Roper used (including relativistic kinematical factors and using Roper's values for the parameters),[1] which we write symbolically as[18]

$$\mathrm{Im} f_{P33} = \Gamma^2/((W_u - m^2)^2 + \Gamma^2). \tag{16}$$

We can go to the narrow-width limit by rewriting the amplitude as

$$\mathrm{Im} f_{P33} = \gamma\Gamma^2/((W_u - m^2)^2 + \gamma^2\Gamma^2), \tag{17}$$

and letting $\gamma \to 0$. By letting γ take on successively smaller values between 0 and 1 we may determine the δ-function limit and thus compute the effect of making the narrow-width approximation. This limit can then be compared with the left-hand cut term due to the N_{33}^* exchange graph as computed by Ball and Wong. We find that the functional dependence on W remains practically the same for all values of $\gamma(0<\gamma<1)$ and the various Born terms (including Ball and Wong N_{33}^* exchange terms) are related by constant factors. The results are shown in Table VI. It is clear that the detailed shape of the resonance reduces its contribution to the unphysical cut by a factor of no smaller than ~ 0.75 as compared to the narrow-width approximation.[16]

It is clear from the tables that the S_{11}, S_{31}, and P_{11} u-channel continua have relatively little effect on the s-channel unphysical cuts, whereas the D_{13} contribution is large. The question now arises as to the effect of these terms on dynamical calculations of the πN system. Figure 1 shows the result of some N/D calculations[19] of the phase shift in the P_{33} partial wave (neglecting small inelastic effects in the D_{33} partial wave). If only the P_{11} continuum is considered, we get a slight improvement of the fit to the data. In fact we get about the same fit as one obtains by using the usual Born terms and including D_{33} inelastic effects. On the other hand, nothing is gained by adding the D_{13} continuum. The cutoff which must be used in the latter case is at a relatively low energy, which is perhaps an indication that the unphysical cut used here is a poorer approximation than for the cases where the D_{13} continuum is neglected. For the P_{11} and other partial waves we were unable to find any value of the cutoff which gave experimentally reasonable behavior when the D_{13} continuum is added.

We must conclude that this, presumably better, treatment of the πN u-channel forces does not give better agreement for the calculated πN phase shifts. The usual approximation in which only N and N_{33}^* exchange is considered in the u channel is much better as far as dynamical calculations are concerned.

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Regge Poles and Unequal-Mass Scattering Processes*

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It is not clear from the Regge representation that the asympotic form $s^{\alpha(u)}$ holds in the backward scattering of unequal-mass particles, because the cosine of the u-channel scattering angle remains small as s increases. In this paper we use a representation for the scattering amplitude first suggested by Khuri to show that the form $s^{\alpha(u)}$ is valid throughout the backward region. However, in order to ensure the analyticity of the amplitude defined by the Khuri representation at $u=0$, it is necessary that Regge trajectories occur in families whose zero-energy intercepts are spaced by integers. Denoting the leading or parent trajectory by $\alpha_0(u)$, we find that daughter trajectories $\alpha_k(u)$ must exist, of signature $(-1)^k$ relative to the parent, satisfying $\alpha_k(0) = \alpha_0(0) - k$. We then study Bethe-Salpeter models and find that this daughter-trajectory hypothesis is satisfied for any Bethe-Salpeter amplitude which Reggeizes in the first place. This fact follows elegantly from the four-dimensional symmetry of Bethe-Salpeter equations at zero total energy. Some phenomenological implications of the daughter-trajectory hypothesis are discussed. We have also characterized the behavior of partial-wave amplitudes in unequal-mass scattering at $u=0$ and find the hitherto unsuspected result $a(u,l) \sim u^{-\alpha(0)}$, where $\alpha(u)$ is the leading u-channel Regge trajectory.

I. INTRODUCTION

The characteristic features of the Regge pole description of high-energy scattering processes are the asymptotic forms $s^{\alpha(t)}$ or $s^{\alpha(u)}$. However, in the scattering of unequal-mass particles, the question of whether the Regge form $s^{\alpha(u)}$ holds in the backward region has never been settled because there is a cone about the backward direction in which $\cos\theta_u$ does not become large with increasing s. There has been general uneasiness[1,2] about applying the Regge asymptotic form in this region.

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[1] For example see S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962), Ref. 15.

[2] D. A. Atkinson and V. Barger, Nuovo Cimento **38**, 634 (1965).

Our investigations show that the simple Regge form holds throughout the backward region. This conclusion is obtained by establishing a representation for the scattering amplitude which explicitly exhibits the Regge behavior in the region in question. Further we suggest very strongly that as a general consequence of Lorentz invariance, Regge trajectories occur in families, the leading parent trajectory $\alpha_0(t)$ occurring with a set of daughter trajectories $\alpha_n(t)$ with zero-energy intercepts $\alpha_n(0)=\alpha_0(0)-n$. The daughter trajectories play a minor role in equal-mass situations, but for unequal-mass scattering their function is to cancel singularities in the asymptotic contribution of the parent trajectory.

As a by-product of this work, we have been able to show that the partial-wave amplitude $a^{\pm}(u,l)$ of an unequal-mass scattering process behaves like $a^{\pm}(u,l)\sim u^{-\alpha_L^{\pm}(0)}$ near $u=0$, where $\alpha_L^{\pm}(0)$ is the leading trajectory of the same signature in the u channel. This behavior is quite different from that usually assumed[3] in approximate dynamical calculations in S-matrix theory.

Usual discussions[1] of the asymptotic behavior in the backward region are based on the application of the Sommerfeld-Watson transformation to expansions of the scattering amplitude in partial waves in the u channel. The high-energy limit is introduced through the variable

$$z_u=\cos\theta_u=-\left[1+\frac{2(su-(m^2-\mu^2)^2)}{u^2-2u(m^2+\mu^2)+(m^2-\mu^2)^2}\right]. \quad (1)$$

This variable is bounded by unity for all s when u is in the backward cone defined by $0\leqq u\leqq u_B=(m^2-\mu^2)^2s^{-1}$, and, since z_u does not become large with increasing s, the conventional Regge representation (i.e., the Sommerfeld-Watson transformed partial-wave expansion) does not furnish an asymptotic limit in this region. Indeed, any representation $A(u,s)=f(u,z_u)$ is suspicious at $u=0$ because the transformation of variables is singular there.

Our discussion is based on work of Khuri[4] who shows that Sommerfeld–Watson transformations and Regge analysis can be applied to representations other than partial-wave expansions. Starting from power series in the Mandelstam variables t and s, we follow Khuri and establish a representation which explicitly exhibits Regge behavior throughout the backward region.

The reader should note that we do not attempt to prove l-plane meromorphy of partial-wave amplitudes in this paper but merely address ourselves to the problem of resolving the kinematic ambiguity in the Regge representation. The resolution of this ambiguity is definitely not trivial and it is not surprising that our investigations have revealed very distinctive features of the unequal-mass scattering problem.

In Sec. II we discuss the Khuri and Regge representations and their connection. We then show that daughter trajectories must exist if the Khuri representation is to define an amplitude with correct analyticity. In Sec. III we give an independent proof of the existence of daughter trajectories based on the four-dimensional symmetry of Bethe-Salpeter equations at $u=0$. In Sec. IV we discuss the phenomenological implications of daughter trajectories and also discuss the kinematics of inelastic two-body processes in which similar ambiguities of the Regge representation occur. In Sec. V we use the preceding results to characterize the behavior of partial wave amplitudes near $u=0$. In Appendix A, we prove that the reduced residue functions of Regge poles have at most isolated singularities at $u=0$, and in Appendix B, we establish a correspondence between the Regge and Khuri representations for $\mathrm{Re}\,l<-\frac{1}{2}$, a result which was thought unlikely in Khuri's original paper.[4]

Our notational convention is always to discuss the effect of Regge poles on the high-energy limit of an s-channel process, and we therefore consider partial-wave expansions in the u channel for the backward scattering problem and in the t channel for forward inelastic processes.

II. KHURI REPRESENTATION

We assume that the scattering amplitude $A(u,t)$ satisfies a fixed u dispersion relation

$$A(u,t)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{A_t(u,t')}{t'-t}+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{A_s(u,s')}{s'-s} \quad (2)$$

and further assume that the corresponding Froissart-Gribov partial-wave amplitudes

$$a^{\pm}(u,l)=\frac{1}{2q^2\pi}\int_{t_0}^{\infty}dt\,A_t(u,t)Q_l\left(1+\frac{t}{2q^2}\right)$$
$$\pm\frac{1}{2q^2\pi}\int_{s_0}^{\infty}ds\,A_s(u,s)Q_l\left(\frac{s-2m^2-2\mu^2+u}{2q^2}-1\right), \quad (3)$$

contain only moving poles in the l plane for $\mathrm{Re}\,l>-\frac{1}{2}$, and that they coincide with even and odd physical partial waves for all non-negative integral l. Hence the subtraction terms which are in general necessary in (2) need not be discussed.

[3] For example, see S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[4] N. N. Khuri, Phys. Rev. Letters **10**, 420 (1963); Phys. Rev. **132**, 914 (1963).

We write an ordinary Regge representation for the amplitude

$$A(u,t,s)=\frac{-1}{4i}\int_{-1/2-i\infty}^{-1/2+i\infty} dl\frac{(2l+1)}{\sin\pi l}\{a^+(u,l)[P_l(z)+P_l(-z)]-a^-(u,l)[P_l(z)-P_l(-z)]\}$$
$$-\pi\sum_{i=0}^{N^+}\beta_i^+(u)[2\alpha_i^+(u)+1]\frac{P_{\alpha_i^+(u)}(z)+P_{\alpha_i^+(u)}(-z)}{2\sin\pi\alpha_i^+(u)}+\pi\sum_{\alpha=0}^{N^-}\beta_j^-(u)[2\alpha_j^-(u)+1]\frac{P_{\alpha_j^-(u)}(z)-P_{\alpha_j^-(u)}(-z)}{2\sin\pi\alpha_j^-(u)}. \quad (4)$$

The case where the background integral can be shifted to the left of $-\frac{1}{2}$ is treated in Appendix B. The Regge pole terms have asymptotic forms $s^{\alpha(u)}$ only for $u\neq 0$, and have logarithmic singularities in u at $u=0$. For s large and positive, the background integral does not converge for complex u,[5] and the representation is not well defined at $u=0$.

Because of these defects of the Regge representation, we are led to consider a new representation based on a power series in the Mandelstam variables t and s

$$A(u,t,s)=\sum_{\nu=0}^{\infty} b(u,\nu)t^\nu+\sum_{\nu=0}^{\infty} c(u,\nu)s^\nu. \quad (5)$$

The common region of convergence of the two series is the domain $|t|<4\mu^2$, $|s|<m^2$, for πN kinematics. Continuation to other regions is made after the Sommerfeld–Watson transformation.

The power-series coefficients are given by

$$b(u,\nu)=\pi^{-1}\int_{t_0}^{\infty} dt\, A_t(u,t)t^{-\nu-1},\quad c(u,\nu)=\pi^{-1}\int_{s_0}^{\infty} ds\, A_s(u,s)s^{-\nu-1}. \quad (6)$$

Actually the integrals defining $b(u,\nu)$ and $c(u,\nu)$ converge only for $\mathrm{Re}\nu>M$ and $\mathrm{Re}\nu>N$, respectively, where M and N are the number of subtractions in the t and s channel contributions to the dipsersion relation (2), and must be defined by analytic continuation to the left of these lines. We note that $b(u,\nu)$ and $c(u,\nu)$ are analytic in a neighborhood of $u=0$ and are in this respect much simpler than partial wave amplitudes.

To investigate the continuation of $b(u,\nu)$ and $c(u,\nu)$ into the region where their defining integrals diverge, we use the Regge representation (4) to compute the absorptive parts

$$A_t(u,t)=D_t(u,t)+\tfrac{1}{2}\pi\sum_i \beta_i^+(u)[2\alpha_i^+(u)+1]P_{\alpha_i^+(u)}(z)+\tfrac{1}{2}\pi\sum_j \beta_j^-(u)[2\alpha_j^-(u)+1]P_{\alpha_i^-(u)}(z)\,, \quad (7)$$

$$A_s(u,s)=D_s(u,s)+\tfrac{1}{2}\pi\sum_i \beta_i^+(u)[2\alpha_i^+(u)+1]P_{\alpha_j^+(u)}(z)-\tfrac{1}{2}\pi\sum_j \beta_j^-(u)[2\alpha_j^-(u)+1]P_{\alpha_i^-(u)}(-z)\,. \quad (8)$$

Here $D_t(u,t)$ and $D_s(u,s)$ are the discontinuities of the Regge background integrals for positive and negative z, respectively, and we are to use for z the expressions

$$z=1+(t/2q^2) \quad (9)$$

in (7), and

$$z=1-[(s+u-2m^2-2\mu^2)/2q^2] \quad (10)$$

in (8). For real $u\neq 0$, $D_t(u,t)=O(t^{-1/2})$, and $D_s(u,s)=O(s^{-1/2})$ so that their contributions to $b(u,\nu)$ and $c(u,\nu)$ through (6) are analytic for $\mathrm{Re}\nu>-\frac{1}{2}$.

The contribution of the Regge-pole terms can be found from the integrals

$$\int_{t_0}^{\infty} dt\, P_{\alpha(u)}(1+t/2q^2)t^{-\nu-1} \quad (11)$$

and

$$\int_{s_0}^{\infty} ds\, P_{\alpha(u)}\left(\frac{s-2m^2-2\mu^2+u}{2q^2}-1\right)s^{-\nu-1}. \quad (12)$$

Khuri[4] has shown that (11) is regular for $\mathrm{Re}\nu>-\frac{1}{2}$ except for simple poles at $\nu=\alpha(u), \alpha(u)-1, \cdots, \alpha(u)-n$ where $\frac{1}{2}>\mathrm{Re}(\alpha(u)-n)>-\frac{1}{2}$. This result follows from the truncated asymptotic expansion

$$P_\alpha(x)=g_0(\alpha)x^\alpha+g_1(\alpha)x^{\alpha-2}+\cdots+g_u(Q)x^{\alpha-2m}+G_\alpha(x)\,. \quad (13)$$

[5] We are grateful to Professor M. Froissart for pointing this out to us.

The integer m is determined by the condition $\frac{3}{2} > \mathrm{Re}(\alpha-2m) > -\frac{1}{2}$, so that $G_\alpha(x) = O(x^{-1/2})$ and its contribution to the integral (11) is analytic in $\mathrm{Re}\nu > -\frac{1}{2}$. An identical technique works for (12) and we again find that to each Regge pole $\alpha(u)$, there correspond Khuri poles at $\nu=\alpha(u)$, $\alpha(u)-1$, $\cdots$, $\alpha(u)-n$, with $\frac{1}{2} > \mathrm{Re}\alpha(u)-n > -\frac{1}{2}$. It is useful to speak of the pole at $\nu=\alpha(u)$ as the principal Khuri pole, and the poles displaced to the left by integers as satellite poles. The reader should be careful to distinguish these satellite ν-plane poles from the daughter Regge poles which we discuss later.

The residues of the Khuri poles can easily be computed, and we obtain

$$b(u,\nu) = \frac{1}{\sqrt{\pi}} \sum_i \frac{\beta_i^+ \Gamma(\alpha_i^+ + \frac{3}{2})}{q^{2\alpha_i^+} \Gamma(\alpha_i^+ + 1)} \left[\frac{1}{\nu - \alpha_i^+} + \frac{2q^2 \alpha_i^+}{\nu - \alpha_i^+ + 1} + \cdots + \frac{\rho_{ni}^+}{\nu - \alpha_i^+ + n_i} \right]$$
$$+ \frac{1}{\sqrt{\pi}} \sum_j \frac{\beta_j^- \Gamma(\alpha_j^- + \frac{3}{2})}{q^{2\alpha_j^-} \Gamma(1 + \alpha_j^-)} \left[\frac{1}{\nu - \alpha_j^-} + \frac{2q^2 \alpha_j^-}{\nu - \alpha_j^- + 1} + \cdots + \frac{\rho_{ni}^-}{\nu - \alpha_j^- + n_j} \right] + \bar{b}(u,\nu); \quad (14)$$

and

$$c(u,\nu) = \frac{1}{\sqrt{\pi}} \sum_i \frac{\beta_i^+ \Gamma(\alpha_i^+ + \frac{3}{2})}{q^{2\alpha_i^+} \xi \Gamma(\alpha_i^+ + 1)} \left[\frac{1}{\nu - \alpha_i^+} - \frac{(2q^2 + 2m^2 + 2\mu^2 - u)\alpha_i^+}{\nu - \gamma_i^+ + 1} + \cdots + \frac{\sigma_{ni}^+}{\nu - \alpha_i^+ + n_i^+} \right]$$
$$- \frac{1}{\sqrt{\pi}} \sum_j \frac{\beta_j^- \Gamma(\alpha_j^- + \frac{3}{2})}{q^{2\alpha_j^-} \xi \Gamma(\alpha_j^- + 1)} \left[\frac{1}{\nu - \alpha_j^-} + \frac{2q^2 \alpha_j^-}{\nu - \alpha_j^- + 1} + \cdots + \frac{\sigma_{nj}^-}{\nu - \alpha_j^- + n_j^-} \right] + \bar{c}(u,\nu), \quad (15)$$

where $\bar{b}(u,\nu)$ and $\bar{c}(u,\nu)$ are regular in $\mathrm{Re}\nu > -\frac{1}{2}$. We have omitted the argument u of the residue and trajectory functions, and have written explicitly only the residues of the principal and first-satellite Khuri poles. Residues of the higher satellites are given in Appendix B. The significant property of these residues is that the residue of the jth satellite pole contains the term $(2q^2)^j$, and therefore has a pole of order j at $u=0$.

So far we have established that $b(u,\nu)$ and $c(u,\nu)$ are meromorphic functions of ν for $\mathrm{Re}\nu > -\frac{1}{2}$ and for u real, $u \neq 0$. It follows from the definition (6) that $b(u,\nu)$ and $c(u,\nu)$ are analytic in u in the whole cut u plane for $\mathrm{Re}\nu > M$ and $\mathrm{Re}\nu > N$, respectively. However, to the left of these lines, the analyticity (meromorphy, to be more exact) of $b(u,\nu)$ and $c(u,\nu)$ at $u=0$ (or for complex u), cannot be inferred rigorously from the definition (6) because the defining integrals diverge or from the Regge representation (4) since the latter fails to furnish the asymptotic behavior of $D_t(0,t)$ and $D_s(0,s)$. It seems impossible to avoid this difficulty, which we regard as a failure of the Regge representation rather than as any genuine defect of the Khuri amplitudes. Therefore we *assume* that the Khuri amplitudes $b(u,\nu)$ and $c(u,\nu)$ as defined by (6) can be continued to $u=0$ or into the complex u plane, and have no singularities for $\mathrm{Re}\nu > -\frac{1}{2}$ other than those given by the finite number of moving poles in (14) and (15). Hence $\bar{b}(u,\nu)$ and $\bar{c}(u,\nu)$ are analytic in the cut u plane and in $\mathrm{Re}\nu > -\frac{1}{2}$.

The next step is to make a Sommerfeld-Watson transformation of the power series (5) obtaining

$$A(u,t,s) = (-2i)^{-1} \int_{-1/2-i\infty}^{-1/2+i\infty} d\nu (\sin\pi\nu)^{-1} (b(u,\nu)(-t)^\nu + c(u,\nu)(-s)^\nu)$$
$$- \sqrt{\pi} \sum_i \frac{\beta_i^+ \Gamma(\alpha_i^+ + \frac{3}{2})}{q^{2\alpha_i^+} \Gamma(\alpha_i^+ + 1) \sin\pi\alpha_i^+} \left[(-t)^{\alpha_i^+} + (-s)^{\alpha_i^+} - 2q^2 \alpha_i^+ (-t)^{\alpha_i^+ - 1} \right.$$
$$\left. + (2q^2 + 2m^2 + 2\mu^2 - u)\alpha_i^+ (-s)^{\alpha_i^+ - 1} + \cdots + (-1)^{n_i} \rho_{n_i}^+ (-t)^{\alpha_i^+ - n_i} + \sigma_{n_i}^+ (-s)^{\alpha_i^+ - n_i} \right]$$
$$- \sqrt{\pi} \sum_j \frac{\beta_j^- \Gamma(\alpha_j^- + \frac{3}{2})}{q^{2\alpha_j^-} \Gamma(\alpha_j^- + 1) \sin\pi\alpha_j^-} \left[(-t)^{\alpha_j^-} - (-s)^{\alpha_j^-} - 2q^2 \alpha_j^- (-t)^{\alpha_j^-} \xi^1 \right.$$
$$\left. - (2q^2 + 2m^2 + 2\mu^2 - u)\alpha_j^- (-s)^{\alpha_j^- - 1} + \cdots + (-\tfrac{1}{2})^{n_j} \rho_{n_j}^- (-t)^{\alpha_j^- - n_j} + \sigma_{n_j}^+ (-s)^{\alpha_j^- - n_j} \right]. \quad (16)$$

The Khuri background integral converges and defines a function which falls off at least as fast as an inverse square root as s or t become large with u fixed. Each square bracket in (16) gives the contributions of the principal and satellite Khuri poles coming from one Regge pole of definite signature.

We now examine the pole terms in the limit appropriate to high-energy backward scattering in the s channel, by substituting $t = 2m^2 + 2\mu^2 - s - u$ in (16), expanding powers of this quantity in binomial series, and considering some large positive s. Each square bracket becomes

$$(1 \pm e^{i\pi\alpha^\pm}) [s^{\alpha^\pm} + \alpha^\pm (u - 2m^2 - 2\mu^2 - 2q^2) s^{\alpha^\pm - 1}$$
$$+ f_2(\alpha,u) s^{\alpha^\pm - 2} + \cdots + f_n(\alpha,u) s^{\alpha^\pm - n}] + \bar{f}_\alpha(u,s). \quad (17)$$

The functions appearing here are written explicitly in

Appendix B. The function $\bar{f}_\alpha(u,s)$ is of order $s^{-1/2}$ and comes from the convergent tails of the binomial series, while the first n terms correspond exactly to the first n terms of the expansion of

$$(2q^2)^{-\alpha}[P_\alpha(-z)\pm P_\alpha(z)], \tag{18}$$

which is what one would have obtained from the ordinary Regge representation.

To proceed further it is necessary to discuss the analytic properties of the Regge residue functions. In Appendix A it is shown that the reduced residue functions defined by $\bar{\beta}(u)=q^{-2\alpha(u)}\beta(u)$ have no cuts in the vicinity of $u=0$. However, the proof does allow finite-order poles or essential singularities at this point.

We consider the analyticity properties of (16) at $u=0$. The background integral is analytic there, and so is the full amplitude. The contribution of each principal Khuri pole has the same analyticity as the reduced residue of the Regge pole to which it corresponds, and the jth satellite contribution has an additional singular polynomial of order j in u^{-1}. The sum of all the Khuri-pole contributions must be analytic at $u=0$, and this can occur only if the singularities of the individual contributions cancel because of cooperation among the Regge trajectories.

Let $\alpha_0^+(u)$ be the leading Regge trajectory near $u=0$, assumed for definiteness to be of positive signature. Its reduced residue must be analytic at $u=0$, since a singularity there could not otherwise be cancelled. The first Khuri satellite contribution then has a pole at $u=0$ whose residue can be computed from (16) and (17). To cancel this pole there must be another Regge trajectory $\alpha_1^-(u)$, of opposite signature, satisfying $\alpha_1^-(0)=\alpha_0^+(0)-1$, which we call the first daughter trajectory.[6] Its reduced residue $\bar{\beta}_1^-(u)$ has a pole at $u=0$, fixed so that the singular part of its principal Khuri contribution exactly cancels that of the first Khuri satellite of the leading parent Regge pole.

In general there will be a series of daughter trajectories $\alpha_k(u)$ in the l plane, of alternating signature, satisfying

$$\alpha_k(0)=\alpha_0(0)-k\,,$$
$$k=1,\cdots,n\,,\quad \tfrac{1}{2}>\mathrm{Re}\alpha_0(0)-n>-\tfrac{1}{2}. \tag{19}$$

The corresponding reduced residues $\bar{\beta}_k(u)$ will have poles of order k at $u=0$, with everything arranged so that singularities of the individual Khuri-pole contributions cancel among themselves upon summation. The cancellation requirement imposes conditions on the first $k-1$ derivatives of the daughter trajectory functions and on derivatives of the reduced residue function as well. It should be noted that the daughter poles need satisfy (19) only at $u=0$ and will in general not be integrally spaced for $u\neq 0$.

There may, of course, be more than one Regge trajectory with reduced residue analytic at $u=0$. Each such parent trajectory will have a series of daughters with the properties discussed above.

In Appendix B we show that the whole discussion above can be generalized to include the case where the Regge background integral contour can be shifted to the line $\mathrm{Re}l=-L$ with $L>\frac{1}{2}$. In this case the Khuri amplitudes will be meromorphic for $\mathrm{Re}\nu>-L$, and there will be correspondingly more Regge daughters in each family of trajectories.

The following prescription for the high-energy contribution of a family of Regge trajectories sums up our work on the Khuri representation. The contribution of a parent Regge trajectory in the Regge representation is well defined for $u\neq 0$, and involves Legendre function of argument z. Obtain the asymptotic series in powers of s of the Legendre functions keeping only the finite number of terms which grow faster than background. Introduce one by one the daughter trajectories with reduced residues chosen to cancel the singularities which occur in the term by term continuation to $u=0$ of the contributions of the parent trajectory and the higher lying daughters. All this can be done in a finite number of steps and results in a finite set of powers whose sum is analytic at $u=0$. To take explicit account of the cancellation of singularities, a Taylor expansion about $u=0$ should probably be used in phenomenological data analyses. There is no *a priori* reason why the regular parts of the daughter contributions should not be as important as the parent trajectory contributions in any given order and parameters should be introduced to describe these regular parts.

Although the mechanism of cancellation of singularities by daughter Regge trajectories may seem rather miraculous, it is a rigorous consequence of the assumption that the Khuri amplitudes $b(u,\nu)$ and $c(u,\nu)$ are analytic at $u=0$ except for singularities due to the moving poles in ν. Although not proven, such analytic behavior is suggested by the maximal analyticity concept. Since it does not appear possible to avoid an assumption of this kind, we have sought and obtained additional support for the daughter trajectory hypothesis. This is discussed in the next section.

III. DAUGHTER TRAJECTORIES AND BETHE-SALPETER EQUATIONS

In field theory the scattering amplitude satisfies a Bethe-Salpeter equation which can be written in momentum space as

$$T(p,p';K)=I(p,p';K)+\frac{1}{2\pi^2 i}\int\frac{d^4p''I(p,p,'';K)T(p'',p';K)}{[(\frac{1}{2}K+p'')^2+m^2][(\frac{1}{2}K-p)^2+\mu^2]}. \tag{20}$$

The center-of-mass motion has been separated out, and K is the total energy-momentum four-vector, while p and p' are the relative energy-momentum vectors of the

[6] This possibility was first suggested to us by Professor S. Mandelstam.

particles in the initial and final states. The interaction kernel $I(p,p';K)$ is defined in formal field theory as the sum of *all* graphs which are two-particle irreducible, although it is usual in practice to approximate the kernel by a small number of irreducible graphs. Equation (20) defines an off-mass-shell extension of the T matrix, and the physical scattering amplitude is obtained by evaluating at

$$|\mathbf{p}^2|=|\mathbf{p}'^2|=[u^2-2(m^2+\mu^2)u+(m^2-\mu^2)^2](4u)^{-1}=q^2$$
$$p_0=p_0'=(m^2-\mu^2)(2\sqrt{u})^{-1}. \tag{21}$$

Our notation is to use italic letters p to denote four-vectors, and bold letters $\mathbf{p}$ to denote spatial three-vectors, while $|\mathbf{p}|=(\mathbf{p}\cdot\mathbf{p})^{1/2}$, and $P=(p\cdot p)^{1/2}=(|\mathbf{p}|^2-p_0^2)^{1/2}$.

Because of Lorentz invariance, the kernel $I(p,p';K)$ depends at most on the six independent invariants which can be formed from its three four-vector arguments. It is convenient to discuss the properties of the equation in the center of mass frame in which $K=(\sqrt{u},\mathbf{0})$. For $u\neq 0$ the equation is invariant under the group $O(3)$ of three-dimensional spatial rotations, and this invariance permits a separation of the equation via the ordinary partial-wave expansion. For $u=0$ the kernel depends only on the three Lorentz invariants formed from p and p', and the invariance group of the equation is isomorphic to the Lorentz group itself. This extra degree of invariance at $u=0$ ensures the existence of daughter trajectories with exactly the properties described in Sec. II. This phenomenon is analogous to that in Yukawa potential scattering in which the Coulomb degeneracy of bound states is obtained as the potential range becomes infinite because of the extra degree of invariance present.

It is very convenient to make the Wick rotation[7] in which the integration contour of the relative energy variable is moved to the imaginary axis. External relative energies are also continued to imaginary values, and new variables defined by $p_0=ip_4$, $p_0'=ip_4'$, $p_0''=ip_4''$. The resulting integral equation is

$$T(p,p';K)=I(p,p';K)+\frac{1}{2\pi^2}\int\frac{d^4p''I(p,p'';K)T(p'',p';K)}{[(\frac{1}{2}i\sqrt{u}-p_4'')^2+|p''|^2+m^2][(\frac{1}{2}i\sqrt{u}+p_4'')^2+|p''|^2+\mu^2]}, \tag{22}$$

in which the integration space and scalar products are Euclidean. For $K^2<(m+\mu)^2$, the Wick rotation is justified if the Bethe-Salpeter kernel is not too singular on the light cone. Sufficient conditions for its validity are discussed below. We note that at $u=0$, the invariance group of Eq. (22) is 0(4) and it will be very useful to expand the scattering amplitudes using representation functions of this group.

Several authors[8,9] have used the four-dimensional symmetry of the Bethe-Salpeter equation to discuss the high-energy limits of field theory. The only authors who recognize the implications of such a symmetry for Regge trajectories are Domokos and Suranyi.[9] Our discussion resembles theirs in spirit, although the momentum-space approach we use does not appear to have previously been given.

We consider Bethe-Salpeter kernels which possess spectral representations of the form

$$I(p,p';K)=\frac{1}{\pi}\int_{\tau_1}^{\infty}\frac{d\tau\,\sigma(\tau,u)}{\tau+(p-p')^2}+\frac{1}{\pi}\int_{\tau_1}^{\infty}\frac{d\tau\,\rho(\tau,u)}{\tau+(p-p')^2}, \tag{23}$$

where the spectral densities may contain delta functions in τ, but are required not to have two-particle cuts in u. Such kernels are not the most general permitted by Lorentz invariance, but if $\sigma(\tau,u)$ and $\rho(\tau,u)$ vanish as $\tau\to\infty$, they are essentially the only kernels for which the Wick rotation can be justified and l-plane meromorphy proved.[10] We will prove that the pattern of daughter trajectories discussed in the previous section must exist in any Bethe-Salpeter amplitude with l-plane meromorphy. However the symmetry which is responsible for this pattern of daughters is far more general and we briefly discuss its effect in the case of kernels for which meromorphy cannot be proved.

The ordinary partial-wave Bethe-Salpeter equation is obtained by expanding in partial waves

$$T(p,p';K)=\sum_{l=0}^{\infty}(2l+1)T_l(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)P_l(z),$$

$$I(p,p';K)=\sum_{l=0}^{\infty}(2l+1)I_l(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)P_l(z), \tag{23'}$$

where $z=\mathbf{p}\cdot\mathbf{p}'/|\mathbf{p}||\mathbf{p}'|$, and

$$I_l(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=I_l^{(1)}(p_4|\mathbf{p}|;p_4'|\mathbf{p}'|;u)+(-1)^lI_l^{(2)}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u), \tag{24}$$

[7] G. C. Wick, Phys. Rev. **96**, 1124 (1954).
[8] For example, see J. D. Bjorken, J. Math. Phys. **5**, 192 (1964); M. Baker and I. J. Muzinich, Phys. Rev. **132**, 2291 (1963).
[9] G. Domokos and P. Suranyi, Nucl. Phys. **54**, 529 (1964).
[10] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

with

$$I_l^{(1)}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=\frac{1}{2|\mathbf{p}|\mathbf{p}|'|\pi}\int_{\tau_1}^{\infty}d\tau\,\sigma(\tau,u)Q_l\left(\frac{(p_4-p_4')^2+|\mathbf{p}|^2+|\mathbf{p}'|^2+\tau}{2|\mathbf{p}||\mathbf{p}'|}\right),$$

$$I_l^{(2)}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=\frac{1}{2|\mathbf{p}||\mathbf{p}'|\pi}\int_{\tau_2}^{\infty}d\tau\,\rho(\tau,u)Q_l\left(\frac{(p_4+p_4')^2+|\mathbf{p}|^2+|\mathbf{p}'|^2+\tau}{2|\mathbf{p}||\mathbf{p}'|}\right). \quad (25)$$

Separation of the Bethe-Salpeter equation is thus achieved by using the addition theorem for Legendre functions and orthogonality of the spherical harmonics. One obtains

$$T_l^{\pm}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=I_l^{\pm}(p_4|\mathbf{p}|;p_4'|\mathbf{p}'|;u)$$
$$+\frac{1}{2\pi^2}\int\frac{dp_4''|\mathbf{p}''|^2dp''I_l^{\pm}(p_4,|\mathbf{p}|;p_4''|\mathbf{p}''|;u)T_l^{\pm}(p_4'',|\mathbf{p}''|;p_4',|\mathbf{p}'|;u)}{[(\frac{1}{2}i\sqrt{u}-p_4'')^2+|\mathbf{p}''|^2+m^2][(\frac{1}{2}i\sqrt{u}+p_4'')^2+|\mathbf{p}''|^2+\mu^2]}, \quad (26)$$

where to obtain suitable Regge continuations, we have defined $I_l^{\pm}=I_l^{(1)}\pm I_l^{(2)}$ which coincide with physical l amplitudes for even and odd integers respectively.

If the particles are identical then $m=\mu$ and $\sigma(\tau,u)\equiv\rho(\tau,u)$. It is interesting that in this case the odd-signature amplitude vanishes identically on the mass shell, but does not vanish off the mass shell.

The interaction kernels (23) are $O(4)$ invariant for all u, and it is conveneint to express them in terms of four-dimensional spherical harmonics as defined, for example, in the paper of Schwartz.[11] It follows from the generating equation of the Gegenbauer polynomials[12] that

$$\frac{1}{\tau+(p\pm p')^2}=\frac{1}{2PP'}\sum_{n=0}^{\infty}f_n\left(\frac{P^2+P'^2+\tau}{2PP'}\right)C_n^1(\bar{w})(\pm 1)^n, \quad (27)$$

where $\bar{w}=p\cdot p'/PP'$, and

$$f_n(x)=2[x-(x^2-1)^{1/2}]^{n+1}. \quad (28)$$

The functions $f_n(x)$ in the four-space treatment play a role analogous to the Q_l functions in the conventional treatment.

To proceed it is useful to define functions

$$D_\mu^{l+1}(x)=2^l\Gamma(l+1)\left[\frac{2\Gamma(\mu+1)(l+\mu+1)(1-x^2)^l}{\pi\Gamma(\mu+2l+2)}\right]^{1/2}C_\mu^{l+1}(x), \quad (29)$$

which are orthonormal with respect to the lower index on the interval $(-1,\ 1)$ with integration measure $(1-x^2)^{1/2}dx$. In terms of these functions the addition theorem for Gegenbauer polynomials reads[12]

$$C_n^1(\bar{w})=\frac{\pi}{2(n+1)}\sum_{l=0}^{n}D_{n-l}^{l+1}(w)D_{n-l}^{l+1}(w')(2l+1)P_l(z), \quad (30)$$

[11] C. Schwartz, Phys. Rev. **137**, 717 (1965).

[12] *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I. Sec. 3.15.1. This volume will hereafter be referred to as "H.T.F.," and the accompanying volumes, will be referred to as "T.I.T."

where

$$\bar{w}=ww'+(1-w^2)^{1/2}(1-w'^2)^{1/2}z. \quad (31)$$

The variable w is the cosine of the angle between p and the fourth axis, and w' is similarly defined.

One can then expand, after exchanging orders of summation,

$$I(p,p';K)=\pi\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}I_n(P,P';u)[2(n+1)]^{-1}\times D_{n-l}^{l+1}(w)D_{n-1}^{l+1}(w')(2l+1)P_l(z), \quad (32)$$

where

$$I_n(P,P';u)=I_n^{(1)}(P,P';u)+(-1)^nI_n^{(2)}(P,P';u) \quad (33)$$

and

$$I_n^{(1)}(P,P';u)=\frac{1}{2\pi PP'}\int d\tau\,\sigma(\tau,u)f_n\left(\frac{P^2+P'^2+\tau}{2PP'}\right);$$

$$I_n^{(2)}(P,P';u)=\frac{1}{2\pi PP'}\int d\tau\,\rho(\tau,u)f_n\left(\frac{P^2+P'^2+\tau}{2PP'}\right). \quad (34)$$

Comparing (23) and (32) we find

$$I_l^{\pm}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=\pi\sum_{\lambda=0}^{\infty}I_{l,\lambda}^{\pm}(P,P';u)\times[2(l+\lambda+1)]^{-1}D_\lambda^{l+1}(w)D_\lambda^{l+1}(w'), \quad (35)$$

with

$$I_{l,\lambda}^{\pm}(P,P';u)=I_{l+\lambda}^{(1)}(P,P';u)\pm(-1)^\lambda I_{l+\lambda}^{(2)}(P,P';u). \quad (36)$$

As a consequence or $O(4)$ invariance, these kernels depend essentially only on the sum $l+\lambda$, the only separate λ dependence is a simple sign alternation, so that there are two independent kernels for each value of the sum $l+\lambda$.

Completeness of the Gegenbauer polynomials permits us to write a similar expansion for $T_l^{\pm}$

$$T_l^{\pm}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;u)=\pi\sum_{\lambda,\mu=0}^{\infty}T_{l;\lambda\mu}^{\pm}(P,P';u)\times[2(l+\lambda+1)]^{-1}D_\lambda^{l+1}(w)D_\mu^{l+1}(w'). \quad (37)$$

The matrix $T_{l;\,\lambda\mu}^{\pm}$ will not in general be diagonal because the $O(4)$ symmetry of the Bethe-Salpeter equation is broken for $u\neq 0$.

We can use (35) and (37) to rewrite Eq. (26) as

$$T^{\pm}_{l;\,\lambda\mu}(P,P';u)=I^{\pm}_{l,\lambda}(P,P';u)\delta_{\lambda\mu}+(4\pi)^{-1}\sum_{\nu=0}^{\infty}(l+\nu+1)^{-1}\int dP''P''^{3}I^{\pm}_{l,\lambda}(P,P'';u)G_{\lambda\nu}{}^{l}(P'',u)T^{\pm}_{l;\,\nu\mu}(P'',P';u)\,, \tag{38}$$

where

$$G_{\lambda\nu}{}^{l}(P,u)=\int_{-1}^{+1}\frac{dw(1-w^{2})^{1/2}D_{\lambda}{}^{l+1}(w)D_{\nu}{}^{l+1}(w)}{[P^{2}+m^{2}-\frac{1}{4}u-i\sqrt{u}Pw][P^{2}+\mu^{2}-\frac{1}{4}u+i\sqrt{u}Pw]}\,. \tag{39}$$

Equations (26) and (38) are mathematically equivalent. We have merely used orthogonal polynomials to change one of the integrations in (26) into a discrete summation.

Equation (38) is coupled in the discrete indices because the $O(4)$ symmetry is broken for $u\neq 0$ .Indeed

$$G_{\lambda\nu}{}^{l}(P,0)=\delta_{\lambda\nu}[P^{2}+m^{2}]^{-1}[P^{2}+\mu^{2}]^{-1}\,, \tag{40}$$

so that Eq. (38) decouples at $u=0$, and its solution there is diagonal in the indices λ and μ. Substituting $T_{l;\,\lambda\mu}^{\pm}(P,P';0)=\delta_{\lambda\mu}T_{l,\lambda}^{\pm}(P',P)$ in (38) we can write the uncoupled equation

$$T_{l,\lambda}^{\pm}(P,P')=I_{l,\lambda}^{\pm}(P,P';0)+\frac{1}{4\pi(l+\lambda+1)}\times\int\frac{dP''P''^{3}I^{\pm}_{l,\lambda}(P,P'';0)T_{l,\lambda}^{\pm}(P'',P')}{[P''^{2}+m^{2}][P''^{2}+\mu^{2}]}\,. \tag{41}$$

Furthermore, it follows from the remarks after Eq. (36) that there are really only two independent amplitudes $A_{l+\lambda}{}^{(-1)^{\lambda}}(P,P')$ for each value of $l+\lambda$, such that $T_{l,\lambda}{}^{\pm}=A^{\pm}{}_{l+\lambda}$ if λ is even, and $T^{\pm}{}_{l,\lambda}=A^{\mp}{}_{l+\lambda}$ if λ is odd.

Now it has been shown using Fredholm techniques[10] that the solution of Eq. (26) or (38) contains only Regge poles. These appear as zeros of the Fredholm determinant. This determinant factors into an infinite product at $u=0$ corresponding to the decoupling of Eq. (38), with two independent factors for each value of $l+\lambda$, $\lambda=0, 1, 2, \cdots$. The zeros of the factors give the location of poles of $A^{\pm}{}_{l+\lambda}$. Suppose $A^{+}{}_{l+\lambda}$ has a pole at $\alpha^{+}(0)$, so that in its vicinity

$$A_{l+\lambda}{}^{+}(P,P')\approx b(P,P')/[l+\lambda-\alpha^{+}(0)]\,. \tag{42}$$

Then by (37) we find that $T_{l}{}^{+}(p_4,|\mathbf{p}|;p_4',|\mathbf{p}'|;0)$ has Regge poles at $l=\alpha^{+}(0)$, $\alpha^{+}(0)-2$, $\alpha^{+}(0)-4$, etc., $\cdots$, while $T_{l}{}^{-}$ has poles at $l=\alpha^{+}(0)-1$, $\alpha^{+}(0)-3$, etc. Similarly if $A^{-}{}_{l+\lambda}$ has a pole at $\alpha^{-}(0)$, then $T_{l}{}^{-}$ has zero-energy Regge poles at $l=\alpha^{-}(0)$, $\alpha^{-}(0)-2$, etc., and $T_{l}{}^{+}$ has poles at $l=\alpha^{-}(0)-1$, $\alpha^{-}(0)-3$, etc. Since trajectory functions are analytic near $u=0$, a pole in the l plane at $u=0$ must lie on a genuine trajectory. Hence we have proven the existence of daughter trajectories of alternating signature spaced by integers at $u=0$.

The next task is to calculate the residues of the daughter poles using (37) and (42). The calculation is straightforward but messy, and we will merely outline the steps and then give the results. First we define a reduced residue function by

$$b(P,P')=P^{\alpha^{+}(0)}P'^{\alpha^{+}}\bar{b}^{(0)+}(P,P')\,. \tag{43}$$

The factor extracted is analogous to the centrifugal barrier in the conventional treatment, and it is easy to see using the properties of $f_n(x)$ in Eq. (28) that $\bar{b}(P,P')$ has no singularities as its arguments approach zero. The second step is to calculate the $D_{\mu}{}^{l+1}(w)$ explicitly, and then use the relations

$$|\mathbf{p}|=P(1-w^{2})^{1/2}\,,\qquad p_4=Pw\,, \tag{44}$$

to eliminate w and w' from the result. Finally we extrapolate to the mass shell using (21) and remembering that $p_4=-ip_0$ because of the Wick rotation. Denoting the on-shell limit of $\bar{b}(P,P')$ simply by $\bar{b}$, we find that the parent Regge pole and its first- and second-daughter pole terms at $u=0$ are

$$\begin{aligned}T_{l}{}^{+}(0)&\approx\frac{\bar{b}}{l-\alpha^{+}(0)}\,\frac{4^{\alpha^{+}(0)}\Gamma(\alpha^{+}(0)+1)^{2}}{\Gamma(2\alpha^{+}(0)+2)}q^{2\alpha^{+}(0)}+\frac{\bar{b}}{l+2-\alpha^{+}(0)}\,\frac{4^{\alpha^{+}(0)}\Gamma(\alpha^{+}(0))^{2}}{\Gamma(2\alpha^{+}(0))}\left[\alpha^{+}(0)\frac{(m^{2}-\mu^{2})^{2}}{2u}-\frac{m^{2}+\mu^{2}}{2}\right]^{2}q^{2\alpha^{+}(0)-4}\,,\\ T_{l}{}^{-}(0)&\approx\frac{\bar{b}}{l+1-\alpha^{+}(0)}\,\frac{4^{\alpha^{+}(0)}\Gamma(\alpha^{+}(0)+1)^{2}}{\Gamma(2\alpha^{+}(0)+1)}\,\frac{-(m^{2}-\mu^{2})^{2}}{4u}q^{2\alpha^{+}(0)-2}\,.\end{aligned} \tag{45}$$

Corresponding formulas hold if the principal Regge ploe is of negative signature.

We infer from (45) the behavior of the on-shell Regge residue functions near $u=0$. The parent trajectory $\alpha^{+}(u)$ has a reduced residue $\bar{\beta}^{+}(u)$ which is regular near $u=0$. The residue $\bar{\beta}_1{}^{-}(u)$ of the first daughter trajectory can be written as

$$\bar{\beta}_1{}^{-}(u)=\frac{-[2\alpha^{+}(0)+1]\bar{\beta}^{+}(0)(m^{2}-\mu^{2})^{2}}{4u}+h_1(u)\,, \tag{46}$$

where $h_1(u)$ is regular at $u=0$. The pole at $u=0$, discussed in Sec. II, is evident in (46).

Some properties of the residue of the second-daughter trajectory can also be obtained from (45), and there is obviously a double pole at $u=0$. There is also a single pole whose residue cannot be completely calculated from (45) since it involves the solution of the Bethe-Salpeter equation to first order in the symmetry-breaking parameter u. A complete verification of the pole cancellation mechanism for the second and lower lying daughter trajectories requires a perturbation solution of the Bethe-Salpeter equation in the energy u of the type outlined by Domokos and Suranyi.[9] We have not carried out such a calculation because these trajectories lies quite far to the left in the l plane and are unlikely to be of interest in the near future.

We note that in the equal-mass case, the residues on the mass shell of the first and all odd-number daughter trajectories would vanish for all u even in the physically artificial case of nonidentical particles. Statistics, of course, guarantees that the whole on-shell amplitude $T_l^-(u)$ vanishes identically. The second and all other even daughters do contribute at $u=0$ in the equal mass case.

It is easy to use (46) and (16) to compute the contribution of the first daughter to the large-s limit of the amplitude. We write here the resulting large-s contributions of the principal and first satellite Khuri poles of the parent Regge trajectory $\alpha_0^+(u)$ and the principal Khuri contribution of its first daughter $\alpha_1^-(u)$:

$$A(u,s)\approx\frac{-2(\sqrt{\pi})\Gamma(\alpha_0^+(u)+\frac{3}{2})}{\Gamma(\alpha_0^+(u)+1)\sin\pi\alpha_0^+(u)}\bar{\beta}_0^+(u)[1+e^{i\pi\alpha^+{}_0(u)}]\left[s^{\alpha^+{}_0(u)}-\left(m^2+\mu^2-\tfrac{1}{2}u-\frac{(m^2-\mu^2)^2}{2u}\right)\alpha_0^+(u)s^{\alpha^+{}_0(u)-1}\right]$$
$$-\frac{2(\sqrt{\pi})\Gamma(\alpha_1^-(u)+\frac{3}{2})}{\Gamma(\alpha_1^-(u)+1)\sin\pi\alpha_1^-(u)}\left[\frac{-(2\alpha_0^+(u)+1)\bar{\beta}_0^+(u)(m^2-\mu^2)^2}{4u}+h_1(u)\right][1-e^{i\pi\alpha_1^-(u)}]s^{\alpha_1^-(u)}. \quad (47)$$

The condition $\alpha_1^-(0)=\alpha_0^+(0)-1$ ensures that the singular terms in (47) cancel exactly, which is just what was required in Sec. II. There we found that such a cancellation was required in order that the Khuri representation define an amplitude analytic at $u=0$. Here we have shown that the requirement is fulfilled because of the $O(4)$ symmetry of the Bethe-Salpeter equation.

Our proof of the existence and properties of daughter trajectories applies to Bethe-Salpeter kernels of the form (23). This class of kernels can be enlarged by allowing the spectral functions in (23) to depend on the invariants p^2, p'^2, $K\cdot p$, and $K\cdot p'$ with smoothness and asymptotic conditions which ensure that the Wick rotation can be performed and that the resulting integral equation is of Fredholm type. Such kernels also Reggeize, and the existence of daughter trajectories can be proved.

Domokos and Suranyi[9] and other authors have investigated kernels whose spectral functions $\sigma(\mu^2)$ and $\rho(\mu^2)$ are asymptotically constant. A subtraction is then necessary in (23). The Wick rotation is valid in this case, and the resulting marginally singular integral equation yields u-independent square-root branch points in the l plane. These branch points are spaced by integers because of the $O(4)$ symmetry of the equation.

Kernels for which $\sigma(\mu^2)$ and $\rho(\mu^2)$ grow at infinity are highly singular on the light cone in configuration space. Halpern[13] has recently shown that the Wick rotation fails for such kernels, so that previous results in these should be discredited. The Bethe-Salpeter equation in the original Lorentz metric has so far not proved tractable, and it is difficult to give mathematical meaning to the scattering amplitude for such kernels. Since the four-dimensional symmetry of the kernel at $u=0$ follows from Lorentz invariance alone, one might suspect that if the scattering amplitude can be suitably defined, its l-plane singularities would have integer spacing at $u=0$.

It is curious that such an obvious consequence of Lorentz invariance in local field theory as the four-dimensional symmetry has no obivous analog in S-matrix theory. It would be very interesting to formulate and study such a property in the language of analyticity and unitarity. Of course the argument of Sec. II can be viewed as a proof of the daughter trajectory hypothesis in the langauge of analyticity, but the argument there depends on the unequal mass kinematics. In the equal mass case the Khuri representation is manifestly analytic and one would not suspect that daughter trajectories exist.

IV. PHENOMENOLOGY

Our work suggests that Regge trajectories always occur in families whose zero-energy intercepts are integrally spaced in the l plane. Such a picture has very interesting implications for phenomenology. In particular we suggest that each of the presently known particle trajectories is the parent trajectory to a family of daughters of the same baryon number, isospin, hypercharge, and charge conjugation (the latter for $B=0$, $Y=0$, systems). We concentrate on the first-daughter trajectory in this section. This trajectory has opposite signature to the parent, so that if $J^\pm$ is a physically realizable state of angular momentum and parity of the parent trajectory, then $(J-1)^\mp$ is a physically realizable state of the daughter. (Hence, the Pomeranchuk daughter does not give a physical 0^+ meson of zero mass.)

We first discuss briefly backward πN scattering, the experimental configuration which originally motivated our work. The spin kinematics complicates the asymp-

[13] M. Halpern, Ann. Phys. (to be published).

totic formulas in this case, but first daughters of the N and $N_{3/2}^*$ trajectories will be required to cancel the singularities of their parents and a sufficiently accurate analysis of the data should determine the parameters of these trajectories. Of course we must also consider the possibility that the daughter trajectories also rise high enough to make physical particles, and there are possible candidates for such states among the plethora of resonances in this system.

In two-body inelastic processes there are kinematic difficulties similar to those in unequal mass scattering. We consider the s-channel reation

$$1+2\to 3+4\,, \tag{48}$$

in which we allow the four masses m_i^2 to assume arbitrary positive values. The corresponding t-channel process is

$$1+\bar{3}\to \bar{2}+4\,, \tag{49}$$

and the contribution of a t-channel Regge pole at large s is found through the Regge pole term $\beta(t)P_{\alpha(t)}(-z_t)$.

The relevant kinematic formulas are

$$2p_{13}p_{24}z_t=\frac{s-u}{2}+\frac{(m_1^2-m_3^2)(m_2^2-m_4^2)}{4t}\,,$$

$$p_{13}^2=\frac{t^2-2(m_1^2+m_3^2)t+(m_1^2-m_3^2)^2}{4t}\,,$$

$$p_{24}^2=\frac{t^2-2(m_2^2+m_4^2)t+(m_2^2-m_4^2)^2}{4t}\,. \tag{50}$$

As long as one of the mass differences $m_1^2-m_3^2$, $m_2^2-m_4^2$ is nonzero, the value of z_t will not increase with s at $t=0$ and there is an ambiguity in the Regge representation which can be resolved by using a Khuri representation.

The general definition of the reduced Regge residue function is $\beta(t)=(p_{13}p_{24})^{\alpha(t)}\bar{\beta}(t)$, and the function $\bar{\beta}(t)$ will have no cut near $t=0$.

If only one of the mass differences above is nonzero, then the first-Khuri satellite contribution will be regular at $t=0$ and there will be no need for odd-order daughter trajectories. Such trajectories can contribute to the process if their quantum numbers allow them to couple to the external particles involved, and we will discuss this possibility below. The second-Khuri satellite contribution has a single pole at $t=0$ in this mass configuration and even-order daughter Regge trajectories are needed to cancel the ensuing set of singularities. If both mass differences are nonzero all the Khuri satellite contributions are singular, and daughter trajectories of even- and odd-order are required to cancel the singularities.

Consider now the first-daughter trajectory of the Pomeranchuk, called $\alpha_{P1}(t)$. It has $B=0$, $Y=0$, $T=0$, $G=+1$ and odd signature, and its possible couplings can be deduced by studying the possible couplings of a $T=0$, $G=+1$, $J^P=1^-$ meson which could be a physical state on this trajectory. Bose statistics prohibits a coupling to $\pi\pi$, and G-parity conservation rules out $K\bar{K}$. It is easy to see from the formula $G=(-1)^{S+L+T}$ that such a meson could not couple to $N\bar{N}$. Hence $\alpha_{P1}(t)$ does not couple to any of the common two-body equal mass channels and would not be observed in any of the common scattering or reaction processes. It does couple to unequal mass channels and could in principle be observed in double diffractive production processes such as $N+N\to N_{1/2}^*+N_{1/2}^*$ in which two $T=\frac{1}{2}$ nucleon isobars are produced.

We have not studied the behavior of the daughter trajectories away from zero energy, but it is tempting to consider the possibility that they are roughly parallel with the parent trajectories. If so there would be a physical vector meson of mass between 1.0 and 1.5 BeV on the $P1$ trajectory. Such a meson could not decay to two pseudoscalar mesons. It could decay to $K\bar{K}\pi$, with p-wave barriers in the configuration $(K^*\bar{K})$ or in the $(K\bar{K})\pi$ configuration with d-wave angular-momentum barriers in both the $K\bar{K}$ subsystem and in the orbital coupling of $K\bar{K}$ with π. It could decay to four pions in the configuration $\rho\rho$. It is possible that the partial widths of these strong decay modes would be so small that the particle would be identified primarily by its electromagnetic decay modes $\pi^+\pi^-\gamma$ or $\rho^0\gamma$. This argument would also apply to the P' or to any trajectory on which a 2^+ meson lies.

The first daughter of the ρ, $\alpha_{\rho 1}(t)$ has $B=0$, $Y=0$, $T=1$, $G=+1$ and even signature and would create a physical 0^+ meson. This trajectory also cannot couple to $\pi\pi$, $K\bar{K}$ or $N\bar{N}$, and could only be detected at high energies in double production processes. If $\alpha_{\rho 1}(t)$ were roughly parallel with $\alpha_\rho(t)$ then the 0^+ meson it produced would have mass between 700 and 1100 MeV. This time decay into two pseudoscalars and into $K\bar{K}\pi$ would be forbidden by strong interaction conservation laws, the $K\bar{K}\pi$ decay being forbidden even if the meson were sufficiently massive. The lowest allowed strong decay mode would be four pions in the $\sigma\rho$ configuration, so that we might again expect the electromagnetic decay $\pi^+\pi^-\gamma$ to be dominant although one should be more cautious here because a d wave is required in the $\pi^+\pi^-$ system.

It would be interesting to make a more detailed study of the meson daughter trajectories, including estimates of the decay widths of the particles on them, and a comparison with experiment. The present experimental situation, although not conclusive, is not favorable to the existence of such particles, and this would indicate that daughter trajectories have slopes less steep than the parents.

If there really are particles on the daughter trajectories then our notions about the relative importance of the various channels in dynamical calculations need revision because none of the low-mass two-body channels which are thought to be important for the dynamics of the parent particles would be able to communicate with the daughters, and high-mass channels with spin would have to be included. For example in any common dynamical mechanism for the $1^-\rho$ and its 0^+ daughter, the $\sigma\rho$ channel would be important.

V. BEHAVIOR OF PARTIAL-WAVE AMPLITUDES AT $u=0$

It is standard practice in approximate dynamical calculations of partial-wave amplitudes $a(u,l)$ to divide out the zeros at the physical thresholds and consider the reduced amplitudes $\bar{a}(u,l)=q^{-2l}a(u,l)$. There is a question in the unequal mass case whether the zeros of the kinematic factor at $u=0$ lead to corresponding zeros of $\bar{a}(u,l)$. It has been concluded on the basis of a tentative argument in Ref. 3 that $\bar{a}(u,l)$ does not have such kinematic zeros, and this is commonly believed to be the case.

In this section we show that the behavior of $\bar{a}(u,l)$ at $u=0$ is very different from that usually assumed. More precisely we are able to show that the partial-wave amplitudes of definite signature $a^{\pm}(u,l)$ behave like $u^{-\alpha^{\pm}(0)}$ where $\alpha^{\pm}(0)$ is the zero-energy intercept of the leading parent Regge trajectory of the same signature in the direct channel.[14] It is not surprising that the cross-channel asymptotic limit of the full amplitude determines the behavior of partial-wave amplitudes at $u=0$, since the integral from $z_u=-1$ to $z_u=+1$ which defines (physical) partial-wave amplitudes corresponds at $u=0$ to an integral of infinite range over t or s. The proof follows.

We first derive the result under the assumption $A_s(u,s)\equiv 0$, and then discuss the modifications necessary when a third spectral function is included. The absorptive part $A_t(u,t)$ is analytic at $u=0$. Its leading term is easily found from the Khuri representation (16) of the full amplitude,

$$A_t(u,t)\approx 2\gamma(u)t^{\alpha(u)}\,, \tag{51}$$

where $\alpha(u)$ is the leading Regge trajectory, and

$$\gamma(u)=\bar{\beta}(u)\frac{\Gamma[\alpha(u)+\frac{3}{2}]}{\sqrt{\pi}\,\Gamma[\alpha(u)+1]\sin\pi\alpha(u)}\,. \tag{52}$$

In order to write Eq. (51) it is necessary that daughter trajectories exist with the properties we have ascribed to them, since the correction terms to Eq. (51) would otherwise be singular at $u=0$. Actually for any $\epsilon>0$, there exists an $N>0$ such that for $t>N$, the correction to Eq. (1) is bounded by $2\gamma(u)\epsilon t^{\alpha(u)-a}$, where a is some fixed positive number less than the distance from the leading pole to the next singularity in the Khuri ν plane.

We next consider the Froissart-Gribov integral

$$a(u,l)=\frac{1}{2q^2\pi}\int_{t_0}^{\infty}dt\,A_t(u,t)Q_l\times\left(1+\frac{2ut}{(m^2-\mu^2)^2-2u(m^2+\mu^2)+u^2}\right) \tag{53}$$

and divide the interval of integration into a part from t_0 to N and a part from N to ∞, with N chosen as above. For u sufficiently close to zero we may approximate the Legendre function and write the first integral as

$$\begin{aligned}&\frac{1}{2q^2\pi}\int_{t_0}^{N}dt\,A_t(u,t)Q_l\left(1+\frac{2ut}{(m^2-\mu^2)^2}\right)\\&=\frac{-2u}{\pi(m^2-\mu^2)^2}\int_{t_0}^{N}dt\,A_t(u,t)\left[\tfrac{1}{2}\ln\frac{ut}{(m^2-\mu^2)^2}+c\right]\\&=Bu\ln u+Cu\end{aligned} \tag{54}$$

near $u=0$, where B, C, and c are constants.

In the second integral we approximate $A_t(u,t)$ by its leading terms, obtaining near $u=0$

$$\begin{aligned}&\frac{4u\gamma(0)}{(m^2-\mu^2)^2\pi}\int_{N}^{\infty}dt\,t^{\alpha(0)}Q_l\left(1+\frac{2ut}{(m^2-\mu^2)^2}\right)\\&=\frac{4u^{-\alpha(0)}\gamma(0)}{\pi(m^2-\mu^2)^2}\int_{uN}^{\infty}d\xi\,\xi^{\alpha(0)}Q_l\left(1+\frac{2\xi}{(m^2-\mu^2)^2}\right).\end{aligned} \tag{55}$$

In the limit as $u\to 0$ this integral can be evaluated exactly with the help of Eq. (37), Sec. 3.2 and Eq. (4), Sec. 2.4 of H.T.F. One obtains

$$\begin{aligned}a(u,l)&=\frac{2\gamma(0)}{\pi}\frac{\Gamma(\alpha(0)+1)^2\Gamma(l-\alpha)}{\Gamma(\alpha(0)+l+2)}\left[\frac{(m^2-\mu^2)^2}{u}\right]^{\alpha(0)} &&\text{if}\quad \alpha(0)>-1\,,\\&=Bu\ln u+Cu &&\text{if}\quad \alpha(0)\leqq -1\end{aligned} \tag{56}$$

near $u=0$. We note that for $\alpha(0)\leqq -1$ the contribution from the finite part of the integration range is actually more singular at $u=0$ than the infinite contribution. This case is unlikely to be realized in physical situations.

If a third-channel spectral function is present we can write

$$\begin{aligned}A_t(u,t)&\approx\gamma^+(u)t^{\alpha^+(u)}+\gamma^-(u)t^{\alpha^-(u)}\,,\\A_s(u,s)&\approx\gamma^+(u)s^{\alpha^+(u)}-\gamma^-(u)s^{\alpha^-(u)}\,,\end{aligned} \tag{57}$$

where $\alpha^+(u)$ and $\alpha^-(u)$ are the leading Regge trajectories of positive and negative signature. We assume that both are of parent type.[15] The integral over $A_t(u,t)$ in the

[14] E. S. Abers and V. L. Teplitz derived a more primitive form of this result by using the Froissart bound, which is a cruder estimate of the high-energy behavior in the crossed channels than the Regge behavior used here. See Nuovo Cimento **39**, 739 (1965).

[15] The leading trajectory of one signature, say $\alpha^-(u)$, may be the daughter of the other, i.e., $\alpha^-(0)=\alpha^+(0)-1$. In this case $a^+(u,l)$ would have the usual behavior at $u=0$, while it would seem that $a^-(u,l)$ would behave no more singularly than $u^{-\alpha^+(0)+1}\ln u$.

Froissart-Gribov continuation (3) can be handled exactly as above. The integral over $A_s(u,s)$ is slightly messier, but a similar treatment applies, and we find the results

$$a^{\pm}(u,l)=\frac{2\gamma^{\pm}(0)}{\pi}\frac{\Gamma(\alpha^{\pm}(0)+1)^2\Gamma(l-\alpha)}{\Gamma(\alpha^{\pm}(0)+l+2)}\left[\frac{(m^2-\mu^2)^2}{u}\right]^{\alpha^{\pm}(0)} \quad \text{if} \quad \alpha^{\pm}(0)>-1,$$

$$=B^{\pm}u\ln u+C^{\pm}u \quad \text{if} \quad \alpha^{\pm}(0)\leqq -1. \quad (58)$$

We expect that the behavior we have found determines the shape of the partial-wave amplitude in a region in which $|u|\ll(m^2-\mu^2)^2$. As the mass difference vanishes this region would shrink to the origin, and the influence of this behavior on the amplitude at any nonzero value of u would become negligibly small.

It would probably improve the accuracy of approximate dynamical calculations to incoprorate Eq. (58) as a constraint on the calculated amplitudes. The zero-energy intercepts and residues of the trajectories can be taken from high-energy data. For cases such as $T=\frac{1}{2}$ πK scattering where $\alpha^{\pm}(0)\approx\frac{1}{2}$, the divergence at $u=0$ may have an important effect on the low-energy s-wave amplitude in the physical region.

VI. CONCLUSIONS AND DISCUSSION

We have studied and resolved the kinematic ambiguity in the Regge representation in unequal-mass scattering at $u=0$. This ambiguity arises because the transformation from the Mandelstam variables (u,s) to the pair (u,z_u) is singular at $u=0$. Our approach to the problem is through the Khuri representation which involves the Mandelstam variables directly and thus avoids representations involving z_u which are inherently suspicious at $u=0$.

The contribution of a single Regge pole to the Khuri representation has leading term $s^{\alpha(u)}$ at $u=0$, but its lower lying terms have singularities there which must be cancelled since the full amplitude is analytic. The only way this cancellation can occur is for Regge trajectories to exist in families which are spaced by integers at $u=0$. If the leading parent trajectory is $\alpha_0(u)$, then the kth daughter trajectory $\alpha_k(u)$ has signature $(-1)^k$ relative to the parent and satisfies $\alpha_k(0)=\alpha_0(0)-k$. The reduced residue $\bar{\beta}_0(u)$ of the parent trajectory is analytic at $u=0$, while the reduced residue $\bar{\beta}_k(u)$ of the kth daughter has a pole of order k there. It is perfectly consistent with general analytic properties for the reduced residues to have poles at $u=0$.

We have studied Bethe-Salpeter models in order to obtain additional support for the daughter trajectory hypothesis and found that it is satisfied for any Bethe-Salpeter amplitude which Reggeizes in the first place. This property follows elegantly from the four-dimensional symmetry of Bethe-Salpeter equations at $u=0$.

Goldberger and Jones[16] have written a recent paper in which the same subject is approached from a somewhat different point of view. Different results are obtained largely because these authors fail to take into account the mechanism of cancellation of singularities by daughter trajectories. Such a mechanism would eliminate the need for the condition $\alpha(0)<\frac{1}{2}$ which they find necessary for the consistency of their method. This condition would seem to be violated by the Pomeranchuk which certainly couples to unequal-mass channels and in Bethe-Salpeter models (for sufficiently large coupling constant) which have all the analyticity properties used by Goldberger and Jones. Since the daughter trajectory hypothesis is definitely satisfied in Bethe-Salpeter models, we feel that it is the correct mechanism by which the ambiguity in the Regge representation is resolved.

The Regge-pole terms we find are very well adapted to phenomenological data analysis. They are given in Eqs. (16), (17), and (47). The daughter-trajectory hypothesis is obviously rich in phenomenological implications, and we have discussed these briefly here.

Our final result is the elucidation of the behavior of partial-wave amplitudes at $u=0$. The power behavior $u^{-\alpha(0)}$ we find has not hertofore been suspected and its implications for bootstrap calculations in unequal-mass systems deserve further study.

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APPENDIX A: BEHAVIOR OF REGGE RESIDUE FUNCTIONS NEAR $u=0$

We consider the A_t contribution to the reduced partial-wave amplitude $\bar{a}(u,l)=q^{-2l}a(u,l)$ using the Froissart-Gribov definition (3). The A_s contribution can be treated similarly. We use Eq. (37), Sec. 3.2 of H.T.F. to write near $u=0$

$$\bar{a}(u,l)\approx\frac{\Gamma^2(l+1)4^l}{\pi\Gamma(2l+2)}\int_{t_0}^{\infty}dt\, t^{-l-1}A_t(u,t)F \times[l+1,\, l+1;\, 2l+2;\, -(m^2-\mu^2)^2/ut]. \quad (A1)$$

Consider the u discontinuity of Eq. (A1). For $u\gtrsim 0$ the hypergeometric function is analytic and the discontinuity vanishes. For $u\lesssim 0$ we use Eq. (10), p. 400 T.I.T., Vol. 2, to help evaluate the discontinuity,

[16] M. L. Goldberger and C. E. Jones, Phys. Rev. **150**, 1269 (1966); also see Phys. Rev. Letters **17**, 105 (1966).

obtaining

$$\text{Disc}\tilde{a}(u,l)=4^l\int_{t_0}^{(m^2-\mu^2)^2/(-u)} dt\, t^{-l-1}A_t(u,t)F$$
$$\times\{l+1,\,l+1;\,1;\,((m^2-\mu^2)^2/ut)+1\}\,. \quad \text{(A2)}$$

Since the integration range is finite for any fixed $u\neq 0$, the discontinuity is an entire function of l, so that the reduced Regge residue functions $\bar{\beta}_i(u)$ cannot have cuts in the vicinity of $u=0$.

We note that the proof permits isolated singularities of the reduced residues at $u=0$, and we have shown in the text that finite-order poles actually do occur. Of course it follows from the N/D decomposition and the definition of Regge poles as the roots of D that the trajectory functions $\alpha_i(u)$ are analytic at $u=0$ unless two trajectories cross there.

APPENDIX B: KHURI REPRESENTATION FOR Re$\nu<-\frac{1}{2}$

In this Appendix we establish a correspondence between the Khuri and Regge representations for Rel and Reν less than $-\frac{1}{2}$. We also compute in closed form the residues of Khuri satellite poles.

We assume that partial-wave amplitudes $a(u,l)$ are meromorphic for Re$l>-L$. Then Mandelstam's form[17] of the Regge representation can be written

$$A(u,t)=B^+(u,t)+B^-(u,t)+R^+(u,t)+R^-(u,t)\,, \quad \text{(B1)}$$

where

$$B^\pm(u,t)=\frac{1}{4\pi i}\int_{-L-i\infty}^{-L+i\infty} dl(2l+1)a^\pm(u,l)\frac{Q_{-l-1}(-z_u)\pm Q_{-l-1}(z_u)}{\cos\pi l}$$
$$-\frac{1}{2\pi}\sum_{l=\Lambda}^{\infty}(-1)^l(2l+2)a^\pm(u,\,l+\tfrac{1}{2})[Q_{l+1/2}(-z_u)\pm Q_{l+1/2}(z_u)]\,, \quad \text{(B2)}$$

and

$$R^\pm(u,t)=\tfrac{1}{2}\sum_{\alpha_i^\pm(u)>-L}\beta_i^\pm(u)[2\alpha_i^\pm(u)+1]\frac{Q_{-\alpha_i^\pm(u)-1}(-z_u)\pm Q_{-\alpha_i^\pm(u)-1}(z_u)}{\cos\pi\alpha_i^\pm(u)}\,, \quad \text{(B3)}$$

where Λ is the least integer greater than $L-\frac{3}{2}$.

We use (B2) and (B3) to compute the absorptive parts A_t and A_s:

$$A_t(u,t)=B_t^+(u,t)+B_t^-(u,t)+R_t^-(u,t)+R_t^-(u,t)\,,\quad A_s(u,t)=B_s^+(u,t)+B_s^-(u,t)+R_s^-(u,t)+R_s^-(u,t)\,, \quad \text{(B4)}$$

where $B_t^\pm$ and $B_s^\pm$ are the t and s discontinuities of $B^\pm$, and

$$R_t^\pm(u,t)=\tfrac{1}{2}\sum_{\alpha_i+>-L}\beta_i^\pm(2\alpha_i^\pm+1)\tan\pi\alpha_i^\pm Q_{-\alpha_i^\pm-1}(1+(t/2q^2))\,, \quad \text{(B5)}$$

$$R_s^\pm(u,s)=\tfrac{1}{2}\sum_{\alpha_i>-L}\beta_i^\pm(2\alpha_i^\pm+1)\tan\pi\alpha_i^\pm Q_{-\alpha_i^\pm-1}(1+[su-(m^2-\mu^2)^2]/2uq^2)\,, \quad \text{(B6)}$$

where $t>t_0$ in (B5) and $s>(m^2-\mu^2)^2/u$ in (B6).[18]

Only those terms in A_t and A_s which are of order greater than t^{-L} at infinity can contribute singularities to the Khuri power-series coefficients of Eq. (6). Similarly the contribution of the finite interval $s_0<s<(m^2-\mu^2)^2/u$ to $c(u,\nu)$ through (6) is an entire function in the ν plane.

Using truncated asymptotic expansions of the Q_l functions we can evaluate the Khuri pole terms just as in Sec. II and can write in analogy with (14).

$$b(u,\nu)=\delta^\pm(u)\sum_{n=0}^{N}\frac{\Gamma(-\alpha^\pm+n)^2}{\Gamma(-2\alpha^\pm+n)}\frac{(-1)^n}{n!}(4q^2)^n\frac{1}{\nu-(\alpha^\pm-n)}+\bar{b}(u,\nu)\,, \quad \text{(B7)}$$

$$c(u,\nu)=\pm\delta^\pm(u)\sum_{n=0}^{N}(4q^2)^n\frac{1}{\nu-(\alpha^\pm-n)}\sum_{r=0}^{n}\left[1-\frac{(m^2-\mu^2)^2}{4uq^2}\right]^{n-r}\frac{\Gamma(-\alpha^\pm+r)}{\Gamma(-2\alpha^\pm+r)}\frac{1}{r!}\frac{\Gamma(\alpha^\pm-r+1)}{\Gamma(n-r+1)\Gamma(\alpha^\pm-n+1)}\,, \quad \text{(B8)}$$

$$\delta(u)=\frac{1}{\sqrt{\pi}}\beta q^{-2\alpha}\frac{\Gamma(\alpha+\frac{3}{2})\Gamma(-2\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)^2}=-(4\pi)^{-1}\beta(2\alpha+1)\tan\pi\alpha(4q^2)^{-\alpha}\,, \quad \text{(B9)}$$

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[17] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1959).

[18] Strictly speaking, (B5) and (B6) are true only for $(m-\mu)^2>u>0$. For other regions of u, $u\neq 0$, the form of (B5) and (B6) may have to be slightly changed, but the results (B7)–(B11) are still valid.

where $\bar{b}(u,\nu)$ and $\bar{c}(u,\nu)$ are analytic for $\mathrm{Re}\nu > -L$, and N is the largest positive integer satisfying $\mathrm{Re}\alpha - N > -L$. The summation over different Regge trajectories is suppressed in (B7) and (B8) and in similar equations below.

Equations (B7) and (B8) express the meromorphy of $b(u,\nu)$ and $c(u,\nu)$ in $\mathrm{Re}\nu > -L$ and show that the Khuri image of a single Regge pole at α is a set of poles at $\nu = \alpha, \alpha-1, \cdots, \alpha-N$.

The next step is to make a Sommerfeld-Watson transform of the power series (5) to obtain

$$A(u,t,s) = \tfrac{1}{2}i \int_{-L-i\infty}^{-L+i\infty} d\nu (\sin\pi\nu)^{-1}[b(u,\nu)(-t)^\nu + c(u,\nu)(-s)^\nu] + \frac{\pi\delta^\pm}{\sin\pi\alpha^\pm} \sum_{n=0}^{N} \frac{\Gamma(-\alpha+n)^2}{\Gamma(-2\alpha+n)} \frac{1}{n!} (4q^2)^n (-t)^{\alpha-n}$$
$$\pm\pi \frac{\delta^\pm}{\sin\pi\alpha^\pm} \sum_{n=0}^{N} (-s)^{\alpha^\pm - n} \sum_{r=0}^{N} \frac{\Gamma(-\alpha^\pm+r)^2}{\Gamma(-2\alpha^\pm+r)} \frac{1}{r!} \frac{\Gamma(\alpha^\pm - r+1)}{\Gamma(n-r+1)\Gamma(\alpha^\pm - n+1)} (4q^2)^r (u-2m^2-2\mu^2)^{n-r} \quad \text{(B10)}$$

which establishes the correspondence between the Regge and Khuri representations in the half-planes $\mathrm{Re}l > -L$, $\mathrm{Re}\nu > -L$.

Khuri's argument against the possibility of such a correspondence to the left of $\mathrm{Re}l = -\frac{1}{2}$ is based on a counter-example for which the Regge amplitude has fixed poles at the negative integers while the corresponding Khuri amplitude is an entire function. This is not in contradiction to our result, since it is obvious from (B7)–(B9) that the residues of the Khuri poles vanish when the Regge pole is at a negative integer. However such Regge poles do contribute to the full Khuri representation (B10) because of the factor $(\sin\pi\alpha)^{-1}$.

We also see from (B7) and (B8) that the residue of the nth satellite pole contains a factor which is an nth order polynomial in u^{-1}.

To investigate the large-s limit of (B10) it is convenient to substitute $t = 2m^2 + 2\mu^2 - s - u$ in the pole terms there obtaining the representation[19]

$$A(u,t,s) = \tfrac{1}{2}i \int_{-L-i\infty}^{-L+i\infty} d\nu (\sin\pi\nu)^{-1}[b(u,\nu)(-t)^\nu + c(u,\nu)(-s)^\nu]$$
$$+\pi \frac{\delta^\pm}{\sin\pi\alpha^\pm} \sum_{n=0}^{N} \frac{\Gamma(-\alpha^\pm+n)^2}{\Gamma(-2\alpha^\pm+n)} \frac{(u-2m^2-2\mu^2)^{N-n-1} s^{\alpha^\pm - N-1}}{n!} (4q^2)^n \frac{\Gamma(\alpha^\pm - n+1)}{\Gamma(\alpha^\pm - N)\Gamma(N-n+2)}$$
$$\times F\left(N-\alpha^\pm+1,\, 1;\, N-n+2;\, -\frac{u-2m^2-2\mu^2}{s}\right) + \pi(1 \pm e^{-i\pi\alpha^\pm}) \frac{\delta^\pm}{\sin\pi\alpha^\pm} \sum_{n=0}^{N} s^{\alpha^\pm - n}(u-2m^2-2\mu^2)^n$$
$$\times \sum_{r=0}^{n} \frac{\Gamma(-\alpha^\pm+r)^2}{\Gamma(-2\alpha^\pm+r)} \frac{\Gamma(\alpha^\pm - r+1)}{\Gamma(n-r+1)\Gamma(\alpha^\pm - n+1)} \frac{1}{r!} \left(\frac{4q^2}{u-2m^2-2\mu^2}\right)^r . \quad \text{(B11)}$$

The second term in (B11) is of background size.

[19] Reference 12, Vol. 1, p. 101, Eq. (9) is used.