system. The qualitative results reported here should be of assistance in the construction of better models.

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Contribution of $J \leq 3/2$ Resonances to the u-Channel Exchange Potential in πN Scattering

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We calculate the contribution of recently found, highly inelastic πN resonances in the S_{31}, S_{11}, P_{11}, and D_{13} partial waves to the exchange u-channel generalized potential for πN scattering. We use the results of the complex phase-shift analyses for incident pion energies ≤ 1 BeV joined to smooth asymptotic functions to obtain the absorptive parts of the amplitudes. Dispersion integrals are performed over these amplitudes in the physical u region and then related to the direct s channel by crossing. Finally, partial-wave projections in the direct s channel give the partial-wave generalized-exchange u-channel potential input to the N/D equations, which are solved for the phase shifts in the direct channel. The results are as follows: (i) Compared to the exchange of the nucleon and the N^*_{33} (1238 MeV), the S_{11}, S_{31}, and P_{11} u-channel continua have relatively small contributions to the s-channel unphysical cut. (ii) As for the P_{33} partial wave, the D_{13} contribution to the u-channel exchange potential is divergent at large s. Thus a cutoff is needed to solve the N/D equations. The effect of including D_{13} exchange is to make it *harder* to obtain agreement with experiment. (iii) The detailed shape of the P_{33} resonance reduces its contribution to the exchange potential by a factor of 0.75 as compared with the narrow-width approximation.

I. INTRODUCTION

RECENT phase-shift analyses[1–4] of πN scattering (for incident pion kinetic energies $E_L < 1$ BeV) have disclosed many interesting features in the low partial waves. Writing the S-matrix elements

$$S \equiv \eta e^{2i\delta}, \tag{1}$$

we note that the S_{31}, S_{11}, and P_{11} waves have been found to have large δ's and large inelastic production cross sections, i.e., small η's. The D_{13} amplitude displays the same features, in contrast to the familiar P_{33} resonance which shows only small inelastic production at these energies.

It has become increasingly clear that quantitative dynamical dispersion calculations of the low partial waves cannot neglect the right-hand inelastic cuts.[5–7] On the other hand, previous calculations[8–10] have approximated the u-channel exchange πN forces with the N and N^* (1238 MeV) contributions, the latter being treated in the narrow-width approximation.

These previous dynamical calculations have been relatively successful in obtaining low-energy phase shifts in most of the partial waves which agree with the experimental values. However, the P_{11} partial wave is in clear

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[1] L. D. Roper, Phys. Rev. Letters **12**, 340 (1964); L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965); L. D. Roper and R. M. Wright, *ibid.* **138**, B921 (1965).

[2] B. Bransden, P. O'Donnell, and R. Moorhouse, Phys. Letters **11**, 339 (1964).

[3] P. Auvil, C. Lovelace, A. Donnachie, and A. Lea, Phys. Letters **12**, 76 (1964); A. Donnachie, A. Lea, and C. Lovelace, *ibid.* **19**, 146 (1965).

[4] P. Bareyre, C. Brickman, A. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965).

[5] P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).

[6] J. S. Ball, Phys. Rev. **149**, 1191 (1966).

[7] J. S. Ball, G. L. Shaw, and D. Y. Wong, Phys. Rev. in press.

[8] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).

[9] J. S. Ball and D. Y. Wong, Phys. Rev. **133**, B179 (1964); **138**, AB4(E) (1965).

[10] W. Frazer and J. Fulco, Phys. Rev. **119**, 1420 (1960); S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

disagreement[11] (as is the resonant behavior of the D_{13} amplitude). The fundamental question arises of whether the P_{11} resonance is due to more complicated "potentials" than have been used or is essentially an inelastic Castillejo-Dalitz-Dyson (CDD) pole, i.e., it results mainly from forces in one of the inelastic channels and can only be properly treated by performing a multichannel calculation.[12]

The purpose of this paper is to examine the contributions of all the above resonant amplitudes ($J\leq\frac{3}{2}$) to the u-channel forces. The procedure is the following: Let $f\equiv(S-1)/2i\rho$ be the partial-wave amplitude. The contribution[13] ${}^Rf(u)$ to $f(u)$ from the physical scattering region in the u channel is obtained by integrating a dispersion integral of $\mathrm{Im}f$; we fitted various smooth asymptotic values for $\mathrm{Im}f$ onto the phase-shift results below 1 BeV in performing the integrals. Using crossing, ${}^Rf$ provided the generalized u-channel potential (regular in the physical s scattering region). Partial-wave projections of this potential in the direct s channel,[13] ${}^Lf(s)$ were then obtained. These "potentials" ${}^Lf(s)$ provided the left-hand input contributions to the N/D equations[5,14] which are solved, yielding theoretical values for the δ's.

The results of the above calculations are (i) The contributions from the S_{31}, S_{11}, and P_{11} resonances to the u-channel potential are quite small. For example, the P_{11} continuum gives a potential which is less than 10% that coming from the exchange of the nucleon pole. (ii) The detailed shape of the P_{33} resonance (as noted by Dashen and Frautschi[15]) reduces its contribution to the exchange potential by a factor of 0.75 as compared to the narrow-width approximation.[16] (iii) The D_{13} contribution to the potential (as is the case for the P_{33} wave) is divergent at high energy. When a cutoff is introduced and the N/D equations solved for the direct-channel partial-wave scattering amplitudes, the D_{13} contribution to the potential makes it *harder* to obtain agreement with experiment.

Thus we conclude from these results that, unfortunately, a presumably better treatment of the u-channel πN forces does not give better agreement for the calculated πN phase shifts: The low-J resonances (S_{31}, S_{11}, and P_{11}) gave small contributions, whereas the D_{13} contribution when large (depending on the cutoff) made the agreement with experiment worse. Perhaps the contribution from the D_{13} is small or possibly its effects are canceled by the higher partial waves. In any event, we find that the usual approximation for the u-channel forces which considers only N and N^* (1238 MeV) exchange is a good one (in the absence of a proper treatment of the high partial waves). Thus the question of whether the P_{11} resonance (as well as the D_{13} and other of the inelastic resonances) results from a more complicated πN potential or mainly from inelastic channels (which must be treated in a multichannel ND^{-1} calculation[12]) remains open. It seems likely, however, that it is the latter.[7,11]

II. THE u-CHANNEL πN GENERALIZED POTENTIAL

Consider a partial-wave amplitude h in the total energy (W) plane[17]:

$$h_j{}^I(W)=[\eta_j{}^I(W)e^{2i\delta_J{}^I(W)}-1]/2ik\,, \tag{2}$$

where k is the center-of-mass momentum. Here we are interested in the generalized potential[13] ${}^Lh$, i.e., that part of h which is regular in the physical region ($|W|\geq m+1$). In terms of the invariant amplitudes A and B[10]:

$$\begin{aligned}{}^Lh_j(W)=(1/32\pi W^2)\{&[(W+m)^2-1]\\ &\times[{}^LA_{j-(1/2)}(s)+(W-m)\,{}^LB_{j-(1/2)}(s)]\\ &+[(W-m)^2-1][-{}^LA_{j+(1/2)}(s)\\ &+(W+m)\,{}^LB_{j+(1/2)}(s)]\}\,,\end{aligned} \tag{3}$$

where

$$\begin{aligned}{}^LA_l(s)&=\int_{-1}^{1}P_l(x)\,{}^LA(s,t)dx\,,\\ {}^LB_l(s)&=\int_{-1}^{1}P_l(x)\,{}^LB(s,t)dx\,,\end{aligned} \tag{4}$$

with x the cosine of the center-of-mass scattering angle in the direct or s channel. As usual,

$$\begin{aligned}s&=[(k^2+m^2)^{1/2}+(k^2+1)^{1/2}]^2\,,\\ t&=-2k^2(1-x)\,,\\ u+s+t&=2m^2+2\equiv\Sigma\,.\end{aligned} \tag{5}$$

The contributions to ${}^LA$ and ${}^LB$, (4) from the u-channel amplitudes are

$$\begin{aligned}{}^LA^{(1/2,3/2)}(s,t)&=\tfrac{1}{3}[(4,1)\,{}^RA^{3/2}(u,t)+(-1,\,2)\,{}^RA^{1/2}(u,t)]\,,\\ {}^LB^{(1/2,3/2)}(s,t)&=-\tfrac{1}{3}[(4,1)\,{}^RB^{3/2}(u,t)+(-1,\,2)\,{}^RB^{1/2}(u,t)]\,,\end{aligned}$$

[11] The one-channel theoretical calculations which force the nucleon to appear as a bound πN state all give P_{11} phase shifts which stay negative, in violent disagreement with experiment. On the other hand, we note that when the nucleon pole in the direct channel was included as an elementary particle (Refs. 5 and 6), the correct phase-shift behavior was easily obtained. We will assume, however, that the nucleon is a composite particle.

[12] M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 207 (1965). E. Squires, Nuovo Cimento **34**, 1751 (1964); D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N.Y.) **37**, 77 (1966); J. Hartle and C. Jones, *ibid.* **38**, 348 (1966).

[13] In general, we divide a function f into ${}^Rf+{}^Lf$, where ${}^Lf$ contains only the singularities of f outside the physical scattering region.

[14] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1964).

[15] R. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1331 (1965).

[16] This suppression factor 0.75 that we find is not as important as the value 0.6 given by Dashen and Frautschi, Ref. 15.

[17] We use units $\hbar=c=m_\pi=1$ and let m be the nucleon mass.

with

$$ {}^{R}A^I(u,t) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \frac{du'}{u'-u}\,\mathrm{Im}A^I(u',t), \tag{6} $$

$$ {}^{R}B^I(u,t) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \frac{du'}{u'-u}\,\mathrm{Im}B^I(u',t). \tag{7} $$

The functions $A(u,t)$, $B(u,t)$ are expressed in terms of the partial-wave amplitudes as[18]

$$ \begin{aligned} A(u,t) &= 8\pi W_u\left(\frac{W_u+m}{(W_u+m)^2-1}f_1 - \frac{(W_u-m)}{(W_u-m)^2-1}f_2\right), \\ B(u,t) &= 8\pi W_u\left(\frac{1}{(W_u+m)^2-1}f_1 + \frac{1}{(W_u-m)^2-1}f_2\right), \end{aligned} \tag{8} $$

where

$$ \begin{aligned} f_1 &= \sum_{l=0}^{\infty} f_{l+}(u)P_{l+1}'(X_u) - \sum_{l=2}^{\infty} f_{l-}(u)P_{l-1}'(X_u), \\ f_2 &= \sum_{l=1}^{\infty} [f_{l-}(u) - f_{l+}(u)]P_l'(X_u), \end{aligned} \tag{9} $$

with

$$ X_u = 1 + t/2k_u^2. \tag{10} $$

We neglect all $J > \frac{3}{2}$. Thus

$$ \begin{aligned} f_1 &= f_{0+} + 3X_u f_{1+} - f_{2-}, \\ f_2 &= f_{1-} - f_{1+} + 3X_u f_{2-}. \end{aligned} \tag{11} $$

Note from (2) that

$$ \mathrm{Im}h_j^I(W_u) = [1 - \eta_j^I(W_u)\cos 2\delta_j^I(W_u)]/2k_u. \tag{12} $$

Now define

$$ \begin{aligned} a &= 8\pi W_u\{[(W_u+m)/((W_u+m)^2-1)](\mathrm{Im}f_{0+} - \mathrm{Im}f_{2-}) - [(W_u-m)/((W_u-m)^2-1)](\mathrm{Im}f_{1-} - \mathrm{Im}f_{1+})\}, \\ b &= 8\pi W_u\{[(W_u+m)/((W_u+m)^2-1)]3\,\mathrm{Im}f_{1+} - [(W_u-m)/((W_u-m)^2-1)]3\,\mathrm{Im}f_{2-}]\}, \\ c &= 8\pi W_u\{[((W_u+m)^2-1)^{-1}(\mathrm{Im}f_{0+} - \mathrm{Im}f_{2-}) + ((W_u-m)^2-1)^{-1}(\mathrm{Im}f_{1-} - \mathrm{Im}f_{1+})]\}, \\ d &= 8\pi W_u\{[((W_u+m)^2-1)^{-1}3\,\mathrm{Im}f_{1+} + ((W_u-m)^2-1)^{-1}3\,\mathrm{Im}f_{2-}]\}. \end{aligned} \tag{13} $$

Thus

$$ \mathrm{Im}A^I(u,t) = a^I + X_u b^I, \quad \mathrm{Im}B^I(u,t) = c^I + X_u d^I. \tag{14} $$

Using (4), (5), (6), (7), and (14), and interchanging the orders of integration, we obtain

$$ \begin{aligned} {}^{L}A_l^{(1/2,3/2)}(s) &= \frac{1}{3\pi}\int_{(m+1)^2}^{\infty} du' \left\{\int_{-1}^{1} \frac{dx\, P_l(x)}{u'-\Sigma-s-2k_s^2(1-x)} \right. \\ &\quad \left. \times\left[(4,1)a^{3/2} + (-1,\,2)a^{1/2} + \left(1 + \frac{-2k_s^2(1-x)}{k_{u'}^2}\right)[(4,1)b^{3/2} + (-1,\,2)b^{1/2}]\right]\right\} \\ &= -\frac{1}{3\pi k_s^2}\int_{(m+1)^2}^{\infty} du' Q_l\left(1 + \frac{\Sigma-s-u'}{2k_s^2}\right)\left\{[(4,1)a^{3/2} + (-1,\,2)a^{1/2}]\right. \\ &\quad \left. + [(4,1)b^{3/2} + (-1,\,2)b^{1/2}](1 + (\Sigma-s-u')/2k_{u'}^2)\right\}, \end{aligned} $$

and

$$ \begin{aligned} {}^{L}B_l^{(1/2,3/2)} &= ((-1,\,2)/k_s^2)g_{NN\pi}^2 Q_l(1 + (\Sigma-s-m^2)/2k_s^2) + \frac{1}{2\pi k_s^2}\int_{(m+1)^2}^{\infty} du' Q_l\left(1 + \frac{\Sigma-s-u'}{2k_s^2}\right) \\ &\quad \times\{[(4,1)c^{3/2} + (-1,\,2)c^{1/2}] + [(4,1)d^{3/2} + (-1,\,2)d^{1/2}](1 + (\Sigma-s-u')/2k_{u'}^2)\}. \end{aligned} \tag{15} $$

Note that $\delta_{l,0}$ contributions have been ignored. The procedure then for obtaining the generalized potential ${}^{L}h_j^I(W)$ from the u-channel πN resonances $J \leq \frac{3}{2}$ is for η_j^I and δ_j^I given (as a function of energy) determine a^I, b^I, c^I, and d^I from (12) and (13). Then perform the integrals (15) for ${}^{L}A_l^I$ and ${}^{L}B_l^I$ which inserted into (3) gives ${}^{L}h_j^I(W)$.

[18] Wherever there is possible confusion we use the subscripts u and s on W and k to denote the appropriate exhange or direct channel variable, respectively.

III. CALCULATION AND CONCLUSION

The "potential" or unphysical-cut terms in πN scattering are usually computed from single-particle-exchange graphs where the nucleon and N_{33}^* (1238 MeV) appear as poles in the u channel and the ρ meson appears as a pole in the t channel.[5,8,9] The purpose of this paper is to evaluate the contribution of the u-channel continuum to the unphysical cut by using the equations in Sec. II. We consider only those partial waves where

TABLE I. u-channel continuum contribution to S_{11} potential terms. Column 2 is the usual term using the narrow-width approximation for the $N_{33}{}^*$ and including ρ and N exchange and the s-channel nucleon pole. The values of the coupling parameters are given in the caption of Fig. 1 (Ref. 17). The other columns indicate the potential term when the u-channel continuum of the designated partial wave is included. The value of $\delta(\infty)$ is specified below the partial-wave symbol.

W	Usual potential term	S_{11}	S_{31}		P_{11}		D_{13}	
		0	0	π	$-\pi$	0	0	π
7.8	−1.72	−1.71	−1.66	−1.70	−1.80	−1.80	−1.32	−1.42
8.8	−1.57	−1.57	−1.52	−1.55	−1.65	−1.65	−1.02	−1.14
9.8	−1.46	−1.46	−1.42	−1.45	−1.55	−1.55	−0.739	−0.883
10.8	−1.38	−1.38	−1.34	−1.36	−1.47	−1.47	−0.463	−0.631
11.8	−1.31	−1.31	−1.27	−1.30	−1.40	−1.40	−0.183	−0.377
12.8	−1.24	−1.24	−1.22	−1.24	−1.34	−1.34	0.104	−0.118
13.8	−1.18	−1.18	−1.16	−1.18	−1.28	−1.28	0.402	0.150
14.8	−1.12	−1.12	−1.10	−1.12	−1.22	−1.22	0.712	0.427
15.8	−1.06	−1.06	−1.04	−1.06	−1.15	−1.16	1.03	0.714
16.8	−0.999	−0.999	−0.981	−0.994	−1.09	−1.09	1.36	1.01

TABLE II. u-channel continuum contribution to S_{31} potential terms. The notation is the same as in Table I.

W	Usual potential term	S_{11}	S_{31}		P_{11}		D_{13}	
		0	0	π	$-\pi$	0	0	π
7.8	−1.18	−1.18	−1.16	−1.17	−1.02	−1.02	−1.97	−1.76
8.8	−1.25	−1.25	−1.23	−1.34	−1.08	−1.08	−2.34	−2.10
9.8	−1.28	−1.29	−1.27	−1.28	−1.11	−1.11	−2.73	−2.44
10.8	−1.30	−1.30	−1.29	−1.29	−1.12	−1.12	−3.13	−2.79
11.8	−1.29	−1.29	−1.28	−1.29	−1.11	−1.11	−3.54	−3.15
12.8	−1.27	−1.27	−1.26	−1.27	−1.09	−1.09	−3.97	−3.53
13.8	−1.24	−1.24	−1.24	−1.24	−1.06	−1.06	−4.42	−3.91
14.8	−1.21	−1.21	−1.20	−1.21	−1.02	1.02	−4.88	−4.31
15.8	−1.17	−1.17	−1.16	−1.17	−0.985	−0.981	−5.36	−4.72
16.8	−1.12	−1.12	−1.12	−1.12	−0.941	−0.937	−5.85	−5.14

TABLE III. u-channel continuum contribution to P_{11} potential terms. The notation is the same as in Table I.

W	Usual potential term	S_{11}	S_{31}		P_{11}		D_{13}	
		0	0	π	$-\pi$	0	0	π
7.8	−3.17	−3.19	−3.31	−3.24	−3.22	−3.22	−2.82	−2.83
8.8	5.13	5.12	5.01	5.07	5.08	5.08	5.56	5.54
9.8	7.96	7.95	7.84	7.90	7.91	7.91	8.47	8.44
10.8	9.36	9.36	9.26	9.32	9.32	9.32	9.97	9.93
11.8	10.23	10.23	10.13	10.19	10.19	10.19	10.94	10.89
12.8	10.87	10.86	10.77	10.83	10.82	10.82	11.67	11.62
13.8	11.38	11.37	11.29	11.34	11.34	11.33	12.29	12.23
14.8	11.83	11.82	11.74	11.79	11.79	11.78	12.85	12.78
15.8	15.24	12.24	12.16	12.21	12.20	12.20	13.38	13.30
16.8	12.63	12.63	12.55	12.60	12.59	12.59	13.89	13.80

TABLE IV. u-channel continuum contribution to P_{33} potential terms. The notation is the same as in Table I except that the usual potential term only includes M exchange, the dominant effect here.

W	Usual potential term	S_{11}	S_{31}		P_{11}		D_{13}	
		0	0	π	$-\pi$	0	0	π
7.8	3.54	3.54	3.53	3.53	3.46	3.46	4.03	3.99
8.8	2.53	2.53	2.52	2.52	2.44	2.43	3.26	3.20
9.8	2.22	2.22	2.22	2.22	2.12	2.12	3.26	3.18
10.8	2.09	2.09	2.09	2.09	1.98	1.97	3.51	3.40
11.8	2.03	2.03	2.02	2.02	1.90	1.90	3.89	3.75
12.8	1.99	2.00	1.99	1.99	1.86	1.85	4.37	4.19
13.8	1.97	1.98	1.97	1.97	1.83	1.82	4.92	4.70
14.8	1.96	1.96	1.96	1.96	1.81	1.80	5.55	5.27
15.8	1.95	1.96	1.95	1.95	1.79	1.79	6.25	5.91
16.8	1.95	1.95	1.95	1.95	1.78	1.78	7.00	6.59

TABLE V. u-channel continuum contribution to D_{13} potential terms. The notation is the same as in Table I.

W	Usual potential term	S_{11} 0	S_{31} 0	S_{31} π	P_{11} $-\pi$	P_{11} 0	D_{13} 0	D_{13} π
7.8	−5.11	−5.10	−5.06	−5.08	−5.09	−5.09	−5.23	−5.23
8.8	−1.32	−1.32	−1.28	−1.29	−1.31	−1.31	−1.49	−1.49
9.8	−0.764	−0.757	−0.718	−0.734	−0.746	−0.746	−0.981	−0.976
10.8	−1.13	−1.13	−1.09	−1.10	−1.12	−1.12	−1.41	−1.40
11.8	−1.86	−1.86	−1.82	−1.83	−1.84	−1.84	−2.21	−2.20
12.8	−2.74	−2.73	−2.69	−2.71	−2.72	−2.72	−3.16	−3.15
13.8	−3.67	−3.66	−3.61	−3.64	−3.64	−3.64	−4.17	−4.16
14.8	−4.61	−4.60	−4.56	−4.58	−4.58	−4.58	−5.22	−5.20
15.8	−5.55	−5.54	−5.49	−5.52	−5.52	−5.52	−6.27	−6.24
16.8	−6.48	−6.48	−6.42	−6.45	−6.45	−6.45	−7.33	−7.29

resonances with $J \leq \frac{3}{2}$ have been found[1–4] to occur for $E_L < 1$ BeV.

The partial waves which we include in the u-channel continuum are the S_{11}, S_{13}, P_{11}, P_{33}, and D_{13} partial waves. The P_{33} contribution is usually taken into account by making the narrow-width approximation or, equivalently, by computing the $N_{33}{}^*$ exchange graph. The P_{33} contribution to the unphysical cut is divergent at high energy since the $N_{33}{}^*$ has spin $\frac{3}{2}$. This behavior makes it necessary to use a cutoff parameter at high energy in any dynamical calculation of πN scattering. However, such dynamical calculations have been reasonably successful. There is no reason to think the D_{13} partial wave, also with $J=\frac{3}{2}$, cannot be included by the same method with equal success. We make no attempt to include higher spin resonances which have even more divergent high-energy behavior.

Real and imaginary parts of the phase shifts have been determined from phase-shift analyses[1–4] of experimental data $E_L < 1$ BeV. The procedure we follow is to fit the phase-shift analysis[4] up to 1 BeV with simple analytic functions and then join these functions continuously with different functions at higher energies. For example, in the P_{11} partial wave we allow the real part of the phase shift, δ, to approach $-\pi$ (no inelastic CDD pole) or 0 (corresponding to the Roper resonance being due to inelastic effects).[12] Our choice for the asymptotic behavior of η is guided by the fact that we want to use the potential in a dynamical calculation of πN scattering. Thus we must let $\eta \to 1$ at the cutoff in order to avoid divergence in the solution of the N/D equations. We find some differences in the u-channel continuum contributions due to the different asymptotic behavior. In the P_{33} Born terms the S_{31} contributions vary by about a factor of 2 for different asymptotic behavior while the P_{11} contributions vary by less than 3%. The numerical results are shown in Tables I–V. However, as we see, the contributions from the S_{11}, S_{31}, and P_{11} continua are small.

Including the P_{33} continuum in the u channel is the same as saying that we no longer neglect the finite width of the $N_{33}{}^*$ in computing the unphysical cut. We used the same Breit-Wigner form for the P_{33} resonance that

TABLE VI. Effect of the narrow-width approximation on the P_{11} potential terms. The potential term includes only the u-channel P_{33} continuum except for the last column, where Ball and Wong's $N_{33}{}^*$ potential term is used along with their constants. Roper's constants are used for the other columns. γ refers to the quantity in Eq. (17).

W	$\gamma=1$	$\gamma=0.5$	$\gamma=0.25$	$\gamma=0.1$	BW
7.8	6.40	7.16	7.83	7.89	8.54
8.8	6.21	6.97	7.64	7.71	8.31
9.8	6.24	7.01	7.59	7.76	8.35
10.8	6.38	7.16	7.86	7.93	8.52
11.8	6.56	7.37	8.09	8.17	8.76
12.8	6.78	7.62	8.36	8.44	9.05
13.8	7.02	7.88	8.65	8.74	9.36
14.8	7.27	8.17	8.96	9.05	9.68
15.8	7.53	8.46	9.28	9.37	10.02
16.8	7.79	8.75	9.60	9.70	10.36

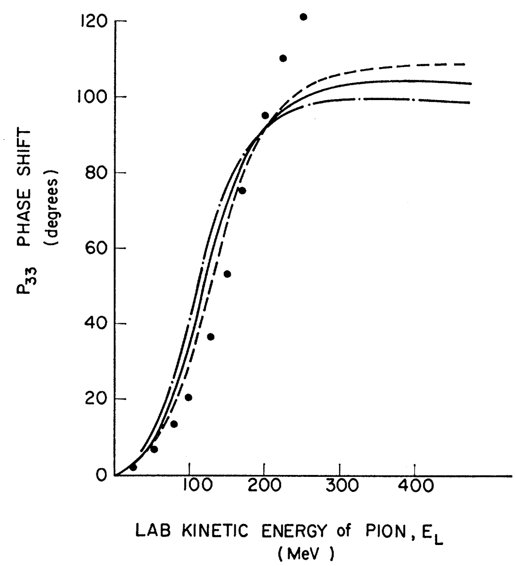

FIG. 1. P_{33} phase shift as a function of the pion laboratory energy for various potential terms (Ref. 19). In the notation of Ref. (5) we used $g_{NN\pi}{}^2/4\pi = 14.6$, $\gamma_{33}=0.06$, $\gamma_2 = 0.27\gamma_1$, and $\gamma_1 = -0.9$ (Ref. 17). The dots represent the phase-shift analysis. The solid curve is the result obtained using the usual potential terms (N, $N_{33}{}^*$, and ρ exchange) and a cutoff at $W=17.3$. The dashed line includes the u-channel P_{11} continuum and has a cutoff at $W=19.0$. The dash-dot curve includes the u-channel contribution from S_{11}, S_{31}, P_{11}, and D_{13} and has a cutoff at $W=12.4$. Elastic unitarity is used in each case.

[19] See Ref. 5 for a discussion of the N/D equations.

Roper used (including relativistic kinematical factors and using Roper's values for the parameters),[1] which we write symbolically as[18]

$$\mathrm{Im} f_{P33} = \Gamma^2/((W_u - m^2)^2 + \Gamma^2). \quad (16)$$

We can go to the narrow-width limit by rewriting the amplitude as

$$\mathrm{Im} f_{P33} = \gamma\Gamma^2/((W_u - m^2)^2 + \gamma^2\Gamma^2), \quad (17)$$

and letting $\gamma \to 0$. By letting γ take on successively smaller values between 0 and 1 we may determine the δ-function limit and thus compute the effect of making the narrow-width approximation. This limit can then be compared with the left-hand cut term due to the N_{33}^* exchange graph as computed by Ball and Wong. We find that the functional dependence on W remains practically the same for all values of $\gamma(0<\gamma<1)$ and the various Born terms (including Ball and Wong N_{33}^* exchange terms) are related by constant factors. The results are shown in Table VI. It is clear that the detailed shape of the resonance reduces its contribution to the unphysical cut by a factor of no smaller than ~ 0.75 as compared to the narrow-width approximation.[16]

It is clear from the tables that the S_{11}, S_{31}, and P_{11} u-channel continua have relatively little effect on the s-channel unphysical cuts, whereas the D_{13} contribution is large. The question now arises as to the effect of these terms on dynamical calculations of the πN system. Figure 1 shows the result of some N/D calculations[19] of the phase shift in the P_{33} partial wave (neglecting small inelastic effects in the D_{33} partial wave). If only the P_{11} continuum is considered, we get a slight improvement of the fit to the data. In fact we get about the same fit as one obtains by using the usual Born terms and including D_{33} inelastic effects. On the other hand, nothing is gained by adding the D_{13} continuum. The cutoff which must be used in the latter case is at a relatively low energy, which is perhaps an indication that the unphysical cut used here is a poorer approximation than for the cases where the D_{13} continuum is neglected. For the P_{11} and other partial waves we were unable to find any value of the cutoff which gave experimentally reasonable behavior when the D_{13} continuum is added.

We must conclude that this, presumably better, treatment of the πN u-channel forces does not give better agreement for the calculated πN phase shifts. The usual approximation in which only N and N_{33}^* exchange is considered in the u channel is much better as far as dynamical calculations are concerned.

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Regge Poles and Unequal-Mass Scattering Processes*

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It is not clear from the Regge representation that the asympotic form $s^{\alpha(u)}$ holds in the backward scattering of unequal-mass particles, because the cosine of the u-channel scattering angle remains small as s increases. In this paper we use a representation for the scattering amplitude first suggested by Khuri to show that the form $s^{\alpha(u)}$ is valid throughout the backward region. However, in order to ensure the analyticity of the amplitude defined by the Khuri representation at $u=0$, it is necessary that Regge trajectories occur in families whose zero-energy intercepts are spaced by integers. Denoting the leading or parent trajectory by $\alpha_0(u)$, we find that daughter trajectories $\alpha_k(u)$ must exist, of signature $(-1)^k$ relative to the parent, satisfying $\alpha_k(0) = \alpha_0(0) - k$. We then study Bethe-Salpeter models and find that this daughter-trajectory hypothesis is satisfied for any Bethe-Salpeter amplitude which Reggeizes in the first place. This fact follows elegantly from the four-dimensional symmetry of Bethe-Salpeter equations at zero total energy. Some phenomenological implications of the daughter-trajectory hypothesis are discussed. We have also characterized the behavior of partial-wave amplitudes in unequal-mass scattering at $u=0$ and find the hitherto unsuspected result $a(u,l) \sim u^{-\alpha(0)}$, where $\alpha(u)$ is the leading u-channel Regge trajectory.

I. INTRODUCTION

THE characteristic features of the Regge pole description of high-energy scattering processes are the asymptotic forms $s^{\alpha(t)}$ or $s^{\alpha(u)}$. However, in the scattering of unequal-mass particles, the question of whether the Regge form $s^{\alpha(u)}$ holds in the backward region has never been settled because there is a cone about the backward direction in which $\cos\theta_u$ does not become large with increasing s. There has been general uneasiness[1,2] about applying the Regge asymptotic form in this region.

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[1] For example see S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962), Ref. 15.

[2] D. A. Atkinson and V. Barger, Nuovo Cimento **38**, 634 (1965).