Cluster Decomposition and the Spin-Statistics Theorem in S-Matrix Theory*

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This paper establishes within S-matrix theory the connection between spin and statistics; namely, that the multiparticle-state vectors are symmetric or antisymmetric for permutation of identical particles according as the particle concerned has integral or half-integral spin. The proof given, which is simpler than previous S-matrix proofs, depends on the cluster-decomposition property, crossing symmetry, and Hermitian analyticity. A considerable part of the paper is concerned with establishing a suitable framework to formulate the first of these properties, cluster decomposition. To this end we develop from first principles the idea of the tensor product $f\otimes g$ which, for any two state vectors f and g, represents the composite state (f and g).

I. INTRODUCTION

THE principal aim of this paper is to establish within an S-matrix framework the results well known in quantum field theory as the connection between spin and statistics. In field theory, where rigorous proofs have been known for some years, these results concern the commutation relations between two fields with space-like separation and may be summarized as follows[1]: First, the theorems of Dell' Antonio and of Lüders, Burgoyne, and Zumino imply that, according as its spin is integral or half-integral, any nonzero field must commute or anticommute both with itself and with its adjoint. Second, the theorem of Araki, which we call the relative statistics theorem, establishes the effect of commuting *different* fields. This theorem depends on the concept of normal commutation relations; namely, that any two fields should anticommute if both are fermion and commute if either or both are boson. The theorem establishes not that all fields must commute normally but that abnormal relations are restrictive in that they imply certain selection rules (not implied by normal relations) and, further, that any abnormal theory is physically equivalent to some normal theory.

We show that the corresponding results in S-matrix theory have a quite different status. In the first place, the concept of a field has no place in S-matrix theory where the primary concerns are the S matrix and the Hilbert spaces of initial and final states, in terms of which S is defined. The operation which corresponds to commuting field operators is the permutation of variables in the multiparticle state vectors—commutation of a field with itself corresponding to permutation of identical particles, commutation of a field with its adjoint to permuting a particle with its antiparticle, and commutation of two distinct fields to permuting two distinct particles. Our principal result is the expected one, that multiparticle state vectors are symmetric or antisymmetric for permutation of *identical* particles according as the particle concerned has integral or half-integral spin. (This result we call the spin-statistics theorem.) On the other hand, permutation of two distinct particles (including particle and antiparticle if these are distinct) is an operation which *a priori* has no meaning. Thus, it is certainly not possible to prove a result analogous to the relative statistics theorem of field theory. However, in the course of proving the spin-statistics theorem we find that it is possible and convenient to define the operation of permuting distinct particles and that the natural way to do this corresponds to the normal commutation relations of the relative statistics theorem; namely, that permutation of two fermions introduces a factor -1 while permutation of two bosons or a boson and a fermion leaves the state vectors unchanged.

Our proof of the spin-statistics theorem is essentially the same as the unpublished proof given by Stapp[2] in 1962 and depends on the cluster decomposition property of the S matrix, on crossing symmetry and on Hermitian analyticity. To the extent that all of these properties are much better understood now than formerly the present proof is both clearer and more convincing.[3] The particular point which we wish to emphasize is that previous discussions of the spin-statistics theorem in S-matrix theory (and of crossing symmetry and Hermitian analyticity) have all started from a certain

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[1] For details and original references see either R. F. Streater and A. S. Wightman, *P.C.T., Spin and Statistics and All That* (W. A. Benjamin, Inc., New York, 1964), p. 146 ff.; or R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, Rhode Island, 1965), p. 105 ff.

[2] H. P. Stapp, University of California, Lawrence Radiation Laboratory Report No. 10289, 1962 (unpublished).

[3] Some improvements on Stapp's original proof have been given recently by E. Y. C. Lu and D. I. Olive, Nuovo Cimento **45**, 205 (1966); and H. P. Stapp, University of California, Lawrence Radiation Laboratory Report No. 16816, 1966 (unpublished). We would like to thank Dr. Olive for a helpful correspondence.

postulated form of the cluster decomposition property. And the form in which this property is postulated depends on the validity of the relative statistics theorem which, as we have just argued, is not *a priori* valid or even meaningful. The assumption is of course a permissible hypothesis if one wishes to discuss only crossing symmetry, Hermitian analyticity or even the cluster property, but is obviously inadmissible when one wishes to establish the connection between spin and statistics. Thus, in addition to presenting an improved proof of the spin-statistics theorem it is the purpose of this paper to establish from first principles a framework for the formulation of the cluster property.

The cluster decomposition property expresses the observed fact that experiments which are well separated in space or time are independent.[4,5] Thus, if $[\mathbf{f}'\leftarrow\mathbf{f}]$ and $[\mathbf{g}'\leftarrow\mathbf{g}]$ are any two experimental processes we can consider the two processes $[\mathbf{f}'\leftarrow\mathbf{f}]$ and $[\mathbf{g}_x'\leftarrow\mathbf{g}_x]$ (where $\mathbf{g}_x$ denotes the state $\mathbf{g}$ translated through the four-vector x) as a single composite process $[\mathbf{f}'\&\mathbf{g}_x'\leftarrow\mathbf{f}\&\mathbf{g}_x]$. For large enough separation x the probability of the composite process should be simply the product of the probabilities for the two separate events; i.e.,

$$P(\mathbf{f}'\&\mathbf{g}_x'\leftarrow\mathbf{f}\&\mathbf{g}_x)\underset{x\to\infty}{\longrightarrow}P(\mathbf{f}'\leftarrow\mathbf{f})P(\mathbf{g}'\leftarrow\mathbf{g}). \quad (1.1)$$

To express this condition in quantum-mechanical language it is necessary to answer the following question: If f and g are vectors representing states $\mathbf{f}$ and $\mathbf{g}$, what is the vector (which we denote $f\otimes g$) which represents the single composite state $(\mathbf{f}\&\mathbf{g})$? It is this question which we examine in Sec. III.

It is of course well known that $f\otimes g$ is just the "tensor product" of f and g and that, if f and g are written in the form

$$f=\mathcal{A}^\dagger(f)|\text{vac}\rangle,$$

where $\mathcal{A}^\dagger(f)$ is the appropriate polynomial in the particle creation operators, then[6]

$$f\otimes g=\mathcal{A}^\dagger(f)\mathcal{A}^\dagger(g)|\text{vac}\rangle. \quad (1.2)$$

But it is by no means obvious that the recipe expressed by Eq. (1.2) is a necessary consequence of the physical properties of the composite state $(\mathbf{f}\&\mathbf{g})$. For example, why should the product $f\otimes g$ be bilinear [as (1.2) obviously is]? And is the relation between $f\otimes g$ and $g\otimes f$ as prescribed by (1.2) physically significant, or just a matter of convention? In Sec. III we show that the physical properties of the correspondence of states $\mathbf{f}$, $\mathbf{g}\to(\mathbf{f}\&\mathbf{g})$ determine an essentially unique bilinear map of the representative vectors, f, $g\to f\otimes g$ and that, by suitable choice of the arbitrary phases involved the product $f\otimes g$ can be put in the form (1.2).

Having established a correct form for the product vector $f\otimes g$ we can immediately write down the quantum-mechanical form of the cluster property (1.1). This is given in Sec. IV where we enumerate our assumptions on the S matrix. In addition to the cluster property, these are crossing symmetry and Hermitian analyticity. The recent work of Olive[7] and others[8] has indicated that the latter two properties can be proved on the basis of the cluster property and some general assumptions of analyticity. The proofs which have so far been given all contain somewhat unsatisfactory assumptions concerning the existence of certain analytic continuations. However, for our purposes it is sufficient that, given the correct form of the cluster property and assuming that the relevant continuations are possible, the forms of crossing symmetry and Hermitian analyticity are completely determined.

Finally in Sec. V we use the assumptions of Sec. IV to prove the spin-statistics theorem; namely, that the multiparticle wave functions are symmetric or antisymmetric for permutation of identical particles according as the particle concerned has integral or half-integral spin. Any particle for which this is not true can have no interactions either with itself or with other particles; i.e., all scattering amplitudes involving the particle are identically zero.

II. REPRESENTATION OF STATES

We shall assume that the initial and final states in terms of which the S matrix is defined are represented by wave packets which correspond to vectors in Hilbert spaces $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$. The discussion of this and the next section applies equally to either space and we shall for brevity speak of a single space $\mathcal{H}$. We make the classical assumption that any physically realizable state determines a unique ray (i.e., a vector unique up to a phase) in $\mathcal{H}$ and we assume that $\mathcal{H}$ is divided into orthogonal superselection sectors within which every vector represents a physically realizable state. We shall further assume that the superselection classes are, as appears to be the case, defined by discrete, additive, commuting quantum numbers.

We consider a theory with at most a countably infinite number of stable particles labeled by the integers $t=1, 2, \cdots$, in which case the space $\mathcal{H}$ is the Fock space corresponding to these particles. A state containing definite numbers of the various particles can be characterized by the sequence

[4] E. H. Wichmann and J. H. Crichton, Phys. Rev. **132**, 2788 (1963).

[5] J. R. Taylor, Phys. Rev. **142**, 1236 (1966).

[6] If f and g both contain definite numbers of particles, Eq. (1.2) expresses the familiar result that the wave function for $f\otimes g$ is just the symmetrized product of the wave functions f and g. We must of course establish the form of $f\otimes g$ not only in this simple case but also in the general case when f and g are both superpositions of several number eigenstates.

[7] D. I. Olive, Phys. Rev. **135**, B745 (1964).

[8] J. R. Taylor, J. Math. Phys. **7**, 181 (1966).

$$\mathbf{n}=(n_1,n_2,\cdots),$$

where n_t denotes the number of particles of type t. We shall assume that all particles have positive mass, in which case all such states have only a finite number of particles ($\sum_t n_t<\infty$). Within the sector $\mathcal{H}_\mathbf{n}$ of states with particle numbers $\mathbf{n}$ a state is specified by a wave function

$$f(P^\mathbf{n})=f(p_{11},\cdots,p_{1n_1};p_{21},\cdots,p_{2n_2};\cdots),$$

depending on the momenta and spins (which we leave implicit) of all the particles involved.

The space $\mathcal{H}$ is the direct sum

$$\mathcal{H}=\oplus_\mathbf{n}\mathcal{H}_\mathbf{n}$$

and the general state f is characterized by a sequence of wave functions

$$f=\{f_\mathbf{n}(P^\mathbf{n})\}.$$

The superselection rules imply that the particle numbers $\mathbf{n}$ for which $f_\mathbf{n}$ is nonzero must all lie in a single superselection sector. For scalar product in $\mathcal{H}$ we shall use

$$\langle f|g\rangle=\sum_\mathbf{n}\int\frac{d\Omega_\mathbf{n}}{\mathbf{n}!}f_\mathbf{n}^*(P^\mathbf{n})g_\mathbf{n}(P^\mathbf{n}),$$

where

$$d\Omega_\mathbf{n}=\prod_t\prod_{i=1}^{n_t}d^3p_{ti}/\omega_{ti};$$

ω_{ti} denotes the energy corresponding to the momentum p_{ti} with the appropriate mass, and

$$\mathbf{n}!=\prod_t n_t!.$$

We first remark that the question of permuting variables representing identical particles is one which demands immediate discussion, if analytic properties are to be considered at all. This is because a continuous variation of variables can carry the function $f(\cdots p,q\cdots)$ into $f(\cdots q,p\cdots)$ provided p and q represent identical particles. Indistinguishability of the particles and our assumption that there is a unique ray for a given state[9] imply that these two functions must be equal within a phase. Interchanging variables twice implies that the function must be either symmetric or antisymmetric in p and q. That for a given type of particle the wave functions must be always symmetric or always antisymmetric is not immediately obvious and in fact can only be established after our discussion of the tensor product. However, the argument is straightforward and may be mentioned here: If, for example, two-proton wave functions were antisymmetric while three-proton wave functions were totally symmetric it would be impossible to construct a state made up of a given two-proton state plus a third proton arbitrarily far away. This is contrary to experience and we conclude that in either $\mathcal{H}^{\text{in}}$ or $\mathcal{H}^{\text{out}}$ a given type of particle may be uniquely characterized as a boson (all symmetric wave functions) or a fermion (all antisymmetric wave functions). As yet of course we have no guarantee that a particle could not be a boson in $\mathcal{H}^{\text{in}}$ but a fermion in $\mathcal{H}^{\text{out}}$ or vice versa.

Secondly we would remark that in the formalism so far there is absolutely no meaning attached to the interchange of distinct particles. The particles have been written in an agreed (if arbitrary) order, $t=1$, $2,\cdots$, and state vectors have been defined in this order only. It is clearly unnecessary to define them in any other order, and equally clear that if we wished they could be defined in any new order just as we please. When we have considered the formation of the tensor product we shall see that there is a natural way to define vectors in which distinct particles have been permuted.

Finally, it is sometimes desirable to label initial and final states not by vectors in $\mathcal{H}$ but by a density matrix ρ acting on $\mathcal{H}$. In the first place this is necessary as a matter of principle since experiments are usually done not with pure states but with statistical mixtures, which can only be represented by density matrices. Secondly, even if one is considering a pure state, the density matrix has the marked advantage over the representative vector that it contains no arbitrary phase.

Accordingly any state, or mixture of states, is represented by a density matrix

$$\rho=\{\rho_{\mathbf{mn}}\},$$

where each element

$$\rho_{\mathbf{mn}}=\rho_{\mathbf{mn}}(P^\mathbf{m};Q^\mathbf{n})$$

depends on the two sets of variables $P^\mathbf{m}$ and $Q^\mathbf{n}$, and is symmetrized for interchange of identical p's and of identical q's. The superselection rules imply that only those elements $\rho_{\mathbf{mn}}$ for which $\mathbf{m}$ and $\mathbf{n}$ belong to the same superselection class are nonzero.

III. TENSOR PRODUCT OF STATES

A. Linearity

Given the structure of the Hilbert space of state vectors we can now consider our central preliminary problem; namely, if f and g are vectors representing two states **f** and **g** how one can calculate the vector (written $f\otimes g$) which represents the single state (**f&g**). Clearly the mapping of f, g onto $f\otimes g$ defines a mapping $\otimes$ of $\mathcal{H}\times\mathcal{H}$ into $\mathcal{H}$ and our problem is to discover the properties of this map. To this end we consider first states f and g whose space wave functions (i.e., the Fourier transforms of the momentum space wave functions for some fixed time) are contained in disjoint

[9] It is this assumption which eliminates the possibility of parastatistics. [See O. W. Greenberg and A. M. L. Messiah, Phys. Rev. **136**, B248 (1964).] The argument here is, of course, just the usual one to be found in any textbook on quantum mechanics.

parts of space. For such states we claim that the mapping $\otimes$ must satisfy

$$|\langle f'\otimes g'|f\otimes g\rangle| = |\langle f'|f\rangle\langle g'|g\rangle|\,. \tag{3.1}$$

This condition expresses the idea that if f and f' do not overlap g and g' and so cannot interfere, and if f has probability P of being observed in the state f' while g has probability Q of being observed in the state g', then the state represented by $f\otimes g$ must have probability PQ of being in the state $f'\otimes g'$.

The condition (3.1) fulfills for the map $\otimes$ the role of the usual condition on symmetry transformations to which Wigner's theorem[10] applies. This means, as we show in detail in Appendix A, that the mapping $\otimes$ which is *a priori* a mapping of rays in $\mathcal{H}$, $\mathbf{f}$, $\mathbf{g}\rightarrow(\mathbf{f}\&\mathbf{g})$ can be represented as a *bilinear* map of the representative vectors $f, g\rightarrow f\otimes g\in\mathcal{H}$. For nonoverlapping states this map is also biunitary; i.e., if f and f' do not overlap g and g'

$$\langle f'\otimes g'|f\otimes g\rangle = \langle f'|f\rangle\langle g'|g\rangle. \tag{3.2}$$

Given that the mapping $\otimes$ can be chosen bilinear we have only to determine its effect on a set of basis vectors. Accordingly we consider states f and g lying in the number sectors $\mathcal{H}_{\mathbf{m}}$ and $\mathcal{H}_{\mathbf{n}}$ and require that their image $f\otimes g$ lie in $\mathcal{H}_{\mathbf{m}+\mathbf{n}}$. (For example, if f is a state of $2\pi^0$ while g is π^0+K^+ then $f\otimes g$ is $3\pi^0+K^+$.) Thus, $f\otimes g$ is a vector with particle numbers $\mathbf{m}+\mathbf{n}$ and an appropriately symmetrized wave function. We show in Appendix A that linearity of $\otimes$ together with the requirement that the probabilities of observing given positions, momenta, and spins be the same for $f\otimes g$ as for f and g separately leads to the unique (and expected) result

$$(f\otimes g)(P^{\mathbf{m}+\mathbf{n}}) = a(\mathbf{m},\mathbf{n})\mathcal{S}f(P^{\mathbf{m}})g(P^{\mathbf{n}})\,, \tag{3.3}$$

where $\mathcal{S}$ denotes the usual operation of symmetrization for each particle type separately and $|a(\mathbf{m},\mathbf{n})|=1$. Since the map $\otimes$ is bilinear it follows from (3.3) that for any states f and g the product $f\otimes g$ is given by

$$(f\otimes g)_{\mathbf{m}} = \sum_{\mathbf{n}} a(\mathbf{n},\mathbf{m}-\mathbf{n})\mathcal{S}(f_{\mathbf{n}}g_{\mathbf{m}-\mathbf{n}})\,. \tag{3.4}$$

The coefficients $a(\mathbf{m},\mathbf{n})$ in (3.4) are not completely arbitrary. In addition to satisfying the condition $|a|=1$, they must be such that the two vectors $f\otimes g$ and $g\otimes f$ represent the same physical state and likewise the two vectors $(f\otimes g)\otimes h$ and $f\otimes(g\otimes h)$; i.e.,

$$f\otimes g = \alpha g\otimes f\,,\quad |\alpha|=1 \tag{3.5}$$

and

$$(f\otimes g)\otimes h = \beta f\otimes(g\otimes h)\,,\quad |\beta|=1\,. \tag{3.6}$$

We now show that these two conditions imply that the mapping $\otimes$ is unitary-equivalent to the familiar prescription given in Eq. (1.2).

[10] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), p. 233.

The conditions (3.5) and (3.6) are rather inconvenient because of the undetermined phase factors which they contain. For this reason it is convenient to rewrite the conditions in terms of density matrices. The existence of the bilinear map $\otimes$ on the state vectors implies a corresponding bilinear map on the density matrices. From Eq. (3.4) it follows that for any density matrices ρ and σ

$$(\rho\otimes\sigma)_{\mathbf{m}',\mathbf{m}} = \sum_{\mathbf{n}',\mathbf{n}} A(\mathbf{n}',\mathbf{n},\mathbf{m}'-\mathbf{n}',\mathbf{m}-\mathbf{n}) \times\mathcal{S}(\rho_{\mathbf{n}',\mathbf{n}}\sigma_{\mathbf{m}'-\mathbf{n}',\mathbf{m}-\mathbf{n}})\,, \tag{3.7}$$

where

$$A(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n}) = a(\mathbf{m}',\mathbf{n}')a^*(\mathbf{m},\mathbf{n})\,. \tag{3.8}$$

The superselection rules imply that $A(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})$ is defined only when $\mathbf{m}'$ and $\mathbf{m}$ belong to the same superselection sector and similarly $\mathbf{n}'$ and $\mathbf{n}$.

B. Commutativity of $\otimes$

Because there is a unique density matrix corresponding to any state the condition (3.5) becomes

$$\rho\otimes\sigma = \sigma\otimes\rho\,. \tag{3.9}$$

In order to combine this condition with (3.7) we note first that

$$\mathcal{S}(f_{\mathbf{m}}g_{\mathbf{n}}) = (-1)^{\Sigma^f m_t n_t}\mathcal{S}(g_{\mathbf{n}}f_{\mathbf{m}})\,, \tag{3.10}$$

where $\sum^f$ denotes summation over fermions only. In the same way

$$\mathcal{S}(\rho_{m',m}\sigma_{n',n}) = (-1)^{\Sigma^f(m_t'n_t'+m_tn_t)}\mathcal{S}(\sigma_{\mathbf{n}',\mathbf{n}}\rho_{\mathbf{m}',\mathbf{m}})\,.$$

Thus Eqs. (3.7) and (3.9) imply that

$$A(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n}) = (-1)^{\Sigma^f(m_t'n_t'+m_tn_t)}A(\mathbf{n}',\mathbf{n},\mathbf{m}',\mathbf{m})\,. \tag{3.11}$$

Setting $\mathbf{m}'=\mathbf{n}'$ and $\mathbf{m}=\mathbf{n}$ and recalling that $(-1)^n = (-1)^{n^2}$ for any integer n, we obtain

$$(-1)^{\Sigma^f(m_t'+m_t)} = 1$$

or

$$\textstyle\sum^f m_t' = \sum^f m_t \mod 2\,. \tag{3.12}$$

This means that whenever $\mathbf{m}$ and $\mathbf{m}'$ belong to the same superselection sector they must contain the same number of fermions modulo two. This is just the familiar fermion superselection rule[11] and may be used to simplify the condition (3.11) as follows: Since both the pairs $(\mathbf{m}',\mathbf{m})$ and $(\mathbf{n}',\mathbf{n})$ satisfy (3.12)

$$\sum_{s,t}{}^f m_s'n_t' = \sum_{s,t}{}^f m_sn_t \mod 2\,;$$

[11] The fermion superselection rule is usually proved by invoking the spin-statistics theorem and using the superselection rule separating states of integral and half-integral angular momentum. It is important that we prove it directly here, since it plays an essential role in forming the tensor product, before we prove the spin-statistics theorem.

therefore

$$\sum_t^f (m_t'n_t'+m_tn_t)=\sum_{s\neq t}^f (m_s'n_t'+m_sn_t) \quad \text{mod } 2.$$

Using this identity in the condition (3.11) we find that, if we define

$$B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=(-1)^{\Sigma_{s>t}{}^f(m_s'n_t'+m_sn_t)} \times A(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n}), \quad (3.13)$$

then the numbers B satisfy

$$B(\mu,\nu)=B(\nu,\mu), \quad (3.14)$$

where we have introduced μ and ν to denote the pairs $(\mathbf{m}',\mathbf{m})$ and $(\mathbf{n}',\mathbf{n})$. Thus, commutativity of $\otimes$ implies that $B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})$ is symmetric with respect to the interchange of $(\mathbf{m}',\mathbf{m})$ and $(\mathbf{n}',\mathbf{n})$.

C. Associativity

We next show that by suitable phase changes among the basis vectors the numbers $B(\mu,\nu)$ can all be chosen unity. We do this using the symmetry of $B(\mu,\nu)$, Eq. (3.14), and the associativity condition (3.6) which, in terms of density matrices ρ, σ, and τ, is

$$(\rho\otimes\sigma)\otimes\tau=\rho\otimes(\sigma\otimes\tau).$$

Substituting the expansion (3.7) we find that both the coefficients $A(\mu,\nu)$ and $B(\mu,\nu)$ must satisfy the following equation:

$$B(\lambda+\mu,\,\nu)B(\lambda,\mu)=B(\lambda,\,\mu+\nu)B(\mu,\nu) \quad (3.15)$$

for any λ, μ, and ν.

The solutions to Eqs. (3.14) and (3.15) are studied in Appendix B where it is shown that the numbers $B(\mu,\nu)$ must have the form

$$B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=\frac{\phi(\mathbf{m}'+\mathbf{n}')}{\phi(\mathbf{m}')\phi(\mathbf{n}')}\cdot\frac{\phi^*(\mathbf{m}+\mathbf{n})}{\phi^*(\mathbf{m})\phi^*(\mathbf{n})}, \quad (3.16)$$

where $\phi(\mathbf{m})$ is some function of $\mathbf{m}$ with $|\phi|=1$.

Substitution of this form into the expansion (3.7) makes clear that if we change basis so that the vector $\{\phi(\mathbf{n})f_{\mathbf{n}}(P)\}$ becomes $\{f_{\mathbf{n}}(P)\}$ then in the new basis the coefficients B are all unity. However, our primary concern is with the tensor product of two state vectors, not density matrices, and this may be analyzed using (3.16) in what follows.

D. Final Form for $\otimes$

In analogy with the coefficient B defined by (3.13) we introduce

$$b(\mathbf{m},\mathbf{n})=(-1)^{\Sigma_{s>t}{}^f m_sn_t}a(\mathbf{m},\mathbf{n})$$

in terms of which [cf. (3.8)]

$$B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=b(\mathbf{m}',\mathbf{n}')b^*(\mathbf{m},\mathbf{n}). \quad (3.17)$$

Equation (3.16) implies then that

$$b(\mathbf{m}',\mathbf{n}')\frac{\phi(\mathbf{m}')\phi(\mathbf{n}')}{\phi(\mathbf{m}'+\mathbf{n}')}=b(\mathbf{m},\mathbf{n})\frac{\phi(\mathbf{m})\phi(\mathbf{n})}{\phi(\mathbf{m}+\mathbf{n})}.$$

Since this is true for any $\mathbf{m}'$, $\mathbf{n}'$, $\mathbf{m}$, and $\mathbf{n}$, subject only to the conditions that $\mathbf{m}$ and $\mathbf{m}'$ belong to the same superselection sector and likewise $\mathbf{n}$ and $\mathbf{n}'$, we can separate variables to give

$$b(\mathbf{m},\mathbf{n})=\zeta(M,N)\frac{\phi(\mathbf{m}+\mathbf{n})}{\phi(\mathbf{m})\phi(\mathbf{n})},$$

where the separation constant $\zeta(M,N)$ has modulus one and depends only on the superselection sectors M and N defined by $\mathbf{m}$ and $\mathbf{n}$. The tensor product (3.4) can therefore be written

$$\phi(\mathbf{m})(f\otimes g)_{\mathbf{m}}=\zeta(F,G)\sum_{\mathbf{n}}(-1)^{\Sigma_{s>t}{}^f n_s(m_t-n_t)} \times \mathcal{S}(\phi(\mathbf{n})f_{\mathbf{n}}\phi(\mathbf{m}-\mathbf{n})g_{\mathbf{m}-\mathbf{n}}). \quad (3.18)$$

We first remark that for each pair of superselection sectors F and G there is an overall arbitrary phase in the definition of $f\otimes g$. Thus we can replace the product $f\otimes g$ by the physically equivalent $\zeta(F,G)f\otimes g$, in which case the factor $\zeta(F,G)$ disappears from Eq. (3.18). If we then move to the new basis in which $\{\phi(\mathbf{n})f_{\mathbf{n}}(P)\}$ is denoted $\{f_{\mathbf{n}}(P)\}$, Eq. (3.18) becomes

$$(f\otimes g)_{\mathbf{m}}=\sum_{\mathbf{n}}(-1)^{\Sigma_{s>t}{}^f n_s(m_t-n_t)}\mathcal{S}(f_n g_{m-n}), \quad (3.19)$$

which is the form we want.

E. Comments on the Form of $\otimes$

We first point out that the form (3.19) is precisely that given by the usual creation operator formalism (1.2). For example, if f and g are number eigenstates with particle numbers $\mathbf{m}$ and $\mathbf{n}$ then we denote

$$f=\mathfrak{a}^\dagger(f)|\text{vac}\rangle,$$

where $\mathfrak{a}^\dagger(f)$ is a monomial in the particle creation operators $a_t^\dagger$ with (by convention) $a_t^\dagger$ to the left of $a_s^\dagger$ if $t<s$. In order to write the vector

$$\mathfrak{a}^\dagger(f)\mathfrak{a}^\dagger(g)|\text{vac}\rangle$$

in this conventional order we must commute all operators $a_s^\dagger$ in $\mathfrak{a}^\dagger(f)$ through all operators $a_t^\dagger$, $s>t$, in $\mathfrak{a}^\dagger(g)$. The usual convention that any two fermion operators anticommute, while two bosons or boson and fermion commute, means that these commutations introduce a factor $(-1)^{\Sigma_{s>t}{}^f m_sn_t}$. This means that the operator $\mathfrak{a}^\dagger(f)\mathfrak{a}^\dagger(g)$ creates precisely the state prescribed by Eq. (3.19). Thus our proof that basis vectors can be chosen so that $f\otimes g$ has the form (3.19) provides the justification for the usual creation operator formalism.

If we now examine the relationship between $f\otimes g$ and $g\otimes f$ we can see the sense in which the relative statistics theorem holds in our formalism. From (3.19) it is clear that both products are defined, and from (3.10) and (3.19) that

$$f\otimes g=\epsilon g\otimes f\,,$$

where $\epsilon=-1$ of both f and g contain an odd number of fermions and $\epsilon=+1$ otherwise. Thus, we have shown that the bilinear product $f\otimes g$ can be *chosen* so that the relation of $f\otimes g$ to $g\otimes f$ is *as if* the individual particles permute according to the relative statistics theorem. It must be emphasized that the theorem is substantially a matter of convention. If f and g belong to different superselection sectors the products $f\otimes g$ and $g\otimes f$ contain *independent* arbitrary phase factors and their relationship is entirely a matter of choice.[12] If, on the other hand, f and g lie in the same superselection sector their relationship *is* determined and is that given by the relative statistics theorem.

Finally, since the product $\otimes$ is actually associative, we can, if we wish, use it to define state vectors in which the particles appear in orders different from the conventional order $t=1,\ 2\cdots$. Thus, if $t_1<t_2$, the two-particle vector $|t_1,t_2\rangle$ is defined from the outset, and from (3.19) it is clear that

$$|t_1\otimes t_2\rangle=|t_1,t_2\rangle\,.$$

The vector $|t_2,t_1\rangle$ has no *a priori* meaning but may be defined as

$$|t_2,t_1\rangle=|t_2\otimes t_1\rangle$$

and with this definition distinct particles obviously commute in accordance with the relative statistics theorem.

IV. ASSUMPTIONS ON THE S MATRIX

A. The S Matrix

The existence of a one-to-one correspondence between initial and final states which preserves superposition probabilities implies the existence of a unitary operator S mapping the corresponding spaces $\mathcal{H}^{\text{in}}$ onto $\mathcal{H}^{\text{out}}$. It is the properties of this S operator which are the central theme of S-matrix theory. Before outlining the properties which we use we should mention the relationship between $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$. As we have already said, both are Fock spaces for the set of all stable free particles. We take for granted that the set of all initial particles is the same as that of all final particles and that the superselection operators are the same in $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ and commute with S. This means that the only possible difference between $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ is in their symmetrizations; that is, some particles might be bosons in $\mathcal{H}^{\text{in}}$ but fermions in $\mathcal{H}^{\text{out}}$ or vice versa. In fact this cannot happen because Lorentz invariance implies that S leaves the one-particle states invariant, and so an initial boson could not become a final fermion, or vice versa, without violating the fermion superselection rule. This means that we can now identify $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ and regard S as a mapping of a single space $\mathcal{H}$ onto itself.

The principal concerns in S-matrix theory are not the experimental wave-packet elements $\langle f'|S|f\rangle$ but the momentum-space S-matrix elements, defined as distributions on some appropriate subset of the wave functions of $\mathcal{H}$. For these we use the notation of Ref. 8. An n-particle momentum eigenstate is labeled by the n-tuples

$$P=(p_1,\cdots,p_n)$$

and

$$T=(t_1,\cdots,t_n)\,,$$

where the t_i denote particle types and the p_i the corresponding momenta. The states are denoted either

$$(P,T)$$

or

$$(p_1t_1,\cdots,p_nt_n)\,.$$

As usual we omit spin labels and, whenever convenient, we also omit either P or T. The momentum space S-matrix elements are then denoted $\langle P',T'|S|P,T\rangle$, and since they are defined as distributions on $\mathcal{H}$ they have the same permutation symmetries as the wave functions which define them. In the first instance $\langle P',T'|S|P,T\rangle$ is defined only with the particle types T and T' in conventional order, $t_1\leqslant t_2\leqslant\cdots$; it is usual and convenient to use the formalism of Sec. III to define them with T and T' in any order.

Of the three properties of the S matrix—cluster decomposition, crossing symmetry, and Hermitian analyticity—which we use to prove the spin-statistics theorem, the first determines the form of the other two. It was to establish a suitable framework for this cluster property that the formalism of Sec. III was developed. In fact Sec. III simply justifies the formalism which was assumed without comment in Ref. 5 on the cluster property. Having justified this assumption we may simply quote the results of Ref. 5.

B. Cluster Decomposition

The cluster property as given in Eq. (1.1) expresses the independence of well separated experiments. In terms of the S-matrix elements Eq. (1.1) is

$$|\langle f'\otimes g_x'|S|f\otimes g_x\rangle|\underset{x\to\infty}{\longrightarrow}|S_{f'f}S_{g'g}|\,.$$

In Ref. 5 it is shown, using the superposition principle and linearity of the product $\otimes$, that this limit if suitably generalized to include separation into several inde-

[12] Indeed, when f and g belong to different superselection sectors, it is not even necessary to define both $f\otimes g$ and $g\otimes f$.

pendent experiments,[13] implies a corresponding limit without the modulus signs, namely,[14]

$$\langle f'\otimes g_x'|S|f\otimes g_x\rangle \xrightarrow[x\to\infty]{} S_{f'f}S_{g'g} \qquad (4.1)$$

for any wave packets f, f', g, g'.

To apply the cluster property in S-matrix theory one must translate Eq. (4.1) into its equivalent momentum space form. This translation is given by Eq. (17) of Ref. 5; namely

$$\langle Q|S|P\rangle = \sum_{\text{partitions } \pi} \beta_\pi \prod_{i=1}^{n_\pi} \langle Q_{\pi,i}|T|P_{\pi,i}\rangle. \qquad (4.2)$$

Here Q and P label final and initial momenta and we have omitted particle-type labels. The elements $\langle Q|T|P\rangle$ denote the so-called connected parts and the constant factors $\beta_\pi = \pm 1$ arise only because a general partition of $P=(P_{\pi,1},\cdots,P_{\pi,n_\pi})$ cannot be written in the same order as P. The significance of these momentum-space cluster equations is best understood from the pictorial representation shown in Fig. 1.

The wave-packet limit (4.1) implies more than Eq. (4.2) (which in fact only defines the connected parts $\langle Q|T|P\rangle$); it implies that the connected parts have the form

$$\langle Q|T|P\rangle = i\delta_4(\sum q - \sum p)\langle Q|M|P\rangle,$$

where the distributions $\langle Q|M|P\rangle$ contain no momentum-conserving delta functions.

In S-matrix theory it is postulated that with an appropriate spin basis the amplitude $\langle Q|M|P\rangle$ can be continued to give a unique analytic function of the momenta Q and P. The spin basis for which this is postulated is the so-called spinor basis described for example in Ref. 8, and the amplitudes $\langle Q|M|P\rangle$ evaluated in this spinor basis are known as "M functions" (or rather the connected parts of the M func-

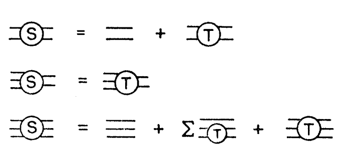

FIG. 1. Three simple examples of momentum-space cluster equations. The bubbles on the left represent momentum-space S-matrix elements $\langle Q|S|P\rangle$; those on the right represent the connected parts $\langle Q|T|P\rangle$, which are the interaction terms. The straight lines are one-particle matrix elements, proportional to $\delta_3(\mathbf{p}-\mathbf{q})$, and represent unscattered particles. The constant factors β_π, which multiply each term [see Eq. (4.2)], are not shown; indeed, for all terms which are shown explicitly, $\beta_\pi = +1$.

[13] This strengthening of the limit excludes the case of Coulomb scattering, which satisfies the limit with modulus signs but not (4.1). The limit (4.1) is almost certainly not true when massless particles are involved, which is one reason why we exclude them from consideration.

[14] It is first shown that this limit holds with a constant phase factor on the right and then that this factor can be removed by suitable redefinition of S.

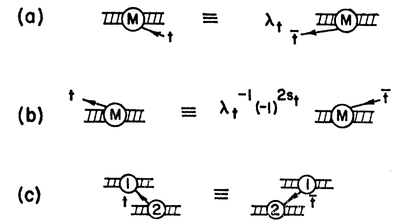

FIG. 2. (a) The crossing identity (4.3) with an initial t particle and a final $\bar{t}$. (b) The identity (4.4) with an initial $\bar{t}$ and a final t. (c) The two poles whose identity leads to the crossing equations shown in (a) and (b). The argument is, briefly, that the amplitudes labeled 1 depend on variables independent of those for the amplitudes 2. This means that corresponding amplitudes must be separately equal within a constant factor λ. The origin of the factor $(-1)^{2s}$ in (b) is explained below.

tions). It is analyticity of these amplitudes which leads to the other two properties which we must describe; namely, crossing symmetry and Hermitian analyticity.

C. Crossing Symmetry and Hermitian Analyticity

Crossing symmetry relates the amplitude for any process $[\cdots\leftarrow\cdots, pt]$ with a particle t in the initial state to that for the "crossed process" $[\cdots, -p\bar{t}\leftarrow\cdots]$ with the antiparticle $\bar{t}$ in the final state. Specifically, it states the following identity between the two analytic functions concerned:

$$\langle\cdots|M|\cdots,pt\rangle \equiv \lambda_t\langle\cdots,-p\bar{t}|M|\cdots\rangle, \qquad (4.3)$$

where λ_t is a constant depending on the particle type t but independent of the momentum p and all other variables concerned. This identity is illustrated in Fig. 2(a).

It is important to recognize that if one amplitude in the crossing identity is evaluated in its physical region then the other is certainly not. Thus if the left-hand amplitude of Eq. (4.3) is physical then the momentum p must be forward time-like; in this case the particle $\bar{t}$ on the right has backward time-like momentum and the right-hand amplitude is certainly not physical. Clearly the very concept of crossing symmetry is meaningless without the idea of analyticity, which allows amplitudes to be defined away from their physical regions.

The crossing identity must therefore be interpreted as follows: The two analytic functions concerned are defined by analytic continuation from their (different) physical regions. The identity (4.3) asserts that there is some path joining these two physical regions along which the identity holds. Strictly speaking this path of continuation should be specified in the statement of Eq. (4.3).[15]

[15] The situation can be compared to the example of two real functions $f(x)=+\sqrt{x}$, $(x>1)$ and $g(x)=+\sqrt{x}$, $(0<x<1)$. Both can be continued to give the same analytic function $\sqrt{z}$, and one can legitimately write $f(z)\equiv g(z)$. However, it is clearly necessary to specify the path of continuation used, since along a path which encircles the origin the identity is simply not true. (In fact, $f\equiv -g$ on such a path.)

The companion equation to Eq. (4.3), which relates any process with an initial $\bar{t}$ to the crossed process with a final t, is

$$\langle\cdots|M|p\bar{t},\cdots\rangle\equiv\lambda_t(-1)^{2s_t}\langle -pt,\cdots|M|\cdots\rangle. \quad (4.4)$$

The interpretation of this equation is the same as that of Eq. (4.3). The important points to note are: (i) The same factor λ_t appears in both equations. (ii) In Eq. (4.4) there is an extra factor $(-1)^{2s_t}$ where s_t is the spin of the crossed particle t. (iii) In Eq. (4.3) the crossed variables are on the right of the states in which they occur; in Eq. (4.4) they are on the left.[16]

The proof given by Olive[7] of the two crossing equations is based on consideration of the one-particle poles shown in Fig. 2(c). The argument is well known and need not be repeated here. Suffice it to say, firstly, that the factor $(-1)^{2s}$ appears in Eq. (4.4) because in passing from the identity of Fig. 2(c) to the separate identities of Figs. 2(a) and 2(b) one must adjust spinor indices so that in each equation they are all of the same type.[17] To achieve this one uses the famous identity

$$x^\alpha y_\alpha = (-1)^{2s} x_\alpha y^\alpha,$$

where x and y are spinors in the representation $D^{0,s}$ of the Lorentz group. Hence the factor $(-1)^{2s}$. Second, the different orderings of variables in Eqs. (4.3) and (4.4) can easily be understood if one traces carefully the origin of the two poles in Fig. 2(c) in the relevant unitarity equations. (For more details see Ref. 8.)

The difficulty with Olive's proof is that it has so far been impossible to guarantee that the desired continuation between the two poles of Fig. 2(c) is actually possible. It is possible that some branch point could intervene and carry the continuation off the physical sheet, or even that a natural boundary of singularities completely prevent continuation. This objection is of course serious. However, for the present purposes it is sufficient to note that if one *assumes* that the necessary paths of continuation exist then the form of the crossing equations is uniquely determined by Olive's argument to be that of Eqs. (4.3) and (4.4). Since it is the form of these equations which leads to the spin-statistics theorem, this is the assumption we make.

The argument which leads to crossing symmetry makes clear that particle and antiparticle must carry opposite values of all additive conserved quantum numbers. In particular this means, because of the fermion superselection rule, that particle and antiparticle must both obey the same statistics; i.e., both are bosons or both are fermions.

[16] Here, we are taking advantage of the convention mentioned in Sec. III which allows particles to be put in any order. Equations (4.3) and (4.4) can, of course, be written with all particles kept in a conventional order, but this introduces further minus signs.

[17] It is important in what follows that, as is easily checked, the crossing equations take the same form for all types of index (upper and lower, dotted and undotted), provided corresponding indices are the same on each side.

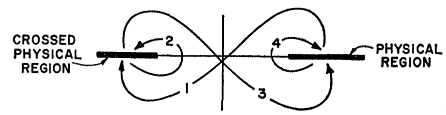

FIG. 3. The complex plane of the energy p^0 of the crossed particle, showing the four paths of continuation: (1) crossing, (2) Hermitian analyticity, (3) crossing, and (4) Hermitian analyticity.

The proof of Hermitian analyticity (HA) is quite similar to that of crossing and is open to the same objections.[18] We assume that the necessary path of continuation exists (leading in this case from any point P to its complex conjugate P^*) and then the form of HA is uniquely determined as

$$\langle P_2,T_2|M|P_1,T_1\rangle\equiv\langle P_1{}^*,T_1|M|P_2{}^*,T_2\rangle^*. \quad (4.5)$$

Here again the identity should really be accompanied by a specification of the path of continuation for which it holds.

The proof of the spin-statistics theorem uses a sequence of four analytic continuations; namely: (1) crossing (4.3), (2) HA, (3) crossing (4.4) and finally, (4) HA. If we consider for a moment spinless bosons (which means that variables can be reordered at will) then the effect of these four continuations for any process $[\cdots\leftarrow\cdot\cdot, pt]$ is

$$\begin{aligned}\langle\cdots|M|\cdot\cdot,pt\rangle&=\lambda_t\langle\cdots,-p\bar{t}|M|\cdot\cdot\rangle\\&=\lambda_t\langle\cdot\cdot|M|\cdots,-p^*\bar{t}\rangle^*\\&=\lambda_t\lambda_t{}^*\langle\cdot\cdot,p^*t|M|\cdots\rangle^*\\&=\lambda_t\lambda_t{}^*\langle\cdots|M|\cdot\cdot,p\bar{t}\rangle. \quad (4.6)\end{aligned}$$

Figure 3 shows the paths of continuation in the plane of the crossed particle's energy p^0. The complete continuation obviously comes back to its starting point. If one *assumes* that it returns to its starting point *on the same sheet*, then Eq. (4.6) obviously implies that $|\lambda_t|=1$. As we shall see, when one takes proper account of the order of variables and of spin, this same argument yields the spin-statistics theorem, and we therefore make this assumption.

The assumptions on crossing and Hermitian analyticity may be summarized as follows: We assume that there exist paths of continuation for crossing and HA. In this case it follows that crossing must take the form of Eqs. (4.3) and (4.4) and HA that of (4.5). We further assume that the sequence of continuations shown in Fig. 3 defines a closed path (returning to its starting point on the same sheet). It is obviously desirable, in the long run, rigorously to verify these properties. In the meantime they may be regarded as quite reasonable assumptions in the framework of S-matrix theory.

[18] The recent unpublished report of H. P. Stapp (Ref. 3) claims to have removed some of the less acceptable features of both proofs.

V. THE SPIN-STATISTICS THEOREM

We first prove the spin-statistics theorem for a particle t on the assumption (discussed below) that there is at least one process[19] $[mt \leftarrow nt]$ with nonzero connected part

$$\langle mt|M|nt\rangle \not\equiv 0\,, \quad \text{some } m \text{ and } n. \tag{5.1}$$

If the process $[2\leftarrow 2]$ has nonzero connected part so does the process $[3\leftarrow 3]$, since the amplitude for $[3\leftarrow 3]$ has a pole whose residue is the product of two amplitudes for $[2\leftarrow 2]$. By a similar argument, if the amplitude for a process $[m'\leftarrow n']$ with $(m'+n')$ odd is nonzero, then there is always some other process $[m\leftarrow n]$ with $(m+n)$ even which also has nonzero connected part. (See Fig. 4.) Thus if the inequality (5.1) holds for any m and n one may suppose that $(m+n)$ is even and greater than 4.

We consider the nonzero amplitude (5.1) which, to focus attention on one particle, we write as $\langle\cdots|M|\cdot\cdot,pt\rangle$, and perform the sequence of continuations described at the end of Sec. IV (see Fig. 3).

$$\begin{aligned}
\langle\cdots|M|\cdot\cdot,pt\rangle &= \lambda_t\langle\cdots,-p\bar{t}|M|\cdot\cdot\rangle, \quad \text{crossing (4.3)}\\
&= \lambda_t\langle\cdot\cdot|M|\cdots,-p^*\bar{t}\rangle^*, \quad \text{HA (4.5)}\\
&= \cdot\cdot(\epsilon_t)^m\langle\cdot\cdot|M|-p^*\bar{t},\cdots\rangle^*\\
&= \cdots\cdots\lambda_t^*(-1)^{2s_t}\langle p^*t,\cdot\cdot|M|\cdots\rangle^*, \quad \text{crossing (4.4)}\\
&= \cdots\cdots\cdots\cdots\langle\cdots|M|pt,\cdot\cdot\rangle, \quad \text{HA}\\
&= \cdots\cdots\cdots\cdots(\epsilon_t)^{n-1}\langle\cdots|M|\cdot\cdot,pt\rangle,
\end{aligned}$$

where $\epsilon_t=+1$ if t is a boson, -1 if t is a fermion.[20] If we assume that the sequence of continuations returns to its starting point on the same sheet, this implies that

$$\lambda_t\lambda_t^*(-1)^{2s_t}(\epsilon_t)^{m+n-1}=1.$$

Since $|\epsilon|=1$ this proves that $|\lambda_t|=1$ and hence, since $(m+n)$ is even, that

$$\epsilon_t=(-1)^{2s_t}, \tag{5.2}$$

which is the normal connection between spin and statistics.

To prove Eq. (5.2) we assumed that at least one connected part $\langle mt|M|nt\rangle$ is nonzero. Thus a particle t which obeys *abnormal* statistics [i.e., which violates Eq. (5.2)] must satisfy

$$\langle mt|M|nt\rangle\equiv 0\,, \quad \text{all } m \text{ and } n.$$

We now prove that this condition implies that *all* connected parts containing one or more abnormal particles are identically zero.

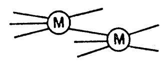

FIG. 4. If, for example, the amplitude for process $[3\leftarrow 2]$ is nonzero, then that for the process $[5\leftarrow 3]$ has a pole of nonzero residue and so cannot be identically zero.

We return to the momentum-space cluster equations shown in Fig. 1. It is clear that if the connected parts for all processes $[mt\leftarrow nt]$ are zero then the corresponding S-matrix elements are the same as those of the unit operator (times an unimportant phase factor)

$$\langle mt|S|nt\rangle=\langle mt|nt\rangle. \tag{5.3}$$

Now if S is any unitary operator and a a vector such that

$$\langle a|S|a\rangle=\langle a|a\rangle,$$

then $Sa=a$.[21] Thus Eq. (5.3) implies that S-matrix elements linking states (nt) to *any other states* are zero.

If the S-matrix elements linking the state (nt) to all other states are zero, so are the corresponding connected parts; and likewise all connected parts related to these by crossing. In particular, we can now assert that for any abnormal particle t

$$\langle mt,m'\bar{t}|M|nt,n'\bar{t}\rangle\equiv 0\,, \quad \text{all } m,\ m',\ n,\ n'.$$

This means that the corresponding S-matrix elements are the same as those of the identity operator and we can repeat the argument of the previous paragraph to prove that the S-matrix elements linking a state $(nt,n'\bar{t})$ with any other state are zero. The same then holds for the corresponding connected parts and we conclude that, because t is abnormal, any connected part with two or more external lines labeled either t or $\bar{t}$ is identically zero.

The only remaining possibility for the abnormal particle t is that there are nonzero amplitudes with just one external line labeled t (or one labeled $\bar{t}$). This possibility is quickly disposed of. We have already seen that all connected parts $\langle t,\cdots|M|t,\cdots\rangle$ are zero, which means that the corresponding S-matrix elements have the form

$$\langle t,T'|S|t,T\rangle=\langle T'|S|T\rangle.$$

A simple extension of the argument used twice above shows that in this case all S-matrix elements linking the state $(t,\cdots)$ to states not containing t are zero. The same is then true for the corresponding connected parts.

This establishes that any connected parts containing any number of abnormal particles t or $\bar{t}$ are identically zero. It follows that abnormal particles can

[19] Here, the notation (nt) denotes a state of n particles of type t. In this last section, we frequently omit all but particle-type labels.

[20] In the third line, we use the fact that t and $\bar{t}$ obey the same statistics and so, according to the conventions of Sec. III, permutation of the $\bar{t}$ variable through the m particles t introduces the factor $(\epsilon_t)^m$.

[21] To prove this, choose an orthogonal basis including a. Then $Sa=a+b$, where b is orthogonal to a and, since S preserves the norm, is actually zero.

never transfer energy or momentum to other particles (either normal or abnormal) or among themselves. They are therefore completely unobservable.[22]

APPENDIX A: LINEARITY OF THE TENSOR PRODUCT

Definition of $\otimes_R$

Let R denote any closed region in configuration space and $\sigma(f)$ the one-particle space support of any state vector f; i.e., the union of the projections of the support of the spatial wave function of f into the one-particle configuration space. We define two nonoverlapping subspaces of $\mathcal{H}$,[23]

$$\mathcal{H}(R)=\{f:f\in\mathcal{H},\sigma(f)\subset R\},$$
$$\mathcal{H}'(R)=\{g:g\in\mathcal{H},\sigma(g)\subset\bar{C}\bar{R}\}.$$

We assume that for any rays $\mathbf{f}\subset\mathcal{H}(R)$ and $\mathbf{g}\subset\mathcal{H}'(R)$ there is a unique ray $(\mathbf{f}\&\mathbf{g})$ corresponding to the state $\mathbf{f}$ *and* $\mathbf{g}$ and to it only; also that the representative vectors satisfy Eq. (3.1)

$$|\langle f'\otimes g'|f\otimes g\rangle|=|\langle f'|f\rangle\langle g'|g\rangle| \qquad \text{(A1)}$$

for f, $f'\in\mathcal{H}(R)$ and g, $g'\in\mathcal{H}'(R)$. We show first that these assumptions lead to an essentially unique bilinear (and biunitary) map $\otimes_R$ of $\mathcal{H}(R)\times\mathcal{H}'(R)$ into $\mathcal{H}$.

For fixed f in $\mathcal{H}(R)$ the correspondence $\mathbf{f}$, $\mathbf{g}\rightarrow(\mathbf{f}\&\mathbf{g})$ gives a map of $\mathcal{H}'(R)$ into $\mathcal{H}$ which satisfies the conditions of Wigner's theorem. Thus for each f in $\mathcal{H}(R)$ the correspondence can be represented by a *linear*[24] map $\otimes_f$ on $\mathcal{H}'(R)$

$$\otimes_f:g\rightarrow f\otimes g,$$

where

$$f\otimes(g_1+g_2)=f\otimes g_1+f\otimes g_2,$$

and (A2)

$$f\otimes(zg)=z(f\otimes g).$$

As usual each $\otimes_f$ is unique up to an over-all phase factor. We may trivially extend $\otimes_f$ so that it is also linear on the subspace $\{zf:z\in\mathbf{C}^1\}$.

We next focus attention on some fixed g_0 in $\mathcal{H}'(R)$. As f ranges over $\mathcal{H}(R)$ the vector $f\otimes g_0$ defines a map of $\mathcal{H}(R)$ into $\mathcal{H}$ which also satisfies the criteria of Wigner's theorem. Thus, by adjusting the phases of the $\otimes_f$, we can make this map linear,

$$(f_1+f_2)\otimes g_0=f_1\otimes g_0+f_2\otimes g_0. \qquad \text{(A3)}$$

The image $f\otimes g$ of every pair f, g in $\mathcal{H}(R)\times\mathcal{H}'(R)$ is now completely determined (apart from one over-all phase) and we must prove that the mapping so defined is automatically bilinear. To this end it is obviously sufficient to verify that

$$(f_1+f_2)\otimes g=f_1\otimes g+f_2\otimes g$$

for any f_1 and f_2 in $\mathcal{H}(R)$ and any g in $\mathcal{H}'(R)$. Since this is already known to hold for $g=zg_0$, it is sufficient to prove it for g orthogonal to g_0. It is also sufficient to prove it for f_1 and f_2 orthogonal. In this case, we consider

$$\begin{aligned}(f_1+f_2)\otimes(g_0+g)&\\ &=(f_1+f_2)\otimes g_0+(f_1+f_2)\otimes g \quad \text{(by A2)}\\ &=f_1\otimes g_0+f_2\otimes g_0+(f_1+f_2)\otimes g \quad \text{(by A3).} \qquad \text{(A4)}\end{aligned}$$

But by (A1) and the orthogonality of f_1, f_2 and g_0, g the left-hand side is equal to

$$\begin{aligned}&\alpha f_1\otimes(g_0+g)+\beta f_2\otimes(g_0+g), \quad |\alpha|=|\beta|=1\\ &\quad=\alpha f_1\otimes g_0+\alpha f_1\otimes g+\beta f_2\otimes g_0+\beta f_2\otimes g,\\ &\qquad\qquad \text{(by A2).} \qquad \text{(A5)}\end{aligned}$$

Because all vectors concerned are orthogonal we can compare coefficients in (A4) and (A5) and deduce first that $\alpha=\beta=1$ and hence that

$$(f_1+f_2)\otimes g=f_1\otimes g+f_2\otimes g,$$

as required.

Continuation of $\otimes_R$ to All Regions

For each closed region R the correspondence $\mathbf{f}$, $\mathbf{g}\rightarrow(\mathbf{f}\&\mathbf{g})$ is now represented by the bilinear map $\otimes_R$ on $\mathcal{H}(R)\times\mathcal{H}'(R)$. We now show that the various $\otimes_R$ can be joined together to give a unique bilinear image for any nonoverlapping states f and g. To this end we must show that the phases of the $\otimes_R$ can be adjusted so that for any regions R and S the maps $\otimes_R$ and $\otimes_S$ coincide on their common domain (if any); i.e.,

$$f\otimes_R g=f\otimes_S g \qquad \text{(A6)}$$

for all

$$f\in\mathcal{H}(R\cap S), \quad g\in\mathcal{H}'(R\cup S). \qquad \text{(A7)}$$

The two vectors in (A6) represent the same physical state and so are certainly equal within a phase,

$$f\otimes_R g=\gamma_{RS}f\otimes_S g, \quad |\gamma_{RS}|=1.$$

Since both maps are bilinear, the factor γ_{RS} is the same for all f and g satisfying (A7). This means that if $\otimes_R$ and $\otimes_S$ coincide on any nonzero pair f, g then they coincide on their whole common domain.

We first divide space by an arbitrary plane into two regions I and II and choose an arbitrary compact region III which intersects both I and II. (See Fig. 5.) We now fix the phase of $\otimes_{\text{III}}$ arbitrarily and adjust $\otimes_{\text{I}}$ so that

$$\otimes_{\text{I}}=\otimes_{\text{III}} \quad \text{on} \quad \mathcal{H}(\text{I}\cap\text{III})\times\mathcal{H}'(\text{I}\cup\text{III}),$$

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[22] Of course, if one envisages an S-matrix theory of strong interactions only, then the result is that abnormal particles cannot interact strongly. It would not preclude their having electromagnetic or weak interactions.

[23] The region $\bar{C}\bar{R}$ is the closure of the complement of R. If there are superselection rules, we must suppose that $\mathcal{H}(R)$ and $\mathcal{H}'(R)$ are contained in any two fixed superselection sectors. This allows us to ignore superselection rules throughout the Appendix.

[24] That the map is linear, not antilinear, follows, because we naturally choose the relationship between configuration and momentum spaces to be the same for all sectors.

Similarly we adjust $\otimes_{\text{II}}$ to coincide with $\otimes_{\text{III}}$ on their common domain.

We now define the product $\otimes_R$ for any other closed region R in terms of the three products $\otimes_{\text{I}}$, $\otimes_{\text{II}}$, and $\otimes_{\text{III}}$. If R is compact there are just four possibilities:

(1) $R\subset\text{I}$. We define $\otimes_R$ so that $\otimes_R=\otimes_{\text{I}}$ on their common domain. This definition ensures that if R and S are both contained in I, then $\otimes_R$ and $\otimes_S$ coincide on their common domain, if any.

(2) $R\subset\text{II}$. We define similarly $\otimes_R$ so that $\otimes_R=\otimes_{\text{II}}$ where applicable. If $R\subset\text{II}$ and $S\subset\text{I}$ then $\mathcal{H}(R\cap S)$ is empty. Thus, so far our definitions guarantee that $\otimes_R$ and $\otimes_S$ coincide on their common domain whenever this exists.

If R is contained in neither I nor II, then it overlaps both. There are two cases to consider:

(3) $\text{III}\subset R$. We define $\otimes_R=\otimes_{\text{III}}$ on $\mathcal{H}(\text{III})\times\mathcal{H}'(R)$. If R and S both contain III they clearly coincide on $\mathcal{H}(\text{III})\times\mathcal{H}'(R\cup S)$ and hence everywhere on their common domain. Further $\otimes_R$ and $\otimes_{\text{I}}$ coincide on their common domain since they do on $\mathcal{H}(\text{I}\cap\text{III})\times\mathcal{H}'(R\cup\text{I})$. Thus if R contains III while $S\subset\text{I}$, then $\otimes_R=\otimes_S$ on $\mathcal{H}(R\cap S)\times\mathcal{H}'(R\cup S)$. Similarly if $\text{III}\subset R$ and $S\subset\text{II}$.

(4) $\text{III}\not\subset R\not\subset\text{I}$ or II. We choose any R' containing both[25] R and III and define $\otimes_R=\otimes_{R'}$ on $\mathcal{H}(R)\times\mathcal{H}'(R')$. It may be checked that this definition is independent of the choice of R' and also that $\otimes_R$ coincides with $\otimes_S$ on their common domain for S in any of the categories (1) to (4).

The map $\otimes_R$ is now defined and satisfies (A6) for all compact regions R. Finally if R is closed but infinite it is easy to see that $\otimes_R$ can be defined to coincide with $\otimes_{R'}$ where R' is any compact region contained in R. This definition is independent of the particular R' used and satisfies the consistency condition (A6). It completes the definition of a universal bilinear product $f\otimes g$ for all nonoverlapping states.

Finally we note that condition (A1) together with bilinearity implies that the map $\otimes$ is actually biunitary; i.e.

$$\langle f'\otimes g'|f\otimes g\rangle=\langle f'|f\rangle\langle g'|g\rangle$$

provided f' and f do not overlap g' and g.

Linear Extension to Overlapping States

The bilinear product $f\otimes g$ has so far been defined only for nonoverlapping states and it is natural to seek a bilinear extension onto more general pairs of states. If we anticipate for a moment the well-known answer to this problem we can see immediately that the product cannot be extended onto the whole space $\mathcal{H}\times\mathcal{H}$. This is because the map $\otimes$, although biunitary on nonoverlapping states, is in general unbounded. Thus if f is a state of one boson, it is well known that $\|f\otimes f\|=\sqrt{2}\|f\|^2$; and states of n identical bosons can be chosen so that $\|f\otimes f\|=[(2n)!/(n!)^2]^{1/2}\|f\|^2$. Thus, the map is unbounded and one can construct states $f=\{f_n\}$ for which $f\otimes f$ does not exist (as a vector in $\mathcal{H}$) at all. However, all states of physical interest are contained in the manifold $\mathcal{H}'$ of states which contain only a finite number of particles, and on $\mathcal{H}'\times\mathcal{H}'$ the map does have a bilinear extension. We show that on any pair $\mathcal{H}_m\times\mathcal{H}_n$ the map $\otimes$ has a unique bilinear extension which is in fact bounded.

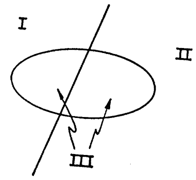

FIG. 5. The three regions I, II, and III used to define the product $\otimes$.

For simplicity we consider only the case of a one-particle sector $\mathcal{H}_1$ say. The generalization to higher sectors, though rather complicated, is quite straightforward. Accordingly we let f and g be any one-particle state vectors (of the same type of particle) with bounded wave functions of compact space support. Let $\{I_i^1\}_{i=1}^{\infty}$ denote a partition of space into unit cubes. By repeatedly halving the dimensions of all cubes we obtain a sequence of partitions $\{I_i^\alpha\}$, $\alpha=1, 2, \cdots$ with each I_i^α contained in some I_j^β for all $\alpha>\beta$, and $I_i^\alpha\to 0$ as $\alpha\to\infty$. For each α, we can write

$$f=\sum_{i=1}^{\infty} f_i^\alpha,$$

where $\sigma(f_i^\alpha)\subset I_i^\alpha$. This is in fact a finite sum since $\sigma(f)$ is compact. We now define

$$f\otimes^{(\alpha)}g=\sum_{i\neq j} f_i^\alpha\otimes g_j^\alpha, \tag{A8}$$

(this sum certainly exists since it is actually finite) and

$$f\otimes' g=\lim_{\alpha\to\infty} f\otimes^{(\alpha)}g.$$

We note first that this limit certainly exists since[26]

$$\|f\otimes^{(\alpha)}g-f\otimes^{(\alpha+1)}g\|^2<\text{const}(7/8)^\alpha.$$

Second, if f and g have disjoint one-particle supports, this limit is just $f\otimes g$ as previously defined. Third, the definition is clearly bilinear, and fourth, it is bounded,

[25] It is here that we use the compactness of R. This guarantees that $R\cup\text{III}$ does not occupy all of space, which allows R' to be chosen so that CR' is nonempty.

[26] The factor $\frac{7}{8}$ arises as follows: All terms in $\otimes^{(\alpha)}$ are present in $\otimes^{(\alpha+1)}$. Since each cube of the partition α is divided into eight cubes of the partition $(\alpha+1)$, the number of extra terms in the sum (A8) is multiplied by 8×7 for each successive α. The volume of integration for each term is divided by 8×8 each time. Hence, the factor $\frac{7}{8}$ arises for each successive partition.

since

$$\|f\otimes^{(\alpha)}g\|^2=\sum_{i<j}\|f_i^\alpha\otimes g_j^\alpha+f_j^\alpha\otimes g_i^\alpha\|^2$$

$$=\sum_{i\neq j}\|f_i^\alpha\|^2\|g_j\|^2\pm\langle f_i^\alpha|g_i^\alpha\rangle\langle f_j^\alpha|g_j^\alpha\rangle$$

$$\leqslant\sum_{i,j}\|f_i^\alpha\|^2\|g_j\|^2+\langle f_i^\alpha|g_i^\alpha\rangle\langle f_j^\alpha|g_j^\alpha\rangle$$

$$=\|f\|^2\|g\|^2+|\langle f|g\rangle|^2\leqslant 2\|f\|^2\|g\|^2$$

(which is just the result mentioned above). Since the new operator $\otimes'$ is defined on a dense set in $\mathcal{H}_1\times\mathcal{H}_1$ it can now be extended as a bounded bilinear operator onto the whole of $\mathcal{H}_1\times\mathcal{H}_1$.

Finally the operator $\otimes'$ is the only bounded bilinear extension of the original $\otimes$, for if $\otimes''$ were some other extension it is easily seen that

$$f\otimes''g-f\otimes^{(\alpha)}g=\sum_i f_i^\alpha\otimes g_i^\alpha\rightarrow 0 \quad \text{as} \quad \alpha\rightarrow\infty\,,$$

i.e., $\otimes''=\otimes'$. We can now drop the prime from $\otimes'$ since it is the unique extension of $\otimes$.

A similar analysis can be applied to any pair of sectors $\mathcal{H}_{\mathbf{m}}\times\mathcal{H}_{\mathbf{n}}$ with the expected result that $\otimes$ has a unique bounded bilinear extension with the bound

$$\|f\otimes g\|^2\leqslant\frac{(\mathbf{m}+\mathbf{n})!}{\mathbf{m}!\mathbf{n}!}\|f\|^2\|g\|^2.$$

The Product of Two Number Eigenstates

We finally prove for f in $\mathcal{H}_{\mathbf{m}}$ and g in $\mathcal{H}_{\mathbf{n}}$ that $f\otimes g$ has the wave function

$$(f\otimes g)(P^{\mathbf{m}+\mathbf{n}})=a(\mathbf{m},\mathbf{n})\mathcal{S}f(P^{\mathbf{m}})g(P^{\mathbf{n}})\,, \quad \text{(A9)}$$

where $|a|=1$ and $\mathcal{S}$ is the usual symmetrization operator

$$(\mathcal{S}fg)(p_{1,1},\cdots,p_{1,(m_1+n_1)};p_{2,1},\cdots)=\sum_\pi{}'\,\epsilon_\pi f(p_{1,\pi 1},\cdots)g(p_{1,\pi(m_1+1)},\cdots)\,,$$

where $\sum_\pi{}'$ denotes summation over all permutations of identical particles for which the product is not already symmetrized.

For simplicity we consider f and g to be one-particle states (the generalization is completely straightforward) in which case bilinearity of the product $f\otimes g$ implies that

$$(f\otimes g)(p,q)=\int dp'dq'K(p,q,p',q')f(p')g(q')\,. \quad \text{(A10)}$$

We now impose our final physical requirement on the map $\otimes$; namely, that at least for nonoverlapping states, the probability of observing given positions, momenta and spins must be the same for $f\otimes g$ as for f and g, separately. Thus if f and g represent distinct particles we require that

$$|(f\otimes g)(p,q)|^2=|f(p)g(q)|^2\,, \quad \text{(A11)}$$

with an identical condition on the space wave functions. Since $\otimes$ is bilinear this implies that

$$(f\otimes g)^*(p,q)(f'\otimes g')(p,q)=f^*(p)f'(p)g^*(q)g'(q)\,.$$

Substitution of (A10) gives

$$K^*(p,q,p'',q'')K(p,q,p',q')=\delta(p-p'')\delta(p-p')\delta(q-q'')\delta(q-q')\,,$$

from which we conclude, by separation of variables, that

$$K(p,q,p',q')=a(p,q)\delta(p-p')\delta(q-q')\,, \quad \text{(A12)}$$

where $|a(p,q)|=1$. Comparing this with the corresponding equation for the space functions we see that

$$a(p,q)=\text{const}\,. \quad \text{(A13)}$$

Substitution of (A12) and (A13) into (A10) gives the required result (A9).

If f and g represent the same type of particle, then, because of indistinguishability, the condition (A11) becomes

$$|(f\otimes g)(p,q)|^2=|f(p)g(q)|^2+|f(q)g(p)|^2$$

(for nonoverlapping f and g). The result (A12) becomes

$$K(p,q,p',q')=a(p,q)\delta(p-p')\delta(q-q')+b(q,p)\delta(p-q')\delta(q-p')\,,$$

where the symmetry of K with respect to p and q implies that

$$a(p,q)=\pm b(p,q)\,.$$

Once again the corresponding equation in configuration space implies that $a(p,q)$ is actually constant and we obtain the desired result (A9).

APPENDIX B: SOLUTION OF THE ASSOCIATIVITY EQUATIONS

The numbers $B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=B(\mu,\nu)$ have unit modulus and satisfy the equations

$$B(\mu,\nu)=B(\nu,\mu)\,, \quad \text{(3.14)}$$

$$B(\lambda+\mu,\,\nu)B(\lambda,\mu)=B(\lambda,\,\mu+\nu)B(\mu,\nu)\,, \quad \text{(3.15)}$$

and

$$B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=b(\mathbf{m}',\mathbf{n}')b^*(\mathbf{m},\mathbf{n})\,. \quad \text{(3.17)}$$

The variables $\mathbf{m}$, $\mathbf{n}$, etc., run over the infinite, positive, unit lattice of all possible particle numbers. The pairs $\mu=(\mathbf{m}',\mathbf{m})$ lie on a lattice $\mathcal{L}$ defined by the condition that $\mathbf{m}$ and $\mathbf{m}'$ belong to the same superselection sector; this lattice is neither rectangular nor positively generated, but is closed under addition. We choose any basis g_1, g_2, $\cdots$ for the lattice $\mathcal{L}$ and first solve Eq.

(3.15) for the lattice points $\mu=ng_i$ (for n a positive integer). A special case of Eq. (3.15) is

$$B([m+1]g,\, ng)B(g,mg)=B(g,\,[m+n]g)B(mg,ng).$$

Iterating m times we find

$$B(mg,ng)=\frac{B(g,\,[m+n-1]g)B(g,\,[m+n-2]g)\cdots B(g,ng)}{B(g,\,[m-1]g)\cdots B(g,g)}=\frac{\Phi([m+n]g)}{\Phi(mg)\Phi(ng)},\quad \text{(B1)}$$

if we define the phase factors[27]

$$\Phi(mg)=\prod_{n=1}^{m-1}B(g,ng),\quad \Phi(g)=1,\quad \Phi(0)=1/B(0,g).$$

Equation (B1) is the general solution to Eq. (3.15) on the sublattices generated by a single generator g. We first extend this solution onto the lattice $\mathcal{P}$ generated by positive multiples of g_1, g_2, $\cdots$ and then onto the whole lattice $\mathcal{L}$. On the lattice $\mathcal{P}_{k-1}$ positively generated by $g_1, \cdots, g_{k-1}$ we assume that the solution of Eqs. (3.14) and (3.15) has the form

$$B(\mu,\nu)=\frac{\Phi(\mu+\nu)}{\Phi(\mu)\Phi(\nu)},\quad \mu,\,\nu\in\mathcal{P}_{k-1}.\qquad \text{(B2)}$$

Now for $\mu\in\mathcal{P}_{k-1}$ we define

$$\Phi(\mu+ng_k)=B(\mu,ng_k)\Phi(\mu)\Phi(ng_k);$$

then from the identity [which follows from Eq. (3.15)]

$$B(\lambda+\mu,\,\nu+\sigma)=\frac{B(\lambda+\nu,\,\mu+\sigma)B(\lambda,\nu)B(\mu,\sigma)}{B(\lambda,\mu)B(\nu,\sigma)},\quad \text{(B3)}$$

we immediately verify that for $\mu,\,\nu\in\mathcal{P}_{k-1}$

$$B(\mu+mg_k,\,\nu+ng_k)=\frac{\Phi(\mu+\nu+[m+n]g_k)}{\Phi(\mu+mg_k)\Phi(\nu+ng_k)};$$

i.e., the solution (B2) holds on $\mathcal{P}_k$. Hence, by induction, the solution (B2) holds on $\mathcal{P}$.

[27] This definition of $\Phi(0)$ is independent of the choice of g. In fact, as can be verified from (3.15), $B(0,\mu)$ is independent of μ.

If $\mu\notin\mathcal{P}$, we choose any $\nu\in\mathcal{P}$ such that $\mu+\nu\in\mathcal{P}$ and we define

$$\Phi(\mu)=\Phi(\mu+\nu)/\Phi(\nu)B(\mu,\nu).$$

It is easily checked using (3.15) and (B2) that the left-hand side is independent of the choice of ν and from the definition it is clear that $B(\mu,\nu)$ is given by (B2) for any μ and ν provided ν and $\mu+\nu$ are in $\mathcal{P}$. Finally for any μ, $\nu\in\mathcal{L}$ we can always choose ρ, $\sigma\in\mathcal{P}$ such that $\mu+\rho$, $\nu+\sigma\in\mathcal{P}$; expanding $B(\mu+\rho,\,\nu+\sigma)$ first by (B2) and second by using the identity (B3) we find that

$$B(\mu,\nu)=\frac{\Phi(\mu+\nu)}{\Phi(\mu)\Phi(\nu)},\quad \text{all}\quad \mu,\,\nu\in\mathcal{L}.\qquad \text{(B4)}$$

Equation (B4) is the most general solution to Eqs. (3.14) and (3.15). We can simplify the solution using (3.17), which implies that

$$B(\mathbf{m}'',\mathbf{m}',\mathbf{n}'',\mathbf{n}')B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=B(\mathbf{m}'',\mathbf{m},\mathbf{n}'',\mathbf{n}).\quad \text{(B5)}$$

Substitution of (B4) into (B5) gives

$$\frac{\Phi(\mathbf{m}''+\mathbf{n}'',\,\mathbf{m}'+\mathbf{n}')\Phi(\mathbf{m}'+\mathbf{n}',\,\mathbf{m}+\mathbf{n})}{\Phi(\mathbf{m}''+\mathbf{n}'',\,\mathbf{m}+\mathbf{n})}=\frac{\Phi(\mathbf{m}'',\mathbf{m}')\Phi(\mathbf{m}',\mathbf{m})}{\Phi(\mathbf{m}'',\mathbf{m})}\cdot\frac{\Phi(\mathbf{n}'',\mathbf{n}')\Phi(\mathbf{n}',\mathbf{n})}{\Phi(\mathbf{n}'',\mathbf{n})},$$

whence

$$\frac{\Phi(\mathbf{l},\mathbf{m})\Phi(\mathbf{m},\mathbf{n})}{\Phi(\mathbf{l},\mathbf{n})}=\exp i(\mathbf{a}\cdot\mathbf{l}+\mathbf{b}\cdot\mathbf{m}+\mathbf{c}\cdot\mathbf{n}),$$

for some real vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Setting $\mathbf{l}=\mathbf{m}$ and then $\mathbf{m}=\mathbf{n}$ we see that $\mathbf{a}=\mathbf{c}=0$. Thus

$$\Phi(\mathbf{l},\mathbf{m})=\Phi(\mathbf{l},\mathbf{n})\Phi^*(\mathbf{m},\mathbf{n})\exp(i\mathbf{b}\cdot\mathbf{m}).\qquad \text{(B6)}$$

If from each superselection sector we choose a representative lattice point $\mathbf{n}_0$ and define

$$\phi(\mathbf{m})=\Phi(\mathbf{m},\mathbf{n}_0)$$

then (B6) becomes, with $\mathbf{n}=\mathbf{n}_0$

$$\Phi(\mathbf{l},\mathbf{m})=\phi(\mathbf{l})\phi^*(\mathbf{m})\exp(i\mathbf{b}\cdot\mathbf{m}).$$

Substitution in (B4) gives

$$B(\mathbf{m}',\mathbf{m},\mathbf{n}',\mathbf{n})=\frac{\phi(\mathbf{m}'+\mathbf{n}')}{\phi(\mathbf{m}')\phi(\mathbf{n}')}\frac{\phi^*(\mathbf{m}+\mathbf{n})}{\phi^*(\mathbf{m})\phi^*(\mathbf{n})}$$

which is precisely the solution quoted in (3.16).