varied from as high as the resonance itself to as low as two full widths below the resonance, the results vary from $\mu a_0=0.28$, $\mu a_2=+0.01$, $\sigma(\infty)=54$ mb to $\mu a_0=0.21$, $\mu a_2=-0.01$, $\sigma(\infty)=20$ mb. In other words, if the resonance begins to dominate roughly the same way in the $\pi^+\pi^0$ and $\pi^0\pi^0$, channels, the actual point where the resonance begins to dominate can influence the numerical value of $\sigma(\infty)$ but not the scattering lengths. In fact this is true of any uncertainties present in this calculation which affect both channels, in roughly the same way. For instance, the point ω_{inel} is a measure of where the scattering becomes asymptotic. Varying ω_{inel}

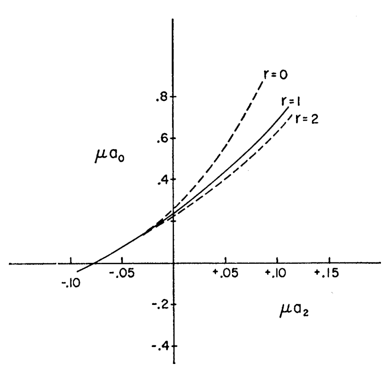
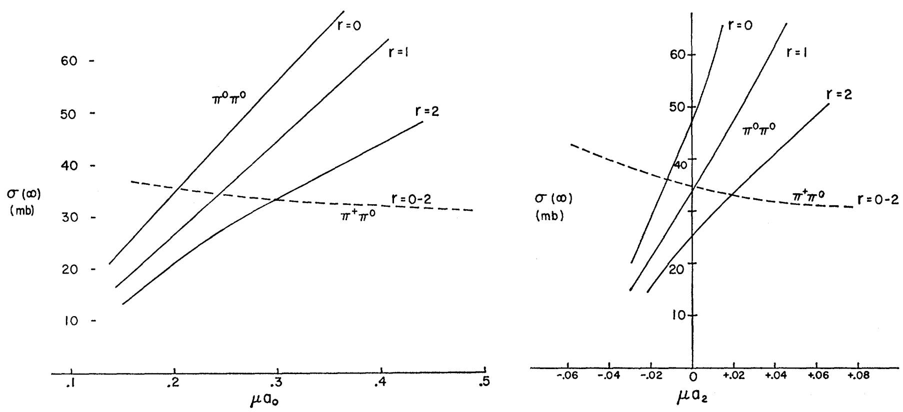


FIG. 2. The relation between a_0 and a_2 provided by the unsubtracted dispersion relation (6) for various effective ranges.



(a) (b)

FIG. 3. $\sigma(\infty)$ as a function of the scattering lengths a_0 or a_2. Either (a) a_0 or (b) a_2 can be chosen as the independent variable, for they are related as shown in Fig. 2.

changes the high-energy correction $1/(1-\gamma)$ in (15) and (16). Dropping this completely can decrease $\sigma(\infty)$ by 30% but leaves the scattering lengths essentially unchanged. Varying ω_{inel} arbitrarily for the ρ and f so that the two channels are treated differently can produce an uncertainty $\mu\Delta a_0=0.09$, $\mu\Delta a_2=0.02$, and $\Delta\sigma(\infty)=6$ mb. Also, if the phase approaches $\frac{1}{2}\pi$ much more slowly than can be accommodated by the above parametrization, $\sigma(\infty)$ for both channels can be significantly increased or decreased, but as long as the approach is comparable in both channels, as one would expect, the scattering lengths would not be appreciably affected.

As a summary, the scattering length changes roughly from $0.20\,\mu^{-1}$ to $0.30\,\mu^{-1}$ when one varies the effective range and the energies $\bar{\omega}$ and ω_{inel}. Since these are to be considered as independent sources of uncertainties, we conclude that $\mu a_0=0.25\pm0.08$ within the range of the parametrizations considered in the present work. Our final results (17) are obtained this way.

Two other resonances[3] have been conjectured in the $I=0$ channels, the ϵ ($J=0$, $M\simeq760$ MeV, $\Gamma\simeq100$ MeV) and the σ ($J=0$, $M\simeq400$ MeV, $\Gamma\simeq80$ MeV). Present experimental evidence[2] indicates that neither exists and we take this point of view, so that only the ρ and f contribute to this calculation. Nevertheless, we discuss briefly what would be the case if either did exist as a true $\pi\pi$ resonance. Only the $\pi^0\pi^0$ channel would be affected. Since these resonances are fairly low-energy resonances with substantial widths, one expects a large change in our determination of a_0 and a_2. In fact, a preliminary calculation indicates that the dependence of $\sigma_{00}(\infty)$ on a_0 and a_2 would be entirely different from what is shown in Fig. 3. Moreover, our calculation indicates, though in a preliminary way, that the two curves for $\sigma_{+0}(\infty)$ and $\sigma_{00}(\infty)$ do not seem to cross within the parametrizations considered in the present work. We found solutions which gave the same limits for all $\sigma(\infty)$ only when we gave up the relation between a_0 and a_2 implied by the unsubtracted dispersion relation (6).

V. PCAC SUM RULE

By assuming the equal-time commutation relations proposed by Gell-Mann[7] for the weak axial-vector currents and the partially conserved axial-vector current[8] (PCAC) hypothesis which states that the divergence of the axial-vector current is proportional to the pion field, Adler[5] and Muzinich and Nussinov[9] have derived a sum rule involved the off-the-mass-shell pion-pion scattering amplitude. By making a continuation in mass from zero to the physical pion mass, they attempt to place limits on the physical pion-pion scattering lengths. Although they use slightly different mass continuation, both authors conclude that the lower limit for a_0 is approximately one pion Compton wavelength (μ^{-1}), which disagrees completely with our results (17).

We show in this section that the sum rule is actually consistent with our results (17). We also show that the different conclusion reached by the other authors[5,9] is due to inadequate use of the sum rule.

The exact sum rule is

$$\frac{2}{g_A{}^2}=\frac{2M_N{}^2}{g_r{}^2K_{NN\pi}{}^2(0)}F(0,0,0,0)\,, \tag{18}$$

where the notation is that used by Adler.[5] The amplitude F is defined as

$$F(\nu,t,-p_1{}^2,-p_2{}^2)=\frac{i(q_{10}q_{20})^{1/2}}{2\mu\nu}\int d^4x\,e^{ix\cdot p_1}(-\Box_x+\mu^2)$$
$$\times\langle\pi^+(q_2)|\,T\{j^-(0)\varphi^+(x)-j^+(0)\varphi^-(x)\}\,|\pi^+(q_1)\rangle, \tag{19}$$

where $j(x)=(-\Box_x+\mu^2)\varphi(x)$, $2\mu\nu=-p_1\cdot(q_1+q_2)$, $t=-(q_1-q_2)^2$, $q_1{}^2=q_2{}^2=-\mu^2$. The amplitude F is normalized so that $\operatorname{Im}F(\nu,0,\mu^2,\mu^2)=q(\nu)(\sigma_-(\nu)-\sigma_+(\nu))/2\nu$, where $q(\nu)=(\nu^2-\mu^2)^{1/2}$ and $\sigma_\pm(\nu)$ are the total

[7] M. Gell-Mann, Physics **1**, 63 (1964).

[8] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).

[9] I. J. Muzinich and S. Nussinov, Phys. Letters **19**, 248 (1965).

cross sections for a physical π^+ scattering from a physical $\pi^\pm$. For forward scattering, ν reduces to the laboratory pion energy ω used in this paper and $F(\omega,0,\mu^2,\mu^2)=f(\omega)/12\omega$, where $f(\omega)$ is the amplitude defined in Sec. II, and hence satisfies the unsubtracted dispersion relation (5). The sum rule (18) involves the amplitude with two pions off the mass shell. In order to use this sum rule to obtain information on the scattering lengths, one has to know first how to continue in mass from zero to μ. As yet, there is no convincing way to do this. Hence we replace $F(0,0,0,0)/K_{NN\pi}{}^2(0)$ by $F(0,0,\mu^2,\mu^2)$, i.e., we take the pions on the mass shell. One can then use the dispersion relation for F, which is (5), to give

$$\frac{2}{g_A{}^2}\simeq\frac{2M_N{}^2}{g_r{}^2\pi}\int_0^\infty dq\,\frac{q^2}{\omega^3}(\sigma_-(\omega)-\sigma_+(\omega)). \tag{20}$$

This approximate sum rule is used by Muzinich and Nussinov,[9] while Adler[5] breaks the cross section into partial waves and replaces q^2/ω^3 by

$$q(\omega+\tfrac{1}{2}\mu)^{2l-1}/\omega(\omega^2-\mu^2)^l.$$

If the main contributions to the integral come from the S waves and the known resonances, then the two different forms should give effectively the same result. In order to extract information about the scattering lengths from (20), both authors[5,9] attempted to evaluate the integral directly by parametrizing the cross section. However, one can use the exact dispersion relation (6) to relate the integral in (20) to the scattering lengths. By constructing the difference between (5) evaluated at $\omega=0$ and (6), one can rewrite (20) as

$$\frac{2}{g_A{}^2}=\frac{M_N{}^2}{g_r{}^2}\frac{8\pi}{3\mu}(2a_0-5a_2)-\frac{M_N{}^2}{g_r{}^2}\frac{2\mu^2}{\pi}\int_0^\infty\frac{dq}{\omega^3}[\sigma_-(\omega)-\sigma_+(\omega)], \tag{21}$$

where $\sigma_-(\omega)-\sigma_+(\omega)=\frac{1}{6}(2\sigma_0+3\sigma_1-5\sigma_2)$. Compared to the integral in (20), the integral in (21) is much less sensitive to the high-energy behavior and the parametrization of the S wave. It is in fact possible to estimate the integral in (21) in a simple manner in order to show that the second term is actually small compared with the first term in (21) for scattering lengths which are not too large. Thus assuming $|a_2|\ll a_0$, one obtains from the sum rule (21) that

$$\mu a_0\simeq\frac{3}{8\pi}\frac{g_r{}^2}{g_A{}^2}\frac{\mu^2}{M_N{}^2}=0.35. \tag{22}$$

We interpret the fact that (22) differs significantly from the results obtained by Adler[5] or Muzinich and Nussinov[9] to mean that their parametrization of the $\pi\pi$ cross section is not compatible with the unsubtracted dispersion relation (6). Though we do not know the uncertainty in the sum rule (20) arising from mass continuation, we are inclined to conclude that our scattering lengths (17) are consistent with the Adler sum rule (18).

Note added in proof. S. L. Adler (private communication) observed that the second term on the right-hand side of (21) is small compared with the left-hand side of (21) even when the σ and ϵ resonances are assumed and also for the S-wave scattering lengths which are as large as one pion Compton wavelength.

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APPENDIX

Let $g(s)$ be the S-wave amplitude $(e^{2i\delta}-1)/2ip$, where s is the total c.m. energy squared and p is the c.m. momentum, $4p^2=s-4\mu^2$. Let $h(s)$ be the inverse amplitude $1/g(s)$. The physical cut for $g(s)$ extends from $4\mu^2$ to $+\infty$. Then $h(s)$ has this physical cut, a left-hand cut, and possible poles arising from the zeros of $g(s)$. We take the kinematical cut from $\sqrt{s}$ as part of the left-hand cut. Hence, one has

$$h(s)=h(4\mu^2)+\frac{(s-4\mu^2)}{\pi}\int_{4\mu^2}^\infty\frac{ds'}{s'-4\mu^2}\frac{\mathrm{Im}h(s')}{s'-s}+(\text{left-hand integral})+(\text{poles}). \tag{A1}$$

In the elastic region, $h(s)=p\cot\delta-ip$. The integral in (Al) with $\mathrm{Im}h(s')$ replaced by its elastic value $-p(s')=-\frac{1}{2}(s'-4\mu^2)^{1/2}$ can be evaluated exactly. [The simplest way is to recognize it as the subtracted dispersion relation for $-ip(s)$.] The result is $-i\frac{1}{2}(s-4\mu^2)^{1/2}$ with the cut from $4\mu^2$ to $+\infty$. Inserting the above integral on the left-hand side of (Al), one obtains

$$h(s)+\frac{i(s-4\mu^2)^{1/2}}{2}=\frac{(s-4\mu^2)}{\pi}\int_{s_0}^\infty\frac{ds'}{s'-4\mu^2}\frac{[\mathrm{Im}h(s')+p(s')]}{s'-s}+(\text{left-hand integral})+(\text{poles}), \tag{A2}$$

where s_0 is the inelastic threshold. The function on the left-hand side of (A2) becomes $p\cot\delta$ in the elastic region. Hence, $p\cot\delta$ can be expanded in a power series in $s-4\mu^2$, or equivalently p^2, in some vicinity of $s=4\mu^2$ as

$$p\cot\delta=1/a+\tfrac{1}{2}rp^2+0(p^4). \tag{A3}$$

The radius of convergence is the distance to the inelastic threshold or to the left-hand cut or to the nearest zero of the S-wave amplitude, whichever is the smallest.

The above argument alone hardly indicates any magnitude for the effective range, nor does it indicate how far above threshold the first two terms in (A3) provide a good approximation to the true phase shift.

In order to get some idea about the magnitude of the effective range, let us assume that $\pi\pi$ scattering is described, at least in some region above threshold, in terms of an effective energy-dependent potential whose range is that of 2π exchange. The simplest such potential which permits the effective-range expansion is $V_0(r) + (p^2/\mu)V_1(r)$, where $V_0(r)$ and $V_1(r)$ are appreciable only within a range of order $0.5\,\mu^{-1}$. Let the S-wave component of the wave function be $u(r)/r$, normalized so that $u(r) \sim \bar{u}(r) \equiv \sin(pr+\delta)/\sin\delta$ for large r and let u_0 and $\bar{u}_0$ denote the zero-energy limits of these functions. By standard arguments,[10] one has the identity

$$p\cot\delta = -\frac{1}{a} + p^2 \int_0^\infty dr[\bar{u}_0\bar{u} - u_0 u + u_0 V_1(r) u]. \quad \text{(A4)}$$

Hence, expanding in powers of p^2 one obtains the effective-range formula with

$$r = 2\int_0^\infty dr[\bar{u}_0^2 - u_0^2 + V_1(r) u_0^2]. \quad \text{(A5)}$$

Assuming that $V_0(r)$ is sufficiently strongly attractive, it is known that the first two terms in (A5) give a contribution very close to the force range.[10] The last term in (A5) gives a positive contribution to the effective range when $V_1(r)$ is positive (which makes the effective potential more attractive below threshold). However, some upper bound for $V_1(r)$ is provided by the requirement that the effective potential must not generate a $\pi\pi$ bound state below threshold. One can see that a sufficient (but not necessary) condition for this to be satisfied is that $V_1(r)$ be of the order of one or smaller within the force range. In this case the last term in (A5) is of the order of half the force range or smaller. Though it is not possible to set a precise upper limit, one can see this way that too large an effective range would violate the condition that there are no $\pi\pi$ bound states.

Experimental information is available on a process similar to $\pi\pi$ scattering, namely, KN scattering. In both cases, the longest range force arises from the exchange of 2π, giving a force range of $0.5\,\mu^{-1}$. The effective-range expansion provides in this case an excellent fit[11] to the S-wave scattering up to at least 642 MeV incident K kinetic energy, with an effective range $r \sim 0.4\,\mu^{-1}$. This value is only slightly less than the force range due to 2π exchange.

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Off-the-Mass-Shell Correction in Pion-Pion Scattering*

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A discussion is given of the off-the-mass-shell correction to the PCAC (partially conserved axial-vector current) relation for pion-pion scattering. In particular, special attention is drawn to the fact that the usual s and u cuts of the pion-pion amplitude give rise to cuts in the external pion mass extended far below the cut due to the three-pion intermediate state. An estimate is made of the off-the-mass-shell correction due to these induced cuts. It is shown that this correction is not likely to be significant as long as the S-wave pion-pion interaction is relatively weak, implying that the use of the PCAC relation for pion-pion scattering without the off-the-mass-shell correction is not expected to be any worse than that for any other PCAC relation.

I. INTRODUCTION AND SUMMARY

The algebra of current commutators proposed by Gell-Mann[1] and the PCAC hypothesis[2] have been used by Adler and Weisberger[3] to express the axial-vector coupling constant renormalization in β decay, g_A, in terms of the pion-nucleon scattering amplitude evaluated at some unphysical point. The success of this calculation[3] prompted several authors to relate g_A to other strong-interaction amplitudes to extract information about the lesser known strong interaction, such as $\pi\pi$ or πK.[4–7]

In principle, the only obstacle to these PCAC relations is the external mass continuation. The exact PCAC relation refers to the scattering amplitude in which two of the external pion masses are set equal to zero. In order to relate this to a physical amplitude, a continuation has to be done in the external pion mass

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