Structure of $K_{3\pi}$ Decay*

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Using the partially conserved axial-vector current and the commutation relations of the vector and axial-vector currents with themselves and with the Hamiltonian responsible for weak nonleptonic decays, we demonstrate how to expand weak-decay amplitudes in the momenta of pions involved in the decays. The resulting expansion gives the on-mass-shell decay amplitude up to first order in pion momenta. We apply these techniques to the three-pion decays of kaons and show that they quantitatively reproduce the experimental situation of an amplitude linearly dependent on the "odd"-pion kinetic energy. All three pions are treated on an equal footing, and on-the-mass-shell quantities are dealt with everywhere.

I. INTRODUCTION

THE gathering of extensive experimental data[1] on the three-pion decay of the kaon has revealed the simple, but remarkable, fact that the amplitude for this weak-decay process behaves as a constant term, the so-called "phase-space" term, plus a small *linear* dependence on the energy of the unlike, or "odd," pion. This deviation from the phase space can be expressed quite compactly as[1]

$$A(K(k)\rightarrow \pi_1(q_1)+\pi_2(q_2)+\pi_3(q_3)) =C\left(1-\frac{a}{m_\pi^2}(S_{\rm odd}-S)\right), \quad (1)$$

with a and C two constants, m_π the pion mass, $S_i=-(k-q_i)^2$, and $S=(S_1+S_2+S_3)/3$. The linear form (1) was first suggested by Weinberg in 1960,[2] and provides a quite acceptable fit to the data.[1]

It is not surprising that such a simple form as (1) has attracted considerable theoretical attention,[1] for the explanation of it should certainly reveal quite a bit about the nonleptonic weak decays. There are at least two lines of attack to this problem, and they have been summarized by Khuri and Treiman[3] as the "intrinsic" and the "final-state-interaction" approaches. In the latter,[4] which is by far the more popular, one assumes that the structure of the decay amplitude comes entirely from interactions among the three pions in the outgoing final state, and that only the magnitude of the decay amplitude is determined by the transformation of the kaon into three pions. The "intrinsic" viewpoint is that somehow final-state interactions are negligible for the energies of the pions involved, namely a maximum pion kinetic energy ≈ 50 MeV, and that everything comes from the "core" of the K-3π vertex. As Khuri and Treiman emphasize, this separation of "final-state" and "intrinsic" properties of the amplitude is really imprecise; nevertheless, one knows what it means in an operational sense.

In this paper we shall adopt the "intrinsic" attitude and freely neglect the interactions among the final pions. In some, again imprecise, sense, therefore, we ignore what occurs "after" the kaon has become three pions and concentrate on the essence of the transformation. The technique we shall use is an expansion of amplitudes (decay or scattering) in the momenta of pions. These techniques have been developed by Weinberg,[5] and applied by him to the cases of multiple pion emission in a scattering process and K_{l4} decay. The fundamental tools involved in this procedure are the commutation relations (CR) of the vector and axial-vector currents with themselves[6] and with the weak-decay Hamiltonian[7] and the partially conserved axial-vector current (PCAC). The outstanding advantage to the approach developed by Weinberg is that one always works with pions *on the mass shell*[8] and thus never needs to concern himself with corrections due to off-mass-shell effects. Naturally enough, the techniques have far wider application than weak-decay processes.[5,9]

Before outlining the order of presentation of our results, let us make two comments. (1) The $K_{3\pi}$ decay process has been considered by others[7,10] also using the CR's and PCAC. It is worthwhile to note how our work differs from theirs. Suzuki and Callan and Treiman investigate these decays at zero four-momentum for a selected one of the pions. They cannot, therefore, say anything about the structure of the linear dependence

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[1] G. H. Trilling, in *Proceedings of the International Conference on Weak Interactions* (Argonne National Laboratory, Chicago, Illinois, 1965), p. 115.

[2] S. Weinberg, Phys. Rev. Letters **4**, 87 (1960); **4**, 585(E) (1960).

[3] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).

[4] There is a large number of authors who have contributed to this approach beginning with Khuri and Treiman, Ref. 3, and B. S. Thomas and W. G. Holladay, Phys. Rev. **115**, 1329 (1959). The work of L. M. Brown and P. Singer [Phys. Rev. **133**, B812 (1964)] is also characteristic of this method.

[5] S. Weinberg, Phys. Rev. Letters **16**, 879 (1966); **17**, 336 (1966).

[6] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[7] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966).

[8] This has been realized also in the work of S. L. Adler, Phys. Rev. **139**, B1638 (1965). See especially his discussion of induced pseudoscalar terms.

[9] C. G. Callan has in preparation an extensive discussion of our basic equation, Eq. (9), and its use in K_{l3} decay and π-N scattering.

[10] M. Suzuki, Phys. Rev. **144**, 1154 (1966); Y. Hara and Y. Nambu, Phys. Rev. Letters **16**, 875 (1966).

of the decay amplitude but can only ask how the magnitude (phase-space term) compares to $K_{2\pi}$ decay. Their conclusions are that sometimes it is justified to extrapolate to zero four-momentum, and sometimes it is not (see especially Callan and Trieman). This ambiguity in their results is not surprising after the demonstration by Weinberg in K_{l4} decay[5] that the form factors there (especially F_3) depend quite sensitively on which pion has zero four-momentum. We differ from these two sets of authors by not having to consider zero pion four-momentum.

Hara and Nambu take a slightly different outlook. They derive a set of conditions on the various $K_{3\pi}$ decays at zero four-momentum for selected pions. They then *assume* a form for the $K_{3\pi}$ amplitude on the mass shell, chosen to give a linear dependence on the energy of the "odd" pion. Their form for the decay matrix element involves certain unknown constants which they fix by requiring their conditions at zero four-momentum to be satisfied. Their results are approximately the same as those presented below, differing by $O((m_\pi/m_K)^2)$. This paper goes further than they in the following respects: (a) We do not have to deal with decay amplitudes at unphysical points; (b) we do not assume a special form for the $K_{3\pi}$ amplitude, but only expand it in pion four-momenta; (c) they treat the three pions quite unsymmetrically by taking one at a time off the mass shell, while we treat all three pions on an equal footing; (d) our approximation scheme is quite clear, while they are very involved with the zero four-momentum limit. The spirit of the work of Hara and Nambu is to use the linear form for the $K_{3\pi}$ matrix element to check the zero four-momentum relations they derive. This in itself is a long extrapolation out of the physical region. Our approach here is to derive the structure of the $K_{3\pi}$ decay amplitude and in that sense it is related to the work of Hara and Nambu but not simply an extension of it. Finally, we are forced to discuss (briefly) the "σ-meson" terms[5] encountered when one has more than one pion he is constructing via PCAC.

(2) An argument often given[1] for the importance of final-state interactions in three-pion final states is that the decay amplitude for the η meson into three pions has the same linear dependence as the $K_{3\pi}$ decay amplitude. Since the only obvious similarity between the two processes is that they each have three-pion final states, one might conclude that interactions among the pions are responsible for the linear dependence observed. This approach has found its strongest advocates among those who would like to use a 0^+ meson, a "sigma" meson, to yield the structure of the K and η decay matrix elements. Trilling[1] has summarized some of the failings of this viewpoint. An alternative approach is to note that in $\eta_{3\pi}$ decay only $\Delta I=1$ transitions are allowed by C conservation. One might imagine that, therefore, the effective Hamiltonian responsible for the decay would have commutation relations with the vector and axial-vector currents similar to those of the weak Hamiltonian H_w. Using the formula derived below, this would give one the structure of $\eta_{3\pi}$ decay. This may be dangerous, however, as we know the electromagnetic Hamiltonian is probably not similar in structure to H_w and have not any *a priori* reason to assume they have similar CR's. Nevertheless, Ramachandran[11] has pursued this idea with quite reasonable success and since he is preparing a report of his work, I shall leave the defense of the idea to him.

The paper is organized in four sections. After this introduction we derive in Sec. II the formulas giving the amplitudes for any two states α and β transforming into one another through the weak Hamiltonian H_w accompanied first by two, then three, pions. In Sec. III we apply these results to $K_{3\pi}$ decay and in the final section we have a discussion of our results.

II. PION-EMISSION FORMULAS

A. Two Pions

We begin by considering the quantity

$$M_{\mu\nu}=\int dxdy\, e^{-iq_a\cdot x}e^{-iq_b\cdot y}\times\langle\beta|T(A_\mu{}^a(x)A_\nu{}^b(y)H_w{}^l(0))|\alpha\rangle, \quad (2)$$

where $A_\mu{}^a(x)$ is the member of an octet of axial-vector currents with quantum numbers a, and $H_w{}^l$ is the Hamiltonian responsible for weak nonleptonic decays. We shall take this Hamiltonian to be a member of an octet of scalar plus pseudoscalar quark densities, thus the index l, to give us a form for its commutation relations with the vector and axial-vector currents. These CR's will then be assumed true beyond the quark model. Now following Weinberg[5] we may write $M_{\mu\nu}$ with all its pion poles removed as $N_{\mu\nu}$:

$$q_{a\mu}q_{b\nu}N_{\mu\nu}=q_{a\mu}q_{b\nu}M_{\mu\nu}-\frac{q_a{}^2F_\pi iq_{b\nu}M_\nu(q_b)}{q_a{}^2+m_\pi{}^2}-\frac{q_b{}^2F_\pi iq_{a\mu}M_\mu(q_a)}{q_b{}^2+m_\pi{}^2}-\frac{F_\pi{}^2q_a{}^2q_b{}^2M}{(q_a{}^2+m_\pi{}^2)(q_b{}^2+m_\pi{}^2)}, \quad (3)$$

where F_π is the pion decay amplitude,

$$M_\nu(q_b)=(q_a{}^2+m_\pi{}^2)\int dxdy\, e^{-i(q_a x+q_b\cdot y)}\cdot\times\langle\beta|T(\phi_{\pi^a}(x)A_\nu{}^b(y)H_w{}^l(0))|\alpha\rangle,$$

and

$$M=(q_a{}^2+m_\pi{}^2)(q_b{}^2+m_\pi{}^2)\int dxdy\, e^{-i(q_a\cdot x+q_b\cdot y)}\times\langle\beta|T(\phi_{\pi^a}(x)\phi_{\pi^b}(y)H_w{}^l(0))|\alpha\rangle.$$

M is equal to the amplitude for the weak decay $\alpha\rightarrow\beta+\pi^a(q_a)+\pi^b(q_b)$.

[11] R. Ramachandran (to be published).

Next we use the identity

$$\frac{\partial}{\partial x^\mu}T(J_\mu(x)B(y)C(z))=T(\partial_\mu J_\mu(x)B(y)C(z))+\delta(x_0-y_0)T([J_0(x),B(y)]C(z))+\delta(x_0-z_0)T([J_0(x),C(z)]B(y)),$$

to bring the momenta q_a and q_b into $M_{\mu\nu}$ and M_μ. Upon using the PCAC hypothesis in the operator form[12]

$$\partial_\mu A_\mu{}^a(x)=m_\pi{}^2F_\pi\phi_\pi{}^a(x)$$

and the commutation relations[13]

$$\delta(x_0-y_0)[A_0{}^a(x),A_\mu{}^b(y)]=if_{abc}V_\mu{}^c(x)\delta^4(x-y)\,,\quad(4)$$

$$\delta(x_0-y_0)[A_0{}^a(x),\phi_\pi{}^b(y)]\equiv\sigma_{ab}(x)\delta^4(x-y)\,,\quad(5)$$

and

$$\delta(x_0-y_0)[A_0{}^a(x),H_w{}^l(y)]=-d_{aln}H_w{}^n(y)\delta^4(x-y)\,,\quad(6)$$

we may observe the cancellation of all explicit pion poles to yield

$$q_{a\mu}q_{b\nu}N_{\mu\nu}=-F_\pi{}^2M-F_\pi(q_a{}^2+q_b{}^2+m_\pi{}^2)\int dx\,e^{-i(q_a+q_b)\cdot x}\langle\beta|T(\sigma_{ab}(x)H_w{}^l(0))|\alpha\rangle+F_\pi\Big[d_{aln}(q_b{}^2+m_\pi{}^2)\int dx\,e^{-iq_b\cdot x}$$
$$\times\langle\beta|T(\phi_\pi{}^b(x)H_w{}^n(0))|\alpha\rangle+d_{bln}(q_a{}^2+m_\pi{}^2)\int dx\,e^{-iq_a\cdot x}\langle\beta|T(\phi_\pi{}^a(x)H_w{}^n(0))|\alpha\rangle\Big]$$
$$-\tfrac{1}{2}(d_{aln}d_{bnp}+d_{anp}d_{bln})\langle\beta|H_w{}^p|\alpha\rangle-i(q_b-q_a)_\mu\frac{if_{abs}}{2}\int dx\,e^{-i(q_a+q_b)\cdot x}\langle\beta|T(V_\mu{}^s(x)H_w{}^l(0))|\alpha\rangle.\quad(7)$$

In the CR (4) we encounter $V_\mu{}^c(x)$, a member of the octet of vector currents.[6] In (5) we have defined the left-hand side by the right for notational convenience. In the σ model[12] the term $\sigma_{ab}(x)$ would be just $\delta_{ab}\sigma(x)$, with $\sigma(x)$ the σ-meson field. In obtaining (6) we have used the assumption that $H_w{}^l$ is an element of scalar plus pseudoscalar quark densities. If we had chosen to represent H_w as a product of currents and assumed "octet dominance" of the resulting product, we should have found f_{aln} instead of d_{aln} in (6). Our results below are unaffected by this choice, so we shall stick to the form (6).[14] The cancellation of the pion poles in (7) is, of course, independent of the CR's since $N_{\mu\nu}$ is constructed to have no pion poles.

Now we follow Weinberg[5] once again in the consideration of Eq. (7). First we shall drop the σ-meson term in the hope that it is negligible in the process we want to consider. There is some justification for this from the treatment of K_{l4} decay by Weinberg and in the paper by Weisberger[15] on the g_A sum rule. If we view this as the contribution of a 0^+ σ meson to $\alpha\to\beta+\pi^a+\pi^b$, then the discussion by Trilling[1] of this type of term in τ decay and K_{e4} decay should lead us to suspect its absence. A really honest statement about the term is that we are quite ignorant with regard to it and will neglect it henceforth in the hope that our final results will provide some *a posteriori* support for its neglect. In fact, this will turn out to be the case; namely, the final results we find for $K_{3\pi}$ decay are in superb accord with the experimental data. Thus we might argue further, though still from a moderately enlightened ignorance, for the neglect of these "σ-meson" terms whenever they appear.

Now we shall make another approximation on (7). We shall drop the $q_{a\mu}q_{b\nu}N_{\mu\nu}$ terms as small in the physical region of the process $\alpha\to\beta+\pi^a+\pi^b$. This is certainly a very good approximation in $K_{3\pi}$ decay where such slow pions are involved. The error made here is approximately $O((k\cdot q_a)(k\cdot q_b)/(k\cdot k)^2)$ or $O((m_\pi/m_K)^2)$. In dropping this term, and its counterpart in the three-pion case below, we are making our expansion in pion momenta. More precisely, we are expanding in invariants involving pion momenta. The neglect of $O(q_aq_b)$ guarantees that no terms of higher order than one in the pion energies will appear in our approximation of the matrix element for $\alpha\to\beta+\pi^a+\pi^b$.

Nothing prevents us from going to the mass shell $q_a{}^2=q_b{}^2=-m_\pi{}^2$ in Eq. (7), so we do that and rewrite it in

[12] M. Gell-Mann and M. Levy, Nuovo Cimento **16**, 705 (1960).

[13] In these CR's we ignore the "Schwinger" terms since they seem only to be important when one considers quantities like $M_{\mu\nu}$ and not $q_{a\mu}q_{b\nu}M_{\mu\nu}$. In the paper of S. L. Adler and Y. Dothan [Phys. Rev. **151**, 1267 (1966)], this point is extensively discussed in Sec. 3.

[14] Moshe Kugler was kind enough to remind me that this point had been mentioned in the work of Hara and Nambu, Ref. 10.

[15] W. I. Weisberger, Phys. Rev. **143**, 1302 (1966).

terms of invariant amplitudes A for the processes encountered:

$$A(\alpha \xrightarrow{H_w{}^l} \beta+\pi^a(q_a)+\pi^b(q_b)) = (i/F_\pi)[d_{aln}A(\alpha \xrightarrow{H_w{}^n} \beta+\pi^b(q_b))+d_{bln}A(\alpha \xrightarrow{H_w{}^n} \beta+\pi^a(q_a))] + (1/2F_\pi{}^2)[(d_{aln}d_{bnp}+d_{bln}d_{anp})\langle\beta|H_w{}^p|\alpha\rangle N_\alpha N_\beta] + \frac{if_{abs}}{2F_\pi{}^2}i(q_b-q_a)_\mu \int dx\, e^{-i(q_a+q_b)\cdot x}\langle\beta|T(V_\mu{}^s(x)H_w{}^l(0))|\alpha\rangle N_\alpha N_\beta. \quad (8)$$

N_α and N_β are normalization factors for the states α and β.

This is our final formula for the emission of two pions in the process $\alpha\to\beta+\pi^a+\pi^b$ which is mediated by the weak Hamiltonian H_w. It is an on-the-mass-shell formula whose dependence on the pion momenta is buried in the first and last terms. The extraction of this dependence in the case of $K_{3\pi}$ decays will be carried out in Sec. III.

Since we shall discuss the three-pion decay of the kaon we must "take out" all three pions involved to treat them on an equal footing. This procedure is more complicated than that used in deriving Eq. (8) only because of the lengthy algebra. As no new concepts arise we shall merely outline the task for the case of three pions.

B. Three Pions

We start by considering

$$M_{\mu\nu\lambda} = \int dxdydz\, e^{-i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle\beta|T(A_\mu{}^a(x)A_\nu{}^b(y)A_\lambda{}^c(z)H_w{}^l(0))|\alpha\rangle$$

and the corresponding $N_{\mu\nu\lambda}$. We then multiply this by $iq_{a\mu}iq_{b\nu}iq_{c\lambda}$ and use the identity

$$(\partial/\partial x^\mu)T(J_\mu(x)B(y)C(z)D(w)) = T(\partial_\mu J_\mu(x)B(y)C(z)D(w)) + \delta(x_0-y_0)T([J_0(x),B(y)]C(z)D(w)) + \delta(x_0-z_0)T([J_0(x),C(z)]B(y)D(w)) + \delta(x_0-w_0)T([J_0(x),D(w)]B(y)C(z))$$

to bring in the momenta. After dropping the "σ-meson" terms there results

$$\begin{aligned} iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda} = {} & F_\pi{}^3M-[d_{aln}M(\alpha \xrightarrow{H_w{}^n} \beta+\pi^b+\pi^c)-d_{bln}M(\alpha \xrightarrow{H_w{}^n} \beta+\pi^a+\pi^c)+d_{cln}M(\alpha \xrightarrow{H_w{}^n} \beta+\pi^a+\pi^b)]\times F_\pi{}^2 \\ & -d_{aln}\frac{if_{bcs}}{2}i(q_c-q_b)_\lambda \int dx\, e^{-i(q_c+q_b)\cdot x}\langle\beta|T(V_\lambda{}^s(x)H_w{}^l(0))|\alpha\rangle \\ & -d_{bln}\frac{if_{acs}}{2}i(q_c-q_a)_\lambda \int dx\, e^{-i(q_c+q_a)\cdot x}\langle\beta|T(V_\lambda{}^s(x)H_w{}^l(0))|\alpha\rangle \\ & -d_{cln}\frac{if_{abs}}{2}i(q_b-q_a)_\lambda \int dx\, e^{-i(q_b+q_a)\cdot x}\langle\beta|T(V_\lambda{}^s(x)H_w{}^l(0))|\alpha\rangle \\ & F_\pi\Big[+i(q_c-q_b)_\lambda\frac{if_{bcs}}{2}\int dx\, e^{-i(q_c+q_b)\cdot x}\langle\beta+\pi^a(q_a)|T(V_\lambda{}^s(x)H_w{}^l(0))|\alpha\rangle \\ & +i(q_c-q_a)_\lambda\frac{if_{acs}}{2}\int dx\, e^{-i(q_a+q_c)\cdot x}\langle\beta+\pi^b(q_b)|T(V_\lambda{}^s(x)H_w{}^l(0))|\alpha\rangle \\ & +i(q_a-q_b)_\lambda\frac{if_{bas}}{2}\int dx\, e^{-i(q_a+q_b)\cdot x}\langle\beta+\pi^c(q_c)|T(V_\lambda(x)H_w{}^l(0))|\alpha\rangle\Big]. \quad (9) \end{aligned}$$

In reaching (9) two further considerations have been made. First, we combined terms like $\langle\beta|H_w|\alpha\rangle$ and $\langle\beta+\pi|H_w|\alpha\rangle$ through the formula, true strictly in the $q_\pi\to 0$ limit,

$$F_\pi(2q_{i0})^{1/2}\langle\beta+\pi^i(q_i)|H_w{}^l|\alpha\rangle = d_{ilp}\langle\beta|H_w{}^p|\alpha\rangle. \quad (10)$$

This introduces errors of the order of $(m_\pi{}^2/m_K{}^2)$ in the case $\beta=0$, $\alpha=K$ which will be considered below. In the case that α and β should contain nucleons, say, this approximation would be quite incorrect. Second, we have removed terms like

$$f_{acs}f_{bst}(q_c-q_a)_\lambda\int dx\, e^{-i(q_a+q_c)\cdot x}\langle\beta|(A_\lambda^t(x)H_w^l(0))_+|\alpha\rangle,$$

since they also are $O((m_\pi/m_k)^2)$ below. The neglect of these terms and the approximation (10) have been made solely for convenience here. They are justified in the case of $K_{3\pi}$ decay and considerably simplify the writing of complicated formulas.

Next we use (8) to extract the dependence on pion momenta in $M(\alpha\xrightarrow{H_w^n}\beta+\pi^a+\pi^b)$ and arrive at our final formula for three-pion emission in a weak decay:

$$\begin{aligned}iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda}={}&F_\pi^3M(\alpha\xrightarrow{H_w^l}\beta+\pi^a(q_a)+\pi^b(q_b)+\pi^c(q_c))-\{d_{aln}(d_{cnm}d_{bmp}+d_{bnp}d_{cmp})\\&+d_{bln}(d_{anm}d_{cmp}+d_{cnm}d_{amp})+d_{cln}(d_{anm}d_{bmp}+d_{bnm}d_{amp})\}\frac{\langle\beta|H_w^p|\alpha\rangle}{2}\\&+F_\pi\Big[\frac{if_{abs}i(q_b-q_a)_\lambda}{2}\int dx\, e^{-i(q_b+q_a)\cdot x}\langle\beta+\pi^c(q_c)|(V_\lambda^s(x)H_w^l(0))_+|\alpha\rangle\\&+\frac{if_{acs}i(q_c-q_a)_\lambda}{2}\int dx\, e^{-i(q_c+q_a)\cdot x}\langle\beta+\pi^b(q_b)|(V_\lambda^s(x)H_w^l(0))_+|\alpha\rangle\\&+\frac{if_{bcs}i(q_c-q_b)_\lambda}{2}\int dx\, e^{-i(q_c+q_b)\cdot x}\langle\beta+\pi^a(q_a)|(V_\lambda^s(x)H_w^l(0))_+|\alpha\rangle\Big].\qquad(11)\end{aligned}$$

III. THE $K_{3\pi}$ DECAYS

In (11) we now take $\alpha=K^p(k)$, a kaon of type p and momentum k, and $\beta=0$. We immediately encounter terms like

$$H_\mu=i\int dx\, e^{-iq\cdot x}\langle\pi(q_\pi)|(V_\mu^s(x)H_w^l(0))_+|K^p(k)\rangle.\qquad(12)$$

If we multiply this by iq_μ, we find

$$iq_\mu H_\mu=if_{sln}\langle\pi|H_w^n|K\rangle,\qquad(13)$$

using the conservation of the vector current and the CR for V_0 and H_w suggested by the quark model. Keeping those terms in H_μ in which the vector current attaches to the pion or the kaon, we have, using (12),

$$H_\mu=if_{sln}\langle\pi(q_\pi)|H_w^n|K^p(k)\rangle\frac{(2k_\mu-q_\mu)}{(m_\pi^2-m_K^2)}.\qquad(14)$$

Equivalently we could have required that $iq_\mu H_\mu$ be independent of q^2, thus making an error of $O(q)$ in H_μ. In either case we make an error $O(q)$ in H_μ and thus an error $O(m_\pi^2/m_K^2)$ in the final expression for the $K_{3\pi}$ amplitude. In (11) we now discard the $iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda}$ term and using (14) and (10) find

$$\begin{aligned}A(K^p(k)\xrightarrow{H_w^l}\pi^a(q_a)+\pi^b(q_b)+\pi^c(q_c))={}&B\{d_{anl}(d_{bnm}d_{cmp}+d_{cnm}d_{bmp})+d_{bnl}(d_{anm}d_{cmp}+d_{cnm}d_{amp})\\&+d_{cnl}(d_{bnm}d_{amp}+d_{anm}d_{cmp})+f_{sln}(d_{anp}f_{bcs}k\cdot(q_c-q_b)/\Delta+d_{bnp}f_{acs}k\cdot(q_c-q_a)/\Delta+d_{cnp}f_{abs}k\cdot(q_b-q_a)/\Delta)\},\qquad(15)\end{aligned}$$

where B is just a constant and $\Delta=(m_\pi^2-m_K^2)/2$. For the various $K_{3\pi}$ decays Eq. (15) yields

$$A(K^+\to\pi_1^0+\pi_2^0+\pi^+)=C\{1+k\cdot[(q_{10}+q_{20})-2q_+]/\Delta\},\qquad(16)$$

$$A(K^+\to\pi_1^++\pi_2^++\pi^-)=2C\{1+k\cdot[2q_--(q_{1+}+q_{2+})]/2\Delta\},\qquad(17)$$

$$A(K_2^0\to3\pi^0)=3C',\qquad(18)$$

and

$$A(K_2^0\to\pi^++\pi^-+\pi^0)=C'\{1+k\cdot[(q_-+q_+)-2q_0]/\Delta\}.\qquad(19)$$

C' and C are two more constants. Rewriting these formulas in the form (1), we immediately find the explicit dependence on the "odd" pion energies and

$$a(+-0)=a(+00)=-2a(++-),\qquad(20)$$

the predictions of the $\Delta I=\frac{1}{2}$ rule,[1] which we have implicitly assumed. For $a(+-0)$, for which Trilling gives the most independent measurements, Eq. (19) predicts

$$a(+-0)=3m_\pi^2/2\Delta=-3m_\pi^2/(m_K^2-m_\pi^2)=-0.26 \quad (21)$$

while the experiments[1] yield

$$a(+-0)_{\rm expt}=-0.24\pm0.02. \quad (22)$$

Needless to say, this is very satisfactory.[16] Since the data on $K_{3\pi}$ decay are compatible with the $\Delta I=\frac{1}{2}$ rule, (20) is also satisfied.

The results (16)$\cdots$(19) also make certain relative rate predictions which are those of the $\Delta I=\frac{1}{2}$ rule. In addition (18) suggests that the spectrum of $K_2^0\to 3\pi^0$ should be flat. There are no data on the pion spectrum in this decay, so we can make no comment on the prediction of a flat spectrum.

IV. DISCUSSION

In the paragraphs above we have shown how one may expand the decay amplitude for $K\to 3\pi$ in the momenta of the pions. We assumed nothing *a priori* about the structure of the matrix element responsible for the decay except that we might hope to find that terms quadratic in pion momenta were indeed negligible. The arguments of Weinberg in Ref. 2 would tell us that if we found any linear energy dependence in the K^+ decays, Bose statistics would force this dependence to be in the energy of the "odd" pion. The $\Delta I=\frac{1}{2}$ rule extends this argument to the K_2^0 decays. Therefore, we should not be amazed that when we did encounter an energy dependence, as in (16), (17), and (19), it was on the "odd" pion energy, although it must be admitted that one did not known beforehand that even the linear dependence would not vanish leaving a constant matrix element. The real tests of the theory, then, once a linear dependence is established, are the slope and sign of the slope of the matrix element. Equation (21) is certainly pleasing in this respect.

[16] In the spirit of our approximations we could neglect the m_π^2 in and then arrive at $a(+-0)=-0.24$! We keep it, however, since it arises quite naturally in Eq. (14).

Our results for the $K_{3\pi}$ decay ampliutdes (16)$\cdots$(19) certainly give further support to the validity of the CR's (4) and (6) and to the correctness of PCAC. In addition, and probably more important here, they are another demonstration of the potentialities of the technique of expanding amplitudes involving "soft" pions in the momenta of those pions. One should not be deceived, however, that all processes will be as easy to treat as those involving only kaons and pions since as soon as α and β contain baryons, one must be very careful about "pole terms" where the axial current connects one baryon to another. The on-the-mass-shell aspect of the technique should also be emphasized as it eliminates much of the confusion which has been present heretofore about "off-the-mass-shell corrections."

Finally let us venture a few more remarks about the σ-meson-like terms we encounter above. The neglect of them has been justified *a posteriori* by the success of our results (21), it is true, but it doesn't really tell us that their neglect should be a general feature of multiple "soft"-pion emission calculations. Instead we can either appeal to other calculations in the same spirit[5,9,15] or to the treatment of pion scattering lengths by Weinberg[17] for support. It remains a conjecture, however, that such terms are negligible whenever they appear. If the proper interpretation of these terms is that they represent the σ-meson, or a $T=0$, 0^+ π-π resonance, then the possibility of ignoring them would be pleasing indeed since one of the "properties" of this elusive "meson" is the necessity to assign it different masses and widths in τ decay, η decay, and K_{e4} decay.

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[17] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).