the second term in (3) cannot contribute to $D$ waves because there are no bilinear expressions in **56*** and **56** with the $C$ and $P$ properties of the parity-violating Hamiltonian. The equal-time commutator in (3) also cannot cause these decays as an explicit calculation of this term with $B_1=\Omega^-$ and $B_2=\Lambda$ or $\Xi$ gives zero. While this result may be a shortcoming of the model itself, the $D$ waves are expected to be small because of the centrifugal barrier. If this amplitude is neglected, the nonleptonic decay rates of $\Omega^-$ can be calculated from the experimental octet decay amplitudes[14] and Eqs. (15) and (17). Using the formulae relating the matrix elements to the decay rates given by Glashow and Socolow,[16] we find

$$R(\Omega^-\to\ \Lambda+K^-)\simeq 14.1\,, \tag{19a}$$

$$R(\Omega^-\to\ \Xi^0+\pi^-)\simeq 5.2\,, \tag{19b}$$

$$R(\Omega^-\to\ \Xi^-+\pi^0)\simeq 2.5\,, \tag{19c}$$

$$R(\Omega^-\to \Xi^{*0}+\pi^-)\simeq 2.2\,, \tag{19d}$$

$$R(\Omega^-\to \Xi^{*-}+\pi^0)\simeq 1.1 \tag{19e}$$

in units of $10^9$ sec$^{-1}$. The last two of these rates are the same as those of Pakvasa and Rosen.[8] These rates imply a lifetime $\tau_{\Omega^-}=0.4\times 10^{-10}$ sec.

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# Faddeev Equations for the Three-Pion Final-State Interaction in $\tau$, $\tau'$, and $\eta$ Decays*†

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The anisotropy in the pion energy spectra for the various three-pion decay modes are assumed capable of providing information concerning the $\pi$-$\pi$ interaction parameters. Such determinations have so far yielded values that differ radically from those obtained from the analysis of low-energy pion-nucleon elastic scattering as well as production phenomena. The problem is reconsidered in the light of the complete three-body scattering amplitude given according to the prescription of Faddeev. The assumptions made regarding the kernel of the integral equations include (i) a simple off-shell extension of the two-body scattering amplitude, separable in the initial and final momenta, (ii) a nonrelativistic framework except for the pion kinematics, and (iii) an attractive $s$-wave $\pi$-$\pi$ interaction in the absence of bound states or resonances. Besides the pion scattering-length parameters, we have a momentum cutoff which, over a wide range of values, has an insignificant effect on the shape of the pion energy spectrum. The integral equations are solved numerically and the parameters that best fit the pion spectra for $\tau$, $\tau'$, and $\eta$ decay are obtained. In each case the $s$-wave scattering length $a_2$ is found to be much larger than $a_0$. This confirms the conjecture that the inclusion of the three-particle effects does not significantly alter the general features of earlier work.

## I. INTRODUCTION

THE three-pion decay modes ($\tau$ and $\tau'$) of the $K$ meson have long been studied[1–7] with the hope of extracting information concerning the basic pion-pion interaction from them. The experimental situation is that in the $\tau$ decay ($K^+\to\pi^+\pi^+\pi^-$), the unlike pion is more likely to be emitted with a higher than average energy,[8] while in the $\tau'$ decay ($K^+\to\pi^0\pi^0\pi^+$), the charged pion is more likely to be emitted with a lower than average energy.[9,10] Attributing this deviation of the pion energy spectrum from pure phase space entirely to the strong final-state interactions of the three pions, several authors[1,3–5] have found (on the basis of a scattering-length approximation) that the $s$-wave pion-pion interaction is more attractive in the $I=2$ channel than

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in the $I=0$ channel. This is contrary to theoretical expectations[11,12] based on the crossing relations, since one anticipates that the $I=0$ channel will be dominant because of the positive contribution it receives from all processes in the crossed channel. Also, the pion-production experiments seem to support the above expectation. In particular, the ABC enhancement[13] would seem to indicate a strong $I=0$ interaction.

Over the years, several suggestions have been made in an attempt to resolve this apparent anomaly. By the inclusion of derivative coupling,[6,7] one could introduce a $p$-wave contribution to adequately explain the pion energy spectrum in $\tau$ and $\tau'$ decay without requiring the strong $I=2$ effects. While such an approach cannot be ruled out, it suffers from the usual objection that the inclusion of so large a $p$-wave contribution with so little energy available to the pions would require the $K$-meson "radius" to be too large ($kR\approx 1$). Brown and Singer[14] and independently Mitra and Ray[15,2] have postulated a resonant $s$-wave $\pi$-$\pi$ interaction in $I=0$. Such a resonance, at about 340 MeV with a width of about 90 MeV, has been shown to be consistant with the $\tau$ and $\tau'$ decay data. While this resonance (usually called the $\sigma$ resonance) could resolve the present situation, there is considerable doubt as to its actual existence, since it has not been seen in $K_{e4}$ decay[16] where its detection should be the easiest. Another proposal is that the apparent discrepancy between the analysis of the production and decay data could be the result of an inadequate treatment of the three-pion multiple-scattering and rescattering effects. It is the purpose of this paper to investigate this latter possibility through an application of the three-particle equations of Faddeev.

Faddeev[17] and Lovelace[18] have shown the existence of a coupled set of integral equations of the Lippmann-Schwinger type for three-particle scattering that avoid the troublesome delta-function singularities in the kernel. Ahmadzadeh and Tjon[19] have used this prescription to study the pion as a bound state of three pions. They find that within a nonrelativistic framework the Faddeev equations do indeed yield a solution provided one can justify the use of a rather large momentum cutoff parameter.[20] In this paper we attempt to apply these techniques to study the three-pion-decay problem. However, since our range of integration variables is quite different, and since we also include both isospin 0 and 2 in the two-body interactions as driving forces, there will be essential differences. The decay amplitude, in the absence of multiple scattering effects, is taken as the inhomogenous part of the Faddeev equations. This input function is assumed to have the same phase as the $\pi$-$\pi$ scattering amplitude in accordance with Watson's final-state theorem.[21] The three-particle effects are included in the kernel through the homogeneous part. We parametrize the pion interaction in terms of the $s$-wave scattering lengths which enter the equations through the kernel and the input functions. The resulting equations are then solved by numerical methods.

In Sec. II the formalism is developed and the integral equations for the decay amplitudes are obtained. The isospin algebra needed to relate the above amplitudes to the actual decay amplitudes is carried out in Sec. III; we also obtain the final expressions for the energy spectrum of the unlike pion. In Sec. IV we compare the results of the calculation to the experimental data and we conclude that, for attractive pion interactions, the $I=2$ channel continues to be far stronger than the $I=0$ channel.[22] While the multiple-scattering terms increase the magnitude of the amplitude, they do not seem to change its energy dependence very much. Thus the qualitative conclusions regarding the pion-scattering lengths arrived at in linear approximations without including the additional rescattering terms prevail.

It is a simple matter to extend this model to the three-pion decay of $\eta$. Here again, the fit to the experimental energy spectrum of the neutral pion suggests a more attractive interaction in the $I=2$ than in the $I=0$ channel. The close similarity between the $K$ and the $\eta$ in their pion energy spectra already suggests that what is observed is indeed due to the strong final-state interactions rather than to the weak or electromagnetic decay vertex.

## II. GENERAL FORMALISM

In this section we briefly indicate the steps in partial-wave decomposition and reduction of the Faddeev equations, the details of which may be found in the literature.[18,19,23,24] Following Faddeev, the full three-body scattering amplitude in the nonrelativistic framework may be written as a sum of three parts;

$$M=M^1+M^2+M^3, \tag{2.1}$$

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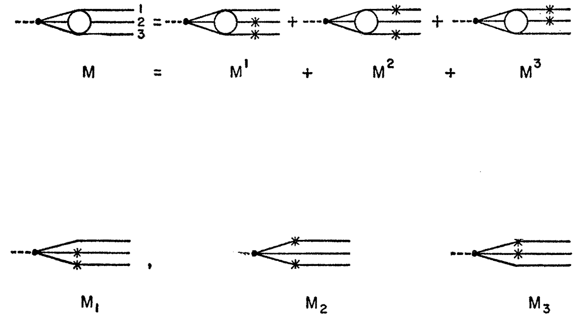


FIG. 1. Symbolic representation of the various amplitudes describing a three-pion decay mode.

and in operator form

$$M^i(Q)=M_i(Q)-M_i(Q)G_0(Q)[M^j(Q)+M^k(Q)]; \quad i,\, j,\, k=1,\,2,\,3. \qquad (2.2)$$

$M^i$ represents that part of the total amplitude where the last two-particle interaction is between particles $j$ and $k$, and $M_i$ represents the corresponding two-particle scattering amplitude in the three-particle Hilbert space. Further, $G_0(=1/[H_0-Q])$ is the three-body Green's function with $H_0$ its kinetic-energy operator, and $Q$ the available kinetic energy for the reaction. These amplitudes are symbolically shown in Fig. 1.

Without loss of generality, we may work in the three-particle center-of-mass frame and introduce the two independent momentum variables $\mathbf{p}_i$ and $\mathbf{q}_i$, where $\mathbf{p}_i$ is the relative momentum between particles $j$ and $k$, and $\mathbf{q}_i$ is the momentum of the $i$th particle relative to the $j$-$k$ subsystem. We now make the usual partial-wave expansion of the states $|\mathbf{p}_i,\mathbf{q}_i\rangle$ and retain only the $s$-wave contributions. At this stage we are left with a set of three coupled integral equations.

The introduction of isospin and Bose statistics for the three-pion state allow the matrix elements $M^j$ and $M^k$ to be simply related to $M^i$. We are thus able to decouple the equations and obtain $M^i$ in terms of the two continuous variables $p_i$ and $q_i$. While there are several ways of labeling the charge states of the three-pion system, it is convenient, at this point, to choose the representation $|I,I_z,I_{jk}\rangle$ where $I_{jk}$ is the isospin of the $j$-$k$ subsystem and $I$ and $I_z$ are the total isospin and its third component. In view of the $\Delta I=\frac{1}{2}$ rule, which continues to be consistent with the most recent experimental evidence,[25] we only need to consider the three-pion states for which the total isospin is unity. Because of Bose statistics, this state can only be a combination of the states for which $I_{jk}=0$ or 2, since the $I_{jk}=1$ state is totally antisymmetric in $s$ wave. We are therefore led to consider equations like

$$M^1(p_1,q_1;I_{23},Q) =M_1(p_1,q_1;I_{23},Q)-\sum_{I_{13}} C_{I_{23},I_{31}} \times\int dp'dq'K_{I_{23}}(p_1,q_1|p',q')M^1(p',q';I_{23}',Q)\,, \qquad (2.3)$$

where $I_{31}=I_{23}'$ and the indices $I$ and $I_z$ have been suppressed. Since the nonrelativistic kinetic-energy operator is given by $H_0=p'^2+\frac{3}{4}q'^2$, we have

$$K_{I_{23}}(p_1,q_1|p'q')=\int\frac{d\Omega}{(4\pi)^2}\frac{p'^2q'^2}{(p'^2+\frac{3}{4}q'^2-Q)} \times\langle\mathbf{p}_1,\mathbf{q}_1|M_1(I_{23},Q)|p'q'\rangle\,, \qquad (2.4)$$

$$M_1(p_1,q_1;I_{23},Q)=\langle p_1,q_1|M_1(I_{23},Q)|K,\text{ at rest}\rangle\,, \qquad (2.5)$$

and $C_{I_{23}'I_{31}}$ are the usual Racah recoupling coefficients for three angular momenta. The kinetic energy is given by $Q=p_1^2+\frac{3}{4}q_1^2$. The matrix elements of $M_1(I_{23},Q)$ are related to the scattering amplitude by

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{q}_1|M_1(I_{23},Q)|\mathbf{p}_1',q_1'\rangle &=\delta^3(\mathbf{q}_1-\mathbf{q}_1')\langle\mathbf{p}_1|M_1(I_{23},\,Q-3q^2/4)|\mathbf{p}_1'\rangle\\ &=\delta^3(\mathbf{q}_1-\mathbf{q}_1')t_1^{(0)}(p_1,\,p_1';I_{23},\,Q-3q^2/4)\,,\end{aligned} \qquad (2.6)$$

where $t_1^{(0)}$ is the off-shell two-body (particles 2 and 3) $s$-wave scattering amplitude that reduces to the physical amplitude when $p_1^2=p_1'^2=\nu(\equiv Q-\frac{3}{4}q^2)$. For the off-shell extensions we choose a separable approximation of the form[26]

$$t_1^{(0)}(p_1,p_1';I_{23},\nu)=-g(p_1)g(p_1')A(I_{23},\nu)/2\pi^2\,, \qquad (2.7)$$

where the $g$'s are the so-called form factors which are expected to go to zero for large values of their argument. $A(I_{23},\nu)$ is the invariant physical amplitude $(e^{i\delta}\sin\delta)/\sqrt{\nu}$ for $s$-wave $\pi$-$\pi$ scattering in the indicated isospin state with center-of-mass momentum squared given by $\nu$. When one integrates $q^2$, the two-body $\nu$ ranges in the region $(Q,-\infty)$, which will be limited by the choice of $g(p)$. In identifying $A(I_{23},\nu)$ with the physical scattering amplitude, Eq. (2.7) satisfies the two-body off-shell unitarity to the extent that $g(\sqrt{\nu})=1$. Using the separable approximation, the $p$ dependence of $M^1$ is factorable and we may write (in a somewhat sloppy notation)

$$M^1(p_1,q_1;I_{23},Q)=g(p_1)M^1(q_1;I_{23},Q)\,, \qquad (2.8)$$

[25] For a summary of experimental indications, see G. Trilling, Proceedings of the International Conference on Weak Interactions, Argonne National Laboratory, 1965, p. 115 (unpublished).

[26] Lovelace has pointed out (Ref. 18) that if the partial-wave amplitude is dominated by a bound state or a resonance, the off-shell amplitude can be well approximated by such a form that is factorable in the initial and final momenta. We assume its validity here.

and with a similar expression for $M_1$ we obtain

$$M^1(q_1;I_{23},Q)=M_1(q_1;I_{23},Q)-\sum_{I_{13}=0,2} C_{I_{23},I_{31}} \times\int dq' K_{I_{23}}(q|q')M^1(q';I_{23}',Q)\,, \quad (2.9)$$

where $I_{31}\equiv I_{23}'$ and

$$K_{I_{23}}(q,q')=\frac{2}{\pi}\int d\cos\theta_{q,q'} \times\frac{q'^2 g(p_1)g(p_2')A(I_{23},\nu)}{[q_1^2+q'^2+q_1q'\cos\theta_{q_1q'}-Q]}\,, \quad (2.10)$$

with

$$p_1'^2=q'^2+q_1^2/4+q_1q'\cos\theta_{q_1q'}\,,$$

and

$$p_1'^2=q_1^2+q'^2/4+q_1q'\cos\theta_{q_1q'}\,.$$

We shall express the amplitude $A$ as the ratio $N/D$ and take $N$ to be the scattering-length parameter $a_{I_{jk}}$. The denominator function $D$ is normalized to unity at $\nu=0$. We thus have

$$A(I_{jk},\nu)=ND^{-1}\,,\quad N=a_{I_{jk}}\,; \quad (2.11)$$

$$D=1+a_{I_{jk}}[\nu/(\nu+1)]^{1/2} \times\{2\ln[(\nu+1)^{1/2}+\nu^{1/2}]+i\pi\}/\pi \quad \text{for } \nu>0\,, \quad (2.12a)$$

$$D=1+2a_{I_{jk}}[(-\nu)^{1/2}/(\nu+1)^{1/2}] \times\tan^{-1}[(\nu+1)^{1/2}/(-\nu)^{1/2}]/\pi \quad \text{for } -1\leq\nu\leq 0\,, \quad (2.12b)$$

$$D=1+2a_{I_{jk}}[\nu/(\nu+1)]^{1/2} \times\ln[(-\nu)^{1/2}+(-\nu-1)^{1/2}]/\pi \quad \text{for } \nu\leq -1\,. \quad (2.12c)$$

Attractive pion interactions in both isospin channels lead to positive scattering lengths in the absence of bound states. The form factor $g(p)$ should assume the value unity at $p^2=\nu$ and also provide sufficient convergence for the integrals to exist. For simplicity we choose the form $g(p)=\theta(p_m^2-p^2)$ with $p_m^2>\nu$, which clearly satisfies the above requirements. The cutoff parameter $p_m$ is related to the range of interaction. The input function $M_1(p,q;I_{23},Q)$ is taken to be proportional to the $\pi$-$\pi$ scattering phase. Further, assuming the ratio of the "Born terms" to be given by the ratio of the scattering lengths in their respective channels ($I_{23}$) we may write

$$M_1(p_1,q_1;I_{23},Q)=g(p_1)A(I_{23},\nu)\,, \quad (2.13)$$

where we have dropped an unimportant over-all proportionality factor. If we now insert (2.10), (2.11), (2.12), and (2.13) into (2.9), we finally arrive at an integral equation of the Fredholm type in a single continuous variable for $M^1$. Note that this is still a $2\times 2$ matrix in the $I_{23}$ isospin space.

Standard techniques are available for solving such equations. The kernel was approximated by an $N\times N$ matrix by choosing finite mesh sizes in the integration. The problem is now reduced to finding the inverse of a nonsingular $2N\times 2N$ matrix.

We now show the relationship between the momenta $p$ and $q$ and the corresponding kinetic energy, since it is the latter variable for which the experimental data are usually presented. In units of the pion mass, the (2-3) center-of-mass momentum $p_1$ is related to the kinetic energy $t_1$ of pion No. 1 (measured in units of the maximum possible kinetic energy) by

$$p_1^2=\rho^2(1-t_1)\,;\quad t_1=T_1/T_{\max}\,, \quad (2.14a)$$

where

$$T_{\max}=[(m_K-1)^2-4]/2m_K\,;\quad \rho^2=m_K T_{\max}/2\,, \quad (2.14b)$$

and

$$q_1^2=4Q/3-\rho^2(1-t_1)\,. \quad (2.14c)$$

With the above linear relation between $q_1^2$ and $t_1$, we now have the amplitude $M^1$ expressed as a function of $t_1$. The amplitudes $M^2$ and $M^3$ are likewise functions of $t_2$ and $t_3$, respectively.

## III. PION SPECTRUM

In this section we will express the decay amplitudes of the various three-pion modes in terms of the functions we obtained upon solving the integral equation (2.9). Since the final three pions are assumed to be in a state of total isospin unity, the amplitude may be written as

$$M=\boldsymbol{\alpha}(\boldsymbol{\beta}\cdot\boldsymbol{\gamma})A(t_1,t_2,t_3)+\boldsymbol{\beta}(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})B(t_1,t_2,t_3) +\boldsymbol{\gamma}(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})C(t_1,t_2,t_3)\,, \quad (3.1)$$

where the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are the isospin functions of pions No. 1, 2, and 3. With a fictitious vector charge index $\rho$ for the $K$ meson, the amplitude is given by

$$M_{\rho\alpha\beta\gamma}=\delta_{\rho\alpha}\delta_{\beta\gamma}A+\delta_{\rho\beta}\delta_{\alpha\gamma}B+\delta_{\rho\gamma}\delta_{\alpha\beta}C\,. \quad (3.2)$$

Taking pion No. 3 as the unlike one in both $\tau$ and $\tau'$ modes, it is easily seen that

$$M_\tau(K^+\to\pi^+(1)\pi^+(2)\pi^-(3)) =A(t_1,t_2,t_3)+B(t_1,t_2,t_3)\,, \quad (3.3a)$$

and

$$M_{\tau^0}(K^+\to\pi^0(1)\pi^0(2)\pi^+(3))=C(t_1,t_2,t_3)\,. \quad (3.3b)$$

Consistent with Bose statistics, the $s$-wave parts of $A$, $B$, and $C$ have the following form[3]:

$$\begin{aligned} A(t_1,t_2,t_3)&=U(t_1)+V(t_2)+V(t_3)\,,\\ B(t_1,t_2,t_3)&=V(t_1)+U(t_2)+V(t_3)\,,\\ C(t_1,t_2,t_3)&=V(t_1)+V(t_2)+U(t_3)\,. \end{aligned} \quad (3.4)$$

Recalling the relation

$$M(t_1,t_2,t_3)=M^1(t_1)+M^2(t_2)+M^3(t_3)\,,$$

we find that in isospin space we have

$$\langle\alpha\beta\gamma|M^1|\rho\rangle=\langle\alpha\beta\gamma|II_zI_{23}\rangle\langle II_zI_{23}|M^1|II_z\rangle\langle II_z|\rho\rangle$$
$$=\langle\alpha\beta\gamma|II_zI_{23}\rangle\langle II_z|\rho\rangle M^1(t_1,I_{23}), \quad (3.5)$$

where $M^1(t_1,I_{23})$ results from solving (2.9). Explicit evaluation of the above coupling coefficients gives

$$M_{\rho\alpha\beta\gamma}{}^1=\delta_{\rho\alpha}\delta_{\beta\gamma}[(\surd 5)M^1(t_1, I_{23}=0)-M^1(t_1, I_{23}=2)]$$
$$+\tfrac{3}{2}(\delta_{\rho\beta}\delta_{\alpha\gamma}+\delta_{\rho\gamma}\delta_{\alpha\beta})M^1(t_1, I_{23}=2). \quad (3.6)$$

Using Bose statistics we get similar expressions for $M^2$ and $M^3$ in terms of $t_2$, $I_{31}$ and $t_3$, $I_{12}$. It is then straightforward to identify

$$U(t_1)=(\surd 5)M^1(t_1, I_{23}=0)-M^1(t_1, I_{23}=2)$$

and

$$V(t_1)=\tfrac{3}{2}M^1(t_1, I_{23}=2). \quad (3.7)$$

For comparison with experiment it is convenient to average over one of the two independent Dalitz variables[27] ($x=t_1-t_2$) with the result

$$|M_\tau(t_3)|^2=\int dx|A(x,t_3)+B(x,t_3)|^2\rho(x,t_3)\Big/\int dx\,\rho(x,t_3), \quad (3.8a)$$

$$|M_{\tau'}(t_3)|^2=\int dx|C(x,t_3)|^2\rho(x,t_3)\Big/\int dx\,\rho(x,t_3), \quad (3.8b)$$

where $\rho(x,t_3)$ is the three-pion phase-space density. For $A$, $B$, and $C$ we use the expressions of (3.7) and (3.4). We also normalize both matrix elements to unity at $t_3=\frac{1}{2}$ (in units of $T_{\max}$).

All the above formulas could be carried over to the three-pion decay mode of the $\eta$ ($\eta\rightarrow\pi^+\pi^-\pi^0$). This decay occurs through the electromagnetic interaction and if charge conjugation $C$ is to be conserved in such interactions, the final three pions must be in an $I=1$ state. Then the formalism developed becomes suitable for treating the $\eta$ decay when the equations are modified with the appropriate $Q$ value for $\eta$. We find

$$M_\eta(\eta\rightarrow\pi^+(1)\pi^-(2)\pi^0(3))=C_\eta(t_1,t_2,t_3), \quad (3.9)$$

where $t_1$, $t_2$, $t_3$ are the kinetic energies of $\pi^+$, $\pi^-$, $\pi^0$, respectively. Analogous to (3.8a) and (3.8b) we have

$$|M_\eta(t_3)|^2=\int dx|C_\eta(x,t_3)|^2\rho_\eta(x,t_3)\Big/\int dx\,\rho_\eta(x,t_3), \quad (3.10)$$

where $\rho_\eta(x,t_3)$ is again the appropriate three-pion phase-space density.

## IV. RESULTS AND DISCUSSION

The numerical solutions obtained for the unlike-pion spectrum [more correctly, the squared matrix elements of (3.8) and (3.10)] are functions of the input parameters $a_0$, $a_2$, and $p_m$. The cutoff parameter $p_m$ was varied over the range from $1.5Q^{1/2}$ up to $6.0Q^{1/2}$ ($\approx 1.2m_\pi$ up to $4.8m_\pi$) and was found to have an insignificant effect on the shape of the spectrum for $0\leq a_{0,2}\leq 10$. It should probably be emphasized that it is the energy dependence of the spectrum that is insensitive to the cutoff parameter, in contrast to the magnitude of the amplitude, whose strong dependence on the cutoff was demonstrated by Ahmadzadeh and Tjon.[19] For values of the cutoff insufficient to produce a "$3\pi$" resonance at the $K$ mass, the shape dependence of our solution for the spectra are found to be reliable. We have chosen to present our results for $p_m=2.0Q^{1/2}$.

In our approximation of the kernel by an $N\times N$ matrix, we wished to take $N$ as small as possible. We found that reliable results could be achieved with $N=48$ and that increasing $N$ to 96 did not change the results. The mesh sizes for the integration were chosen to give more weight to the region where the kernel was most rapidly varying. A search was made through the positive values of the scattering lengths $a_0$ and $a_2$ for the best fit to the experimental-decay distribution. Our results for $\tau$ and $\tau'$ decay are shown in Figs. 2 and 3, while the results for $\eta$ decay are shown in Fig. 4.

Figure 2 shows the experimental $\pi^-$ energy spectrum (with the phase space removed) for the $\tau$ mode.[8] The various curves correspond to the calculated distributions (3.8a) for different values of the parameters $a_0$ and $a_2$. Curve A corresponds to the values $a_0=0.50$ and

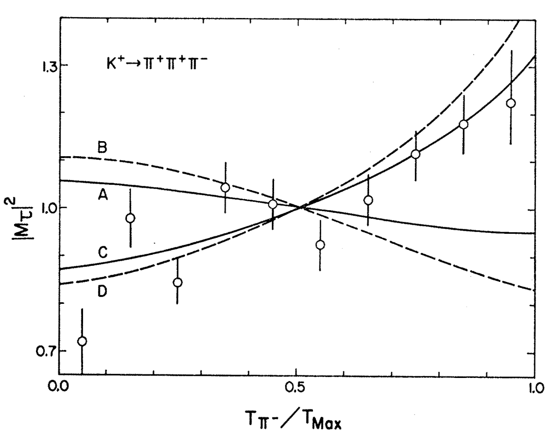


FIG. 2. The $\pi^-$ energy spectrum for $\tau$ decay. The experimental points are those of Huetter *et al.* (Ref. 8). Curve A corresponds to (Ref. 28) $a_0=0.50$ and $a_2=0.16$. Curve B corresponds to (Ref. 29) $a_0=1.30$ and $a_2=0.52$. Curve C is our best fit and corresponds to $a_0=0.4$ and $a_2=1.3$. Curve D is for $a_0=0.3$ and $a_2=1.5$ and corresponds to the minimum total $\chi^2$ for both $\tau$ and $\tau'$.

[27] R. H. Dalitz, Proc. Phys. Soc. (London) **A69**, 527 (1956).

$a_2=0.16$ which are the results of Schnitzer's analysis of the reaction $\pi+N \rightarrow 2\pi+N$ at low energy.[28] Curve B corresponds to the parameters found by Hamilton *et al.* from their analysis of $\pi$-$N$ elastic scattering by means of partial-wave dispersion relations.[29] They determined $a_0=1.3$, and, based on crossing relations and the Chew-Mandelstum equations,[11] this suggests a value of $a_2=0.52$. Clearly, neither of the curves A or B agrees with the experimental spectrum. The parameters that correspond to the ABC phenomenon yield results similar to curves A and B, and therefore cannot be reconciled with the $K$-decay data.

Curve C, with the values $a_0=0.4$ and $a_2=1.3$, provides the best fit to the experimental data presented. This fit corresponds to a $\chi^2$ value of 18.5 for seven degrees of freedom. We may remark that the best

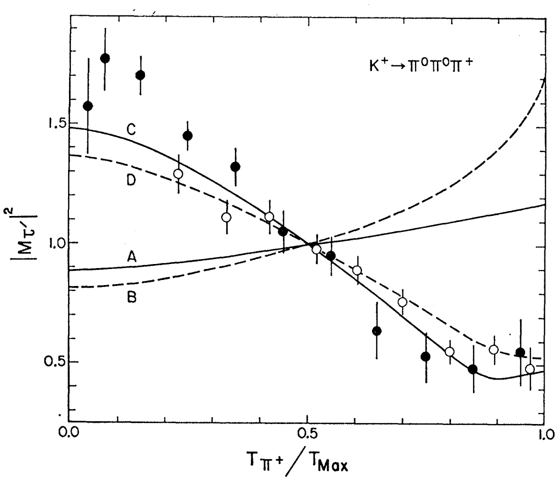


FIG. 3. The $\pi^+$ energy spectrum for $\tau'$ decay. The experimental points are those of Kalmus *et al.* (Ref. 9; open circles) and Bisi *et al.* (Ref. 10; solid circles). Curve A corresponds to (Ref. 28) $a_0=0.50$ and $a_2=0.16$. Curve B corresponds to (Ref. 29) $a_0=1.30$ and $a_2=0.52$. Curve C is our best fit and corresponds to $a_0=0.2$ and $a_2=2.0$. Curve D is for $a_0=0.3$ and $a_2=1.5$ and corresponds to the minimum total $\chi^2$ for both $\tau$ and $\tau'$.

linear fit to the data yields a $\chi^2$ of 18.6 for eight degrees of freedom.[8] Indeed, one does not expect any simple model or theory to give a much reduced value of $\chi^2$ because of the "scatter" of the experimental points in the low-energy region.

The $\pi^+$ energy spectrum for the $\tau'$ mode is plotted in Fig. 3. The experimental points are from a heavy-liquid bubble-chamber experiment at Berkeley[9] and a hydrogen and heavy-liquid chamber experiment at CERN.[10] Here again, the parameters suggested by Schnitzer[28] and Hamilton *et al.*[29] (curves A and B, respectively) are at variance with the experimental $\tau'$ spectrum. The best fit, with $a_0=0.2$ and $a_2=2.0$, is shown as curve C and has a $\chi^2$ value of 31.2 for 17 deg of freedom.

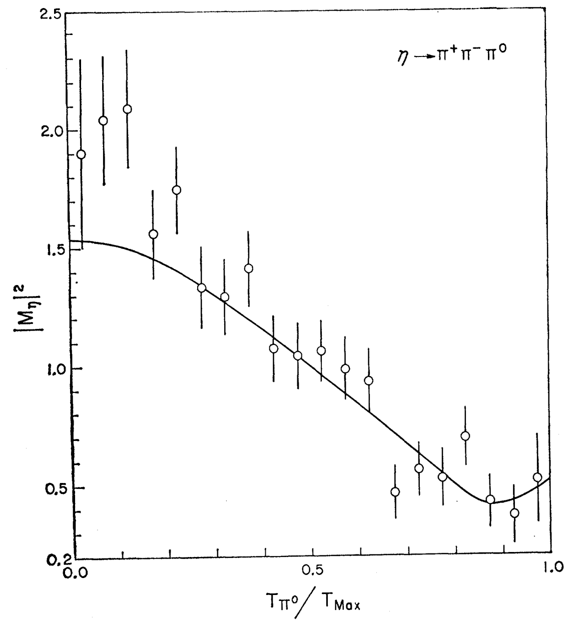


FIG. 4. The $\pi^0$ energy spectrum for $\eta$ decay. The experimental points are those of Crawford *et al.* (Ref. 30). The solid curve is our best fit and corresponds to $a_0=0.1$ and $a_2=1.75$.

The set of parameters that minimizes the total $\chi^2(=\chi_\tau^2+\chi_{\tau'}^2)$ for both $\tau$ and $\tau'$ taken together is $a_0=0.3$ and $a_2=1.5$. The curves labeled D, in both Figs. 2 and 3, correspond to the above values and have a $\chi^2$ of 24.1 for $\tau$ (against a best-fit value of 18.6) and 56.7 for $\tau'$ (against a best-fit value of 31.2). Indeed, it is seen that a large part of the contribution to $\chi^2$ is from the poor fit at the low-energy end for both the $\tau$ and $\tau'$ data.

It may be of interest to point out a few of the general features of our solutions. For both $\tau$ and $\tau'$, the steepness of the curves is roughly determined by the difference $a_0-a_2$, which is reminiscent of the conclusions of Khuri and Treiman.[3] The deviation from a linear energy dependence is pronounced at both the high- and low-energy ends and the fit to the high-energy end is better achieved than for the low-energy region. For both $\tau$ and $\tau'$ we find the high-energy portion is more sensitive to the variation of $a_2$, while $a_0$ is the dominating factor at the low-energy end. As $a_2$ is increased, the deviation from the phase-space value ($|M|^2=1$) at the high-energy end is effected by decreasing $a_0$. As expected by Khuri and Treiman,[3] the nonlinearity becomes accented when both $a_0$ and/or $a_2$ are increased.

Finally, in Fig. 4 we have shown some recent experimental data for the neutral pion energy spectrum in $\eta$ decay[30] ($\eta\rightarrow\pi^+\pi^-\pi^0$). Using Eq. (3.10) we find the best fit to the data with $a_0=0.1$ and $a_2=1.75$. The $\chi^2$ value for this fit is 27.1 for 17 deg of freedom. Notice that these parameters are very close to the best-fit parame-

[28] H. J. Schnitzer, Phys. Rev. **125**, 1059 (1962).

[29] J. Hamilton, P. Mennoti, G. Oades, and L. Vick, Phys. Rev. **128**, 1881 (1962).

[30] Columbia-Berkeley-Purdue-Wisconsin-Yale collaboration, Phys. Rev. **148**, 1573 (1966). We wish to thank Professor F. Crawford for communicating this work to us prior to publication.

TABLE I. Best-fit parameters for $K$ and $\eta$ decay.

| Reaction | $a_0$ | $a_2$ | $\chi^2$ (degrees of freedom) |
|---|---|---|---|
| $K^+ \to \pi^+\pi^+\pi^-(\tau)$ | 0.4 | 1.3 | 18.5(7) |
| $K^+ \to \pi^0\pi^0\pi^+(\tau')$ | 0.2 | 2.0 | 31.2(17) |
| $\tau+\tau'$ | 0.3 | 1.5 | 24.1+56.7(27) |
| $\eta \to \pi^+\pi^-\pi^0$ | 0.1 | 1.75 | 27.1(17) |

ters for the $\tau'$ data and not very far from those for the $\tau$. This similarity between $\tau'$ and $\eta$ decay spectra confirms that they are brought about by the same mechanism. Thus, the weak or electronmagnetic vertex seems to have little influence on the shape of the energy spectrum and the simple mechanism for a strong final-state interaction should satisfactorily explain all these three-pion decays. A summary of our results is shown in Table I.

The striking feature of these solutions is the relative unimportance of the multiple-scattering effects and the dominance of the three-body "Born terms." It is worthwhile to ask how much of this result is implied by our assumptions. The choice of the separable approximation for the off-mass-shell extension and the size of the cutoff parameter may have significant effects. For a wide range of cutoff momenta we find our results unchanged, though it is not quite clear why an unreasonably large parameter cannot be used. However, the cutoff is not a mere mathematical artifice to ensure convergence of the integrals, but signifies the size of the two-body interacting system, and thus cannot assume an arbitrarily large value. It is obvious that the choice of separability for the off-shell amplitude has greatly simplified the calculations. We believe that this has not imposed any drastic peculiarities on the solutions. The calculation can in principle be done without resorting to the separable approximation, but such complexity does not seem to be particularly instructive.

Now, within the above restrictions, the multiple-scattering effects will become significent only when the $Q$ value for the decay is such that the kernel has an eigenvalue approaching unity. This would then correspond to a situation with a pole in the three-particle scattering amplitude and the energy spectrum should then be significantly different. In the absence of such effects, the calculation seems to indicate the final two-body scattering dominates and the energy spectrum is almost entirely given by the inhomogeneous term.

While this work has not ruled out the possibility of fitting the data with negative scattering lengths, we have here adopted the view that the Faddeev equations may have a solution for positive scattering-length parameters obtained on the basis of phenomena other than decay data. This could not be achieved. Indeed, under the assumptions made, the best fit continued to give a stronger $I=2$ effect than $I=0$. It is already known that by adding a negative effective range to the positive scattering length in the $I=0$ channel one can obtain a reasonable fit. Though such a solution (which corresponds to an $s$-wave resonance) is equally well possible in our model, it is in conflict with the $Ke_4$-decay data. The remaining alternative is a negative $I=0$, $s$-wave scattering length which would imply a $\pi$-$\pi$ bound state. Recently, however, Chew[31] has suggested the possibility of a decreasing phase shift[32] with a vanishing residue at the position of the bound-state pole. This simultaneous zero of both $N$ and $D$ could justify a negative scattering length without requiring an observable bound state. We intend to investigate this proposal further.


## ACKNOWLEDGMENTS

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[31] G. F. Chew, Phys. Rev. Letters **16**, 60 (1966).

[32] However, by looking at the sign of the forward-backward asymmetry in the $\pi^-\pi^+$ "scattering angle" distribution in the reaction $\pi^-p \to \pi^-\pi^+n$, L. D. Jacobs and W. Selove, Phys. Rev. Letters **16**, 669 (1966), discount the possibility of a decreasing phase shift. But by the same type of analysis as above, but with much higher incident pion momenta, L. W. Jones *et al.*, Phys. Letters **21**, 590 (1966) find the data could be consistent with Chew's suggestion. We would like to thank Dr. D. O. Caldwell for communicating this work to us prior to its publication.