as a photon does in ep scattering. This intuitive picture is useful in further understanding the conditions of applicability of the present considerations. The mechanism described in this paper can dominate the scattering amplitude if the excitation spectrum in the s channel is a "mild" one, so that at infinite energy there is no appreciable contribution from "compound" reactions. Our theory thus describes a situation just opposite to the one where the application of a statistical model is justified.

Roughly speaking, a statistical model assumes that the spectrum in the s channel is infinitely complicated—"compound-nucleon" formation persists up to the highest energies—whereas according to our basic assumption ours is (if not the best), in a sense, the simplest of all possible worlds. It remains to be seen whether this is indeed the case.

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Bound States of a Relativistic Two-Body Hamiltonian; Comparison with the Bethe-Salpeter Equation*

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The tightly bound states of a simple relativistic two-body Hamiltonian are studied. The coupling constant necessary for obtaining a given binding energy is obtained numerically for a Yukawa-like interaction with variable range. Some fairly general relations restricting the connection between binding energy, coupling constant, and force range, expected to be valid in the tight-binding limit, are derived and tested. A comparison is made with results obtained by Schwartz for the "corresponding" Bethe-Salpeter (B-S) equation. It is concluded that, even in the strong-binding limit, the pair, multimeson, and retardation effects taken into account by the ladder-approximation B-S equation are not very important, at least as far as the relation between coupling constant and binding energy is concerned. These results suggest that a Hamiltonian of the type considered may be a useful tool in exploratory calculations involving quark models. In this connection, we show that a Yukawa-like interaction leads to relativistic motion in the tight-binding limit even if used in a Hamiltonian incorporating relativistic kinematics, and we thereby generalize a result of Greenberg's.

I. INTRODUCTION

THERE has been a considerable revival of interest recently in composite models of elementary particles, mainly in connection with the possibility that heavy triplets such as quarks exist, transforming according to the defining, three-dimensional representation of $SU(3)$. The observed particles might then in some sense be regarded as bound states of the triplet particles and antiparticles.

The purpose of this note is to present some results on the bound states of a relatively simple two-body equation describing the interaction of two particles and to make a comparison with analogous results obtained by Schwartz[1] for the "corresponding" Bethe-Salpeter equation.

The general form of the equation we wish to consider was first suggested by Bakamjian and Thomas,[2] and analyzed from the viewpoint of relativistic scattering theory by Fong and Sucher.[3] For a stationary state ψ of energy E, the equation has the ordinary Schrödinger form

$$H\psi=E\psi\,, \tag{1a}$$

where H is a linear operator in the two-particle Hilbert space $\mathcal{H}$ spanned by plane-wave product states $|\mathbf{p}_1,\mathbf{p}_2\rangle$. For the case of two spinless particles of mass m_1 and m_2, the general form of H is

$$H=[(h^{\mathrm{op}})^2+(\mathbf{P}^{\mathrm{op}})^2]^{\frac{1}{2}}\,, \tag{1b}$$

with

$$\mathbf{P}^{\mathrm{op}}=\mathbf{p}_1^{\mathrm{op}}+\mathbf{p}_2^{\mathrm{op}}\,,$$
$$h^{\mathrm{op}}=E_1(\mathbf{k}^{\mathrm{op}})+E_2(\mathbf{k}^{\mathrm{op}})+v^{\mathrm{op}}\,,$$

and

$$E_i(\mathbf{p})=(m_i^2+\mathbf{p}^2)^{\frac{1}{2}},\quad (i=1,2)\,.$$

Here, the $\mathbf{p}_i^{\mathrm{op}}$ are the single-particle momentum operators, and the interaction operator v^{op} is a more or less arbitrary function of $\mathbf{k}^{\mathrm{op}}$, the c.m. momentum operator, and the operator $\boldsymbol{\varrho}^{\mathrm{op}}$, canonically conjugate to $\mathbf{k}^{\mathrm{op}}$.

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[1] C. Schwartz, Phys. Rev. **137**, B717 (1965). See also C. Schwartz and C. Zemach, *ibid.* **141**, 1454 (1966).

[2] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[3] R. Fong and J. Sucher, J. Math. Phys. **5**, 456 (1964).

As was shown in Ref. 3, the theory described by Eqs. (1a) and (1b) is covariant from the viewpoint of scattering theory. That is, the S matrix transforms appropriately under Lorentz transformations of the three-vector momenta on which it depends. The basic reason is that the two representations of the homogeneous Lorentz group $\{\Lambda\}$ determined respectively by the mappings $\psi_{\mathbf{q}_1,\mathbf{q}_2}{}^{(+)}\rightarrow\psi_{\Lambda\mathbf{q}_1,\Lambda\mathbf{q}_2}{}^{(+)}$ and $\psi_{\mathbf{q}_1,\mathbf{q}_2}{}^{(-)}\rightarrow\psi_{\Lambda\mathbf{q}_1,\Lambda\mathbf{q}_2}{}^{(-)}$ are *identical* when H has the form (1b). (Here $\psi_{\mathbf{q}_1,\mathbf{q}_2}{}^{(+)}$ is an "in" state with asymptotic momenta $\mathbf{q}_1$ and $\mathbf{q}_2$.)

The bound states of Eq. (1) may be studied as follows. Let $\psi_{b,Q}$ denote a bound state of mass m_b, and three-momentum $\mathbf{Q}$; if the theory is to give a covariant description of bound states also, the energy of the state must then be

$$E_b(\mathbf{Q})=(m_b{}^2+\mathbf{Q}^2)^{\frac{1}{2}}.$$

Thus we need

$$H\psi_{b,Q}=E_b(\mathbf{Q})\psi_{b,Q}, \qquad (2a)$$

$$\mathbf{P}^{\mathrm{op}}\psi_{b,Q}=\mathbf{Q}\psi_{b,Q}. \qquad (2b)$$

We now introduce simultaneous eigenstates $|\mathbf{k},\mathbf{P}\rangle$ of $\mathbf{k}^{\mathrm{op}}$ and $\mathbf{P}^{\mathrm{op}}$ as a basis in $\mathcal{H}$, and define a bound-state wave function $\phi_b(\mathbf{k})$ via

$$\langle\mathbf{k},\mathbf{P}|\psi_{b,Q}\rangle=\delta(\mathbf{P}-\mathbf{Q})\phi_b(\mathbf{k}). \qquad (3)$$

With the ansatz (3), Eq. (2b) is satisfied and Eq. (2a) may be seen to be equivalent to

$$\int\langle\mathbf{k}|[(h^{\mathrm{op}})^2+\mathbf{Q}^2]^{\frac{1}{2}}|\mathbf{k}'\rangle\phi_b(\mathbf{k}')d\mathbf{k}'=(m_b{}^2+\mathbf{Q}^2)^{\frac{1}{2}}\phi_b(\mathbf{k}). \qquad (4)$$

A necessary and sufficient condition that Eq. (4) be satisfied for all $\mathbf{Q}$ is that it be satisfied for $\mathbf{Q}=0$, in which case it reduces to

$$[E_1(\mathbf{k})+E_2(\mathbf{k})-m_b]\phi_b(\mathbf{k})=-\int v(\mathbf{k},\mathbf{k}')\phi_b(\mathbf{k}')d\mathbf{k}', \qquad (5)$$

where

$$v(\mathbf{k},\mathbf{k}')\equiv\langle\mathbf{k}|v^{\mathrm{op}}|\mathbf{k}'\rangle.$$

$\phi_b(\mathbf{k})$ must be a square integrable function of $\mathbf{k}$ since the normalization condition

$$\langle\psi_{b,Q}|\psi_{b,Q'}\rangle=\delta(\mathbf{Q}-\mathbf{Q}'),$$

appropriate for the states of an "elementary system" of mass m_b, implies

$$\int|\phi_b(\mathbf{k})|^2d\mathbf{k}=1.$$

Note that the assumption made in (3) that $\phi_b(\mathbf{k})$ is independent of $\mathbf{Q}$ is justified by Eq. (5) in which $\mathbf{Q}$ does not appear.

We shall study the relation between binding energy and coupling constant for the equal-mass case,

$$m_1=m_2=m,$$

for which Eq. (5) reduces, with $E(k)=(\mathbf{k}^2+m^2)^{\frac{1}{2}}$, to

$$[2E(k)-m_b]\phi_b(\mathbf{k})=-\int v(\mathbf{k},\mathbf{k}')\phi_b(\mathbf{k}')d\mathbf{k}', \qquad (6a)$$

and for $v(\mathbf{k},\mathbf{k}')$ of the form

$$v(\mathbf{k},\mathbf{k}')=-\frac{g^2}{(2\pi)^3}\frac{m}{E(k)}\frac{1}{(\mathbf{k}-\mathbf{k}')^2+\mu^2}\frac{m}{E(k')}. \qquad (6b)$$

This choice of v is motivated by two circumstances:

(i) For $|\mathbf{k}|=|\mathbf{k}'|$, $v(\mathbf{k},\mathbf{k}')$ is proportional to the Born approximation to the scattering amplitude for the scattering of the two particles of mass m which would be induced, symmetrization being ignored, by the interaction

$$H_I=G\int:\Phi^\dagger(x)\Phi(x)\phi(x):d\mathbf{x}. \qquad (7)$$

Here the field $\Phi(x)$ describes the quanta of mass m and $\phi(x)$ is a Hermitian spin-zero field describing "mesons" of mass μ. The factors $m/E(k)$ in Eq. (6b) arise automatically in this theory, and the correspondence is exact if we set

$$G=2mg.$$

(ii) The Bethe-Salpeter (B-S) equation[4] for a bound state of four-momentum P is, for the theory described by (7) and in the ladder approximation,

$$[(\tfrac{1}{2}P+p)^2-m^2][(\tfrac{1}{2}P-p)^2-m^2]\phi_P(p)=\frac{-iG^2}{(2\pi)^4}\int\frac{d^4k}{(p-k)^2-\mu^2}\phi_P(k). \qquad (8)$$

This equation has been studied numerically in Ref. 1, so results are available for comparison. Both Eqs. (6) and (8) have the same nonrelativistic limit, the equal-mass Schrödinger equation for a Yukawa potential $V(r)$:

$$V(r)=(-g^2/4\pi)(e^{-\mu r}/r).$$

It is therefore of interest to compare the binding energies predicted by these equations for the same value of the coupling constant, for the case of strong binding (and relativistic motion).

From the viewpoint of the hypothetical underlying field theory determined by Eq. (7), Eq. (6) differs from Eq. (8) in the following sense: The B-S equation sums (exactly) the effects of the ladder diagrams corresponding to repeated meson exchange. Each of these Feynman diagrams corresponds to a set of time-ordered diagrams. Equation (6) may be regarded as summing (approximately) those time-ordered diagrams in which there are no intermediate states containing any extra particle-antiparticle pairs, or more than one meson, and with

[4] H. A. Bethe and E. E. Salpeter, Phys. Rev. 84, 1232 (1951).

retardation neglected. A comparison of these two equations thus gives some insight into the importance of such effects as far as binding energies are concerned.

The method of numerical solution of Eq. (6) is briefly described in Sec. II. Some general features of the relation between coupling constant, binding energy, and force range for a tightly bound system are discussed in Sec. III. The numerical results are described in Sec. IV and compared with the fairly general relations obtained in Sec. III. A comparison is also made with results available for the corresponding B-S equation. Finally, the relevance of the results for models of mesons as quark-antiquark bound states is briefly discussed.

II. NUMERICAL METHOD

Consider any equation of the form

$$[2E(k)-m_b]\phi_b(\mathbf{k})=-\int V(\mathbf{k},\mathbf{k}')\phi(\mathbf{k}')d\mathbf{k}', \qquad (9)$$

with $V(\mathbf{k},\mathbf{k}')$ invariant under rotations:

$$V(\mathbf{k},\mathbf{k}')=V(k,k';\hat{k}\cdot\hat{k}').$$

We may write, with $x=\hat{k}\cdot\hat{k}'$,

$$V(\mathbf{k},\mathbf{k}')=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})V_l(k,k')P_l(x), \qquad (10)$$

where

$$V_l(k,k')=\int_{-1}^{1}V(k,k';x)P_l(x)dx$$

and $P_l(x)$ is the Legendre polynomial. For a bound state of angular momentum l, we set

$$\phi_b(\mathbf{k})=\phi_l(k)Y_{lm}(\hat{k}), \qquad (11)$$

where $Y_{lm}(\hat{k})$ is the usual spherical harmonic. On substitution of (10) and (11) into (9), we get, using

$$\int P_l(\hat{k}\cdot\hat{k}')Y_{lm}(\hat{k}')d\hat{k}'=\frac{4\pi}{2l+1}Y_{lm}(\hat{k}),$$

the reduced equation

$$[2E(k)-m_b]\phi_l(k)=-2\pi\int_0^{\infty}V_l(k,k')\phi_l(k')k'^2dk'. \qquad (12)$$

We restrict ourselves to $l=0$ and the choice for $V(\mathbf{k},\mathbf{k}')$ implied by Eq. (6b). Then

$$V_0(k,k')=-\frac{g^2}{(2\pi)^3}\frac{1}{2\kappa k'}\;\frac{k')^2+\mu^2}{(k-k')^2+\mu^2}\times\frac{m}{E(k)}\times\frac{m}{E(k')}.$$

For the purpose of numerical computation it is convenient to introduce new variables y, y' with

$$k=m\times\frac{1+y}{1-y}$$

and y' similarly defined, and a new function

$$X(y)=k\phi_0(k).$$

Equation (12) may then be rewritten in the form

$$\lambda^{-1}X(y)=\int_{-1}^{1}K(y,y')X(y')dy', \qquad (13)$$

where

$$\lambda=g^2/4\pi \qquad (14)$$

and

$$K(y,y')=[2E(k)-m_b]^{-1}(-8\pi^2/g^2)V_0(k,k')kk'(dk'/dy').$$

Equation (13) may be regarded as an eigenvalue problem for the quantity λ^{-1} and may be solved numerically by standard methods.[5] Use of Gauss's n-point quadrature formula leads to the condition

$$\det(A-\lambda^{-1})=0, \qquad (15)$$

where A is an $n\times n$ matrix, with $A_{ij}=R_jK(y_i,y_j')$, where R_i and the mesh points y_i are chosen according to the method of Gauss. Equation (15) may be written in the form

$$(\lambda^{-1})^n-\sum_{j=1}^{n}P_j(\lambda^{-1})^{n-j}=0, \qquad (16)$$

and the P_j calculated by the method of Leverrier-Faddeev.[6]

For a given value of μ/m and m_b/m, the left-hand side of Eq. (16) was computed for values of λ in the interval (0.05, 20), and the smallest value of λ for which the left-hand side of Eq. (16) vanishes was found. The results are given in Table I and discussed in the final section.

TABLE I. Value of coupling constant $\lambda=g^2/4\pi$ giving an S-wave bound state of mass $m_b=\xi m$ for various values of inverse force range $\mu=\eta m$.

η \ ξ	0.0	0.1	0.2	0.3
0.1	6.24	6.00	5.76	5.52
0.2	7.19	6.92	6.65	6.38
0.3	8.09	7.79	7.50	7.20
0.4	8.99	8.67	8.35	8.03
0.5	9.92	9.57	9.23	8.88

[5] J. B. Scarborough, *Numerical Mathematical Analysis* (The Johns Hopkins Press, Baltimore, Maryland, 1955), 3rd ed.

[6] D. K. Faddeev and V. N. Faddeeva, *Computational Methods of Linear Algebra* (W. H. Freeman and Company, San Francisco, 1963).

III. SOME GENERAL FEATURES OF THE RELATIVISTIC TWO-BODY PROBLEM

The purpose of this section is (1) to derive first an approximate relation restricting the variation of the coupling constant with binding energy for a highly relativistic system, and (2) to obtain a similar relation on the variation of the coupling constant with the range of the force, for the case of a Yukawa-like interaction.

A. Variation of λ with Binding Energy

Consider an equation of the form

$$H\phi = m_b\phi\,, \tag{17}$$

where

$$H = K + \lambda V\,, \tag{18}$$

with λ a variable coupling constant, V a fixed interaction operator, and

$$K = 2[(\mathbf{p}^{\mathrm{op}})^2 + m^2]^{\frac{1}{2}}. \tag{19}$$

We can regard both λ and ϕ as functions of the mass m_b of the bound state. With

$$\langle\phi|\phi\rangle = 1\,, \tag{20}$$

we have

$$\langle\phi|H|\phi\rangle = m_b. \tag{21}$$

Using a prime to indicate differentiation with respect to m_b, we get on differentiation of Eq. (21),

$$\langle\phi|H'|\phi\rangle = 1\,, \tag{21a}$$

the terms involving ϕ' vanishing because Eq. (20) implies

$$\langle\phi|\phi'\rangle + \langle\phi'|\phi\rangle = 0.$$

Since

$$H' = (\partial\lambda/\partial m_b)V \quad \text{and} \quad \langle\phi|V|\phi\rangle = \lambda^{-1}(m_b - \langle\phi|K|\phi\rangle)\,,$$

Eq. (21a) may be rewritten in the form

$$\lambda^{-1}\frac{\partial\lambda}{\partial m_b}(m_b - \langle\phi|K|\phi\rangle) = 1. \tag{22}$$

On defining an *effective* mean momentum p_0 by

$$\langle\phi|2((\mathbf{p}^{\mathrm{op}})^2 + m^2)^{\frac{1}{2}}|\phi\rangle \equiv 2(p_0^2 + m^2)^{\frac{1}{2}}\,, \tag{23}$$

and introducing a dimensionless parameter ξ by

$$\xi = m_b/m\,,$$

Eq. (22) assumes the form

$$\left(\lambda^{-1}\frac{\partial\lambda}{\partial\xi}\right)[\xi - 2(1 + (p_0/m)^2)^{\frac{1}{2}}] = 1. \tag{24}$$

Now in the extreme relativistic limit, with

$$\xi \ll 2\,,$$

we expect, roughly,

$$p_0/m \sim 1.$$

As an example, assuming, in anticipation of the numerical results, that

$$0.8 \leq p_0/m \leq 1\,, \tag{25a}$$

for

$$0 \leq \xi \leq 0.3\,, \tag{25b}$$

the exact Eq. (24) implies that

$$\left(\lambda^{-1}\frac{\partial\lambda}{\partial\xi}\right)(-2\sqrt{2}) \approx 1 \tag{26}$$

to better than 20% accuracy, *regardless of the precise form of V*. Of course, the very requirement that the motion be relativistic for strong binding itself imposes restrictions on the shape of V, as the example of the square well shows.

B. Variation of λ with Range of Force

Suppose that V, in Eq. (18), depends on a parameter μ. On differentiation of Eq. (21) with respect to μ, with m_b and m being kept fixed, we obtain

$$\lambda^{-1}\frac{\partial\lambda}{\partial\mu} = -\left\langle\phi\left|\frac{\partial V}{\partial\mu}\right|\phi\right\rangle \Big/ \langle\phi|V|\phi\rangle. \tag{27}$$

For a Yukawa-like potential V, such as defined by Eq. (6b), we have approximately

$$\frac{\partial V}{\partial\mu} \approx -rV.$$

This suggests that in this case we define an *effective* mean separation r_0 of the constituents of the bound state by

$$\langle\phi|\partial V/\partial\mu|\phi\rangle \equiv -r_0\langle\phi|V|\phi\rangle. \tag{28}$$

On defining

$$\eta = \mu/m\,,$$

Eq. (27) then assumes the form

$$\lambda^{-1}(\partial\lambda/\partial\eta) = mr_0. \tag{29}$$

Although neither p_0 nor r_0 are, respectively, identical with the root-mean-square momentum $\bar{p}$ and separation $\bar{r}$ defined in the usual way, we may expect that $\bar{p}/p_0 \sim 1$ and $\bar{r}/r_0 \sim 1$, so that if $\bar{p}\bar{r} \sim 1$ (from the uncertainty principle) we also have

$$p_0r_0 \sim 1\,, \tag{30}$$

say to within a factor of 2. From Eqs. (29) and (30), we then expect, for relativistic motion ($m/p_0 \sim 1$) that

$$\lambda^{-1}(\partial\lambda/\partial\eta) \sim 1\,, \tag{31}$$

to within a factor of two or so.

The approximate relations (26) and (31) of this section are compared with the numerical results in the next section.

IV. RESULTS AND DISCUSSION

A. Discussion of Numerical Results

The value of the coupling constant $\lambda = g^2/4\pi$ necessary to first produce a bound state of mass $m_b < 2m$ is given in Table I for a variety of values of $\xi = m_b/m$ and $\eta = \mu/m$.

As a check on the method of computation, the value of the coupling constant necessary to bind (scalar) nucleons into the deuteron state was found. Thus, when we put $\mu = m_\pi$, $m = (m_p + m_n)/2$, and $m_b = m_d$, and hence

$$\xi = 1.998, \quad \eta = 0.15,$$

numerical solution of Eq. (6) gave

$$\lambda = 0.376. \qquad (32)$$

This agrees quite well with the value obtained from solution of the ordinary Schrödinger equation,[7] as it should; in the nonrelativistic limit the factors $m/(m^2+p^2)^{\frac{1}{2}}$ included in Eq. (6b) are practically unity for values of p for which the wave function is large.

For the strong-binding situation considered in Table I, the corresponding coupling constant is of course much larger; inspection of Table I shows that

$$5 < \lambda < 10 \qquad (33)$$

for $0 \leq \xi \leq 0.3$, $0 \leq \eta \leq 0.5$. Equation (33) can be roughly understood by noting that for very strong binding, Eqs. (18) and (21) imply, on neglecting m_b compared to $\langle\phi|K|\phi\rangle$ and using Eq. (23), that

$$\lambda \approx -2(p_0^2+m^2)^{\frac{1}{2}}/\langle\phi|V|\phi\rangle. \qquad (34)$$

Since $V \approx -r^{-1}e^{-\mu r}$, we may define another effective separation r_1 by

$$\langle\phi|V|\phi\rangle = -r_1^{-1}\langle\phi|e^{-\mu r}|\phi\rangle \qquad (35)$$

and we may expect that $r_1 \sim r_0$. Since the second factor on the right-hand side of Eq. (35) is certainly less than unity, Eq. (34) assumes the form

$$\lambda > 2(p_0^2+m^2)^{\frac{1}{2}} r_1.$$

Assuming $r_1 \sim r_0 \sim 1/p_0$ and $p_0/m \sim 1$, we get

$$\lambda \gtrsim 3,$$

in agreement with Eq. (33).

TABLE II. Value of $-\lambda^{-1}(\partial\lambda/\partial\xi)2\sqrt{2}$ for a variety of values of $\xi = m_b/m$ and $\eta = \mu/m$.

η \ ξ	0.05	0.15	0.25
0.1	1.11	1.16	1.22
0.2	1.08	1.13	1.18
0.3	1.05	1.10	1.15
0.4	1.02	1.07	1.12
0.5	1.00	1.04	1.09

[7] $\lambda \simeq 0.35$; see, e.g., R. G. Sachs and M. Goeppert-Mayer, Phys. Rev. **53**, 991 (1938).

More interesting than the magnitude of λ is its variation with ξ and η. From Table I it can be seen that, in the domain under consideration, λ is linear in ξ for fixed η, and linear in η for fixed ξ, to a high degree of accuracy. This feature of the numerical results is closely related to the approximate relations derived in Sec. III to which we now turn. According to Eq. (26) we expect that, if Eq. (25a,b) is satisfied,

$$(\lambda^{-1}\partial\lambda/\partial\xi)(-2\sqrt{2}) \approx 1 \qquad (36)$$

to better than 20% accuracy. In Table II, the left-hand side of Eq. (36) is tabulated for a variety of values of ξ and η. It can be seen that the values obtained are indeed very close to unity, confirming the validity of Eq. (36), which, it is to be emphasized, is independent of the form of the interaction V. Conversely, Table I and the exact relation (24) can be used to evaluate $p_0 = p_0(\xi,\eta)$ to good accuracy, without knowledge of the wave function; more simply, the accuracy with which Eq. (36) holds shows that in the domain of interest we indeed have Eq. (25a) satisfied, i.e.,

TABLE III. Value of $\lambda^{-1}(\partial\lambda/\partial\eta)$ for a variety of values of $\xi = m_b/m$ and $\eta = \mu/m$.

η \ ξ	0.0	0.1	0.2	0.3
0.15	1.41	1.42	1.43	1.45
0.25	1.18	1.19	1.20	1.21
0.35	1.06	1.07	1.08	1.09
0.45	0.97	0.98	0.99	1.00

$$p_0 \approx m,$$

and the motion is relativistic.

The linearity in ξ can now be understood since Eq. (36) implies, on regarding it as an approximate differential equation for $\lambda = \lambda(\xi,\eta)$, that

$$\lambda(\xi,\eta) \approx e^{-\xi/(2\sqrt{2})}\lambda(0,\eta),$$

and

$$e^{-\xi/2\sqrt{2}} \approx 1 - \xi/2\sqrt{2}$$

for $0 \leq \xi \leq 0.3$, to better than 1%.

Turning now to the behavior of λ as a function of η, we recall that Eq. (31) should hold to within a factor of 2 or so, or, equivalently,

$$0.5 < \lambda^{-1}(\partial\lambda/\partial\eta) < 2. \qquad (37)$$

Values of $\lambda^{-1}\partial\lambda/\partial\eta$ obtained by interpolation from Table I are shown in Table III. It can be seen that they satisfy the inequality

$$0.9 < \lambda^{-1}(\partial\lambda/\partial\eta) < 1.5,$$

so that (37) is very well satisfied, indeed with surprising accuracy, since the argument leading to (31) was quite rough. The appropriate linearity in η again follows more or less from (37) as the linearity in ξ followed from (36), since, e.g., $e^{+\eta}\approx 1+\eta$ to 8% or better for $0\leq\eta\leq 0.5$.

B. Comparison with the Bethe-Salpeter Equation

In Table IV the value of λ is listed separately for the case $\mu=m(\eta=1)$, for a variety of values of ξ. This facilitates comparison with numerical calculations made by Schwartz[1] for this case for the "corresponding" Bethe-Salpeter equation, Eq. (8) of this paper. As is seen, the values of λ giving the same binding energy are equal to within 10% or better, for $\xi\geq 0.8$, and are not very different, even for $\xi=0$.

These results indicate that even in the strong-binding limit the pair effects, multimeson effects, and single-meson retardation effects taken into account by the ladder-approximation B-S equation are not very important, at least as far as the connection between coupling constant and binding energy is concerned, when $\eta\sim 1$. We do not expect this conclusion to depend very sensitively on the value of η.

C. Implications for Quark Models

There have been a number of proposals to consider hadrons as bound states of massive quarks, the motion being nonrelativistic despite the large binding energy[8] as a result of a sufficiently broad potential well. It has been emphasized by Greenberg[9] that the motion is necessarily relativistic if the quark and antiquark interaction is a Yukawa potential inserted in the nonrelativistic Schrödinger equation. One might raise the question as to whether this continues to be so when a Yukawa-type interaction is used in a relativistic wave equation in which the correct expression for the kinetic energy is used and in which kinematic factors which suppress the importance of high momenta, such as $m/E(p)$, are present as in Eq. (6). The present investigation shows that in fact the relativistic motion persists even in this case. The similarity of the results obtained from Eq. (6) to those from Eq. (8) strongly suggests that relativistic motion will also result from the latter, the B-S equation, in the strong-binding limit.

TABLE IV. Value of coupling constant λ needed to produce an S-wave bound state of mass $m_b=\xi m$ (for $\eta=\mu/m=1$) from Eq. (6) and for Eq. (8), the Bethe-Salpeter equation in ladder approximation.

η \ ξ	0.0	0.8	1.2	1.6	1.8
1.0 (present work)	14.9	11.0	8.86	6.47	5.03
1.0 (Ref. 1)	10.1	9.70	8.55	6.60	5.25

D. Concluding Remarks

The results of this paper suggest that an equation such as Eq. (6), which may be generalized to include spin and spin-dependent interaction, may be a useful tool in the study of highly relativistic bound states. Its main advantage over an equation of the Bethe-Salpeter type is that for bound states one is led in general to an integral equation involving only one variable, which facilitates numerical computations considerably. As has been seen, one can expect the results to be similar to those arising from the "corresponding" B-S equation, so that especially for exploratory, speculative type of calculations (e.g., quark models) the question of "field-theoretic origin" need not trouble one.[10] These remarks have added force for the three-body problem (baryon states in quark models) where a generalization[11] of Eq. (6) seems still to be tractable from a numerical point of view, while use of a three-body B-S equation would present quite formidable difficulties in the strong-binding limit.

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[9] O. W. Greenberg, Phys. Rev. 147, 1077 (1966).

[10] Indeed, if one so chooses, Eqs. (6)–(8) may be regarded simply as different covariant approximations to the same field-theoretic problem.

[11] F. Coester, Helv. Phys. Acta 38, 7 (1965).