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# New Consistency Conditions on Pion-Pion Amplitudes and Their Determination to Fourth Order in External Momenta*

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We derive a set of new consistency conditions for the pion-pion scattering amplitude. These conditions hold for any $s$, $t$, $u$ in the cube $0\leq s, t, u\leq\mu^2$, with the four external mass variables off the mass shell and restricted so that $q_1^2=0$, $q_2^2=s$, $q_3^2=t$, and $q_4^2=u$. Using these consistency conditions, we determine the coefficients of the power-series expansion of the pion-pion amplitude up to and including second-order terms in the variables $s$, $t$, $u$, and $q_i^2$. We use this expansion to calculate the pion-pion $S$-wave scattering lengths and thus check the consistency of Weinberg's recent calculation of these numbers to the next higher order. The final result is within 10% of that obtained by Weinberg.

## I. INTRODUCTION

In a recent paper Weinberg[1] has used current algebra to calculate the pion-pion $S$-wave scattering lengths. The answer he obtained is smaller by at least a factor of 5 from what had been believed to be reasonable estimates of the $\pi\pi$ scattering lengths from dispersion theory or comparison of peripheral models with experiment.

Weinberg's result does not follow from current algebra alone. The restrictions given by current algebra and partially conserved axial-vector current (PCAC) on the $\pi\pi$ amplitude give us information at unphysical points and unphysical external masses. The problem is to extrapolate these results to the physical threshold. This is relatively easy in the case of $\pi N$ scattering where there is a small number, $\mu/M$, and where one neglects terms of order $\mu^2/M^2$, etc. For $\pi\pi$ scattering there is no such number. What Weinberg does to effect an extrapolation is to expand the amplitude in a power series of $s$, $t$, $u$, and the external mass variables, $q_i^2$, $i=1, 2, 3, 4$, and keep terms only up to first order in these variables. One can then determine the three coefficients in the expansion from Adler's consistency condition and a low-energy theorem for $\pi\pi$ scattering. Once the coefficients are known one assumes the expansion is still good up to threshold and calculates the scattering lengths.

Such a method of extrapolation is rather dangerous. It is known that the expansion used is divergent at threshold. One can get around this difficulty by assuming that the unitarity branch point is a weak singularity which allows us to use the expansion at least as an asymptotic expansion up to and maybe a little beyond threshold. Since Weinberg gets small scattering lengths in the end his argument is self-consistent, but it does not in fact prove that the scattering lengths are indeed small. Even if one accepts the asymptotic nature of the expansion one does not *a priori* know at what order it gives a good approximation to the amplitude near threshold. There is no *a priori* reason, for example, to assume that the second-order terms, $s^2$, $st$, $u^2$, etc. are smaller than the first-order terms. One would feel much more at ease with Weinberg's results if one were able to estimate these higher order terms and compare them with the lower order ones. This becomes even more pertinent when we recall that the results of Ref. 1 give much smaller scattering lengths than had been expected from previous arguments.

In this paper we derive a set of new consistency conditions on the $\pi\pi$ amplitude that hold in addition to the Adler[2] consistency condition. We then use these consistency conditions to estimate the coefficients of the expansion of the $\pi\pi$ amplitude to second order in the variables $s$, $t$, $u$, and $q_i^2$. The remarkable result is that the second-order terms turn out to be negligible and Weinberg's results are essentially unchanged within our approximations.

Adler has derived consistency conditions on $\pi N$ and $\pi\pi$ scattering which hold with one pion taken off the mass shell.[2] If one tries to derive a consistency condition for $\pi N$ scattering with two pions off the mass shell, then one has to estimate the matrix element of a scalar density between two nucleon states.[3] This scalar density essentially arises from the equal-time commutator of the axial-vector charge with the divergence of the axial-vector current. Thus as in Ref. 3 one does not get a new consistency condition but a relation between the scalar matrix element and $\pi N$ scattering.

In the case of $\pi\pi$ scattering it turns out that one can essentially eliminate the matrix element of the scalar density between two single-pion states and get new and stronger consistency conditions. The main new tool that one needs to do this is to know the equal-time commutator of the axial-vector charge with the scalar density. There are several ways to do this, all leading to the same answer for our purposes. One can use directly the commutators of the axial-vector charge with the scalar densities, $u_i$, and the pseudoscalar densities, $v_i$, given by Gell-Mann.[4] We can then use

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1 S. Weinberg, Phys. Rev. Letters, **17**, 616 (1966).
2 S. L. Adler, Phys. Rev. **137**, B1022 (1965).
3 K. Kawarabayashi and W. W. Wada, Phys. Rev. **146**, 1209 (1966).
4 M. Gell-Mann, Physics **1**, 63 (1964).

the densities $v_i$, $i=1, 2, 3$, as an interpolating field for the pion. It is reasonable to assume that these pseudoscalar densities are smooth interpolating fields like $\partial_\mu A_i{}^\mu$ and allow us to make extrapolations off the mass shell of the order of the pion mass without introducing large errors. In fact in some models like the quark model (or the $\sigma$ model), $\partial_\mu A_i{}^\mu$ is just proportional to $v_i$. In a general quark model $\partial_\mu A_i{}^\mu$ is proportional to $v_i$ plus $SU(3)$-breaking terms. Anyway, no one ever proved that $\partial_\mu A_i{}^\mu$ was a good interpolating field. This was just verified by experience starting with the success of the Goldberger-Treiman formula. In the same way one can only verify whether $v_i$ are good interpolating fields by the results of using them as such. One can easily see, for example, that the Adler consistency condition for $\pi N$ scattering follows also from using $v_i$ as an interpolating field for one of the pions and $\partial_\mu A_i{}^\mu$ for the other and the commutation relation (1).

If one does not like to introduce a new interpolating field one can get results identical to ours in the following way: First, one uses the commutator of $A_i{}^0(\mathbf{x},t)$ with $\partial_\mu A_j{}^\mu$ to define a scalar density. One assumes this scalar density is a local field. To compute the commutator of the scalar density with the axial charge, one uses the Jacobi identity to get a result essentially identical to our Eq. (2′). In this way, one would just have to replace $v_i$ by $\partial_\mu A_i{}^\mu$ wherever it appears in our paper and the results will be the same.

In Sec. II, we derive a new consistency condition on $\pi\pi$ scattering with two pions taken with zero external mass. We also show how one can get Adler's consistency condition using our methods. These two consistency conditions are used to calculate the coefficients in the Weinberg expansion up to first order, so that we may verify that our method gives the same results.

In Sec. III, starting with a reduction formula for the $\pi\pi$ amplitude in which all four pions are reduced out, we derive a general consistency condition on the amplitude. This consistency condition does not only hold at one point in the six-dimensional space of the off-shell $\pi\pi$ variables, but holds for all $s$, $t$, $u$ in the domain $0\leq s, t, u\leq \mu^2$, with the external masses restricted such that $q_1{}^2=0$, $q_2{}^2=s$, $q_3{}^2=t$, $q_4{}^2=u$. All four external-mass variables are taken off the mass shell.

Finally, in Sec. IV we use this general consistency condition to evaluate all but one of the coefficients of the expansion of the $\pi\pi$ amplitudes up to and including second order in $s$, $t$, $u$, and $q_i{}^2$. We then give arguments to show that the one coefficient left undetermined is small. Our final result is that all the first-order coefficients remain the same as in Ref. 1, and all the second-order ones are negligible within our approximations. Even if we carry over some correction terms to our main approximation we find that they only change Weinberg's value for the scattering lengths by 5%.

## II. A NEW CONSISTENCY CONDITION ON THE PION-PION AMPLITUDE

In order to clarify our method we derive a consistency condition on the $\pi\pi$ amplitude with two pions taken with zero external mass. Our main point is to show how one can get a consistency condition on all three $\pi\pi$ amplitudes which, unlike the $\pi N$ case, does not depend on the matrix elements of the scalar densities. We then show how this consistency condition when coupled with Adler's consistency condition will lead to Weinberg's scattering lengths.

Our starting point is the commutation relations of the axial-vector charge with the scalar and pseudoscalar densities given by Gell-Mann in Ref. 4,

$$[Q_i{}^A(t),v_j(\mathbf{x},t)]=id_{ijk}u_k(\mathbf{x},t)\,, \tag{1}$$

$$[Q_i{}^A(t),u_j(\mathbf{x},t)]=-id_{ijk}v_k(\mathbf{x},t)\,;\qquad i,\ j,\ k=0,\ 1,\ \cdots,\ 8;\tag{2}$$

where

$$Q_i{}^A=\int d^3x\,A_i{}^0(\mathbf{x},t)\,, \tag{3}$$

and $A_i{}^\mu(x)$ is the usual axial-vector current. In a quark model $u_i$ and $v_i$ are given by

$$u_i=\tfrac{1}{2}\bar{t}\lambda_i t;\quad v_i=-(i/2)\bar{t}\gamma_5\lambda_i t,\quad i=0,\ 1,\ \cdots,\ 8. \tag{4}$$

Most of the results obtained from PCAC or current algebra follow from using $\partial_\mu A_\alpha{}^\mu$ as an interpolating field for the pion. The success of PCAC strongly suggests that $\partial_\mu A_\alpha{}^\mu$ is a good interpolating field in the sense that it allows us to go off the mass shell by an amount of the order of the mass of the pion without introducing large errors. One can also use $v_\alpha$, $\alpha=1, 2, 3$, as an interpolating field for the pion. This would not make any fundamental difference for the results derived in this paper, but it will we think make certain points clearer. We would expect $v_\alpha$ to be also a good interpolating field like $\partial_\mu A_\alpha{}^\mu$ since in models like the quark model[4] $\partial_\mu A_\alpha{}^\mu$ is proportional to $v_\alpha$ plus $SU(3)$-breaking terms. (In one specific quark model where the symmetry-breaking Hamiltonian is proportional to $u_8$, $\partial_\mu A_\alpha{}^\mu$ is proportional to $v_\alpha$ for $\alpha=1, 2, 3$.) The only problem with using $v_\alpha$ is that we do not know its normalization to the one-pion state. We shall see how we can get around this problem by using Eqs. (1) and (2) together and getting a relation in which the unknown normalization of the $v_\alpha$'s is canceled by the unknown $\pi\pi$ scalar vertex.

Since in this paper we deal only with pions, $i$, $j$, $k=1, 2, 3$, we simplify Eq. (1) and Eq. (2) by first defining the scalar density $\sigma(x)$ as

$$\sigma(x)\equiv(\sqrt{\tfrac{2}{3}})u_0+(\sqrt{\tfrac{1}{3}})u_8. \tag{5}$$

Then instead of Eq. (1) and Eq. (2) we have

$$[Q_\alpha{}^A(t),v_\beta(\mathbf{x},t)]=i\delta_{\alpha\beta}\sigma(\mathbf{x},t)\,, \tag{1′}$$

$$[Q_\alpha{}^A(t),\sigma(\mathbf{x},t)]=-i\delta_{\alpha\beta}v_\beta(\mathbf{x},t)\,,\quad \alpha,\beta=1,\ 2,\ 3. \tag{2′}$$

These last two commutation relations are the only ones we shall use in this paper. We should perhaps remind the reader that Eqs. (1′) and (2′) are also true in the $\sigma$ model if one identifies $v_\alpha$ with the unrenormalized pion field and $\sigma$ with the unrenormalized $\sigma$ field. We stress here that our final results will not depend on the $\sigma$ field or its matrix elements.

As we mentioned in the introduction, one can avoid using the $v_\alpha$'s and use $\partial_\mu A_\alpha{}^\mu$ in their place in the following way. First, one *defines* a new $\sigma'$ from the commutation relation, $[Q_\alpha{}^A(t),\partial_\mu A_\beta{}^\mu(x,t)]\equiv i\delta_{\alpha\beta}\sigma'(x,t)$, and assumes that this $\sigma'$ is a local field. To calculate the commutator $[Q_\alpha{}^A(t),\sigma'(x,t)]$ one now uses the Jacobi identity and the known commutator $[Q_\alpha{}^A(t),Q_\beta{}^A(t)]$ to get a result similar to Eq. (2), $[Q_\alpha{}^A(t),\sigma'(x,t)] = -i\delta_{\alpha\beta}\partial_\mu A_\beta{}^\mu(x,t)$.[5]

We define the normalization constant $a_\pi$ of the $v_\alpha$ field as

$$\langle 0|v_\alpha(0)|\pi_\beta(q)\rangle=(2\pi)^{3/2}(2q^0)^{1/2}a_\pi\delta_{\alpha\beta}. \tag{6}$$

In our reduction formulas we shall make both the replacements:

$$\partial_\mu A_\alpha{}^\mu(x)\to c_\pi\mu^2\phi_\alpha(x)\,,\quad c_\pi=M_N g^A/G_{\pi NN}\,, \tag{7}$$

$$v_\alpha(x)\to a_\pi\phi_\alpha(x)\,. \tag{8}$$

If we identify $v_\alpha$ with $\partial_\mu A_\alpha{}^\mu$ as in Ref. 3, then in that case $a_\pi=c_\pi m_\pi{}^2$. In that case Eq. (1) and Eq. (2) remain unchanged with $\sigma$ replaced by some $\sigma'$. Since we are only interested in the relative normalizations of $v_\alpha$ and $\sigma$ we do not worry about cases where $a_\pi$ is zero and deal with $a_\pi$ as if it were finite. This does not affect our final answers.

Our first step is to relate $a_\pi$ to the $\sigma\pi\pi$ vertex by the usual Fubini-Furlan trick.[6,7] We write

$$\langle\pi_\alpha(k)|\sigma(0)|\pi_\beta(q)\rangle=(2\pi)^{-3}(4k^0q^0)^{-1/2}\delta_{\alpha\beta}\times f^\sigma(q^2,k^2;(q-k)^2). \tag{9}$$

From Lehmann-Symanzik-Zimmermann (LSZ) we can express $f^\sigma$ as

$$f^\sigma(q^2,k^2;(q-k)^2)\delta_{\alpha\beta}c_\pi\mu^2/(2\pi)^{3/2}(2q^0)^{1/2} =i(\mu^2-k^2)\int d^4x\,e^{ik\cdot x}\times\langle 0|T(\partial_\mu A_\alpha{}^\mu(x)\sigma(0))|\pi_\beta(q)\rangle. \tag{10}$$

This is an identity, as $k^2\to\mu^2$, and the usual PCAC tells us that $f^\sigma$ is a slowly varying function as $k^2$ varies from $k^2=\mu^2$ to $k^2=0$. Integrating Eq. (10) by parts we get the identity

$$\delta_{\alpha\beta}f^\sigma(q^2,k^2;(q-k)^2)c_\pi\mu^2/(2\pi)^{3/2}(2q^0)^{1/2} =k_\mu(\mu^2-k^2)\int d^4x\,e^{ik\cdot x}\langle 0|T(A_\alpha{}^\mu(x)\sigma(0))|\pi_\beta(q)\rangle -i(\mu^2-k^2)\int d^4x\,e^{ik\cdot x}\delta(x_0) \times\langle 0|[A_\alpha{}^0(x),\sigma(0)]|\pi_\beta(q)\rangle. \tag{11}$$

In the limit as $k_\mu\to 0$ the first term on the right is zero. The second term using Eq. (2) gives as $k_\mu\to 0$ and $q$ remains on shell,

$$f^\sigma(\mu^2,0;\mu^2)=-a_\pi/c_\pi\,, \tag{12}$$

where the first two variables in $f^\sigma$ always refer to the external masses of the pions in the $\sigma\pi\pi$ vertex and the third variable is the momentum-transfer variable. The constant $a_\pi$ was defined in Eq. (6) and $c_\pi$ is the pion-decay form factor which, if one uses the Goldberger-Treiman formula, is $c_\pi=M_N g^A/G_{\pi NN}$. Both $a_\pi$ and $f^\sigma$ are in general unknown but the relation (12) helps us eliminate them from our final answers as seen below. (If one chooses $v_\alpha\equiv\partial_\mu A_\alpha{}^\mu$ then in that specific case $f^\sigma=-m_\pi{}^2$.)

To get our consistency condition we define the off-the-mass-shell invariant $\pi\pi$ amplitude by

$$iM(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)[c_\pi\mu^2a_\pi/(2\pi)^3(4q_4{}^0q_2{}^0)^{1/2}] =(\mu^2-q_3{}^2)(\mu^2-q_1{}^2)\int d^4x\,e^{-iq_1\cdot x} \times\langle\pi_\delta(q_4)|T(\partial_\mu A_\alpha{}^\mu(x)v_\gamma(0))|\pi_\beta(q_2)\rangle. \tag{13}$$

Here $q_2{}^2=q_4{}^2=\mu^2$ and is not varied in this section. As $q_3{}^2\to\mu^2$ and $q_1{}^2\to\mu^2$, $M$ as defined in Eq. (13) is guaranteed by the LSZ formalism to give the correct $\pi\pi$ amplitude, assuming we have chosen a $\partial_\mu A_\alpha{}^\mu$ and $v_\gamma$ that are relatively local. (We have factored out the energy-momentum–conserving $\delta$ function and $q_3\equiv q_1+q_2-q_4$.)

Integrating the right-hand side of Eq. (13) by parts, we get

$$iM(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)[c_\pi\mu^2a_\pi/(2\pi)^3(4q_4{}^0q_2{}^0)^{1/2}] =iq_{1\mu}(\mu^2-q_1{}^2)(\mu^2-q_3{}^2)\int d^4x\,e^{-iq_1\cdot x} \times\langle\pi_\delta(q_4)|T(A_\alpha{}^\mu(x)v_\gamma(0))|\pi_\beta(q_2)\rangle -(\mu^2-q_1{}^2)(\mu^2-q_3{}^2)\int d^4x\,e^{-iq_1\cdot x} \times\langle\pi\ (q_4)|[A_\alpha{}^0(x),v_\gamma(0)]|\pi_\beta(q_2)\rangle\delta(x_0). \tag{14}$$

We now let both $q_1\to 0$ and $q_3\to 0$. The first term is

[5] We thank S. Weinberg, W. Weisberger, and M. Nauenberg for stressing this point. One assumes here that $\partial_\mu A_\alpha{}^\mu$ is part of a chiral quadruplet to get $\sigma'$ multiplied by $\delta_{\alpha\beta}$; see Ref. 1.

[6] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965). See also V. Alessandrini, M. A. B. Bég, and L. S. Brown, Phys. Rev. **144**, 1137 (1966), and Ref. 7.

[7] W. Weisberger, Phys. Rev. **143**, 1303 (1966).

zero and the second term, after using Eq. (2) and Eq. (9), gives us

$$\lim_{q_1\to 0}\lim_{q_3\to 0} M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)c_\pi\mu^2a_\pi = -\mu^4 f^\sigma(\mu^2,\mu^2;0)\delta_{\alpha\gamma}\delta_{\beta\delta}. \quad (15)$$

Let us assume that $f^\sigma$ is a slowly varying function of the external-pion masses as $q^2$ varies between zero and $\mu^2$ and the same for the transfer variable, and write

$$f^\sigma(\mu^2,0;\mu^2)\cong f^\sigma(\mu^2,\mu^2;0). \quad (16)$$

We justify this approximation in detail at the end of the next section. As far as varying the external-mass variables are concerned this is just the usual PCAC assumption. Varying the third variable—i.e., the one in the $\sigma$ channel—could be more dangerous and we study it in detail later.

With Eq. (16) we can use Eq. (12) to eliminate $f^\sigma$ and $a_\pi$ from Eq. (15) and get

$$\lim_{q_1\to 0}\lim_{q_3\to 0} M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)=\frac{\mu^2}{c_\pi^2}\delta_{\alpha\gamma}\delta_{\beta\delta}. \quad (17)$$

We recall the isospin decomposition of $M$ into the three amplitudes $A$, $B$, and $C$ given by

$$M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)=A\delta_{\alpha\beta}\delta_{\gamma\delta}+B\delta_{\alpha\gamma}\delta_{\beta\delta}+C\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (18)$$

where

$$A=A(s,t,u;q_1^2,q_2^2,q_3^2,q_4^2)\text{, etc.}, \quad (19)$$

and

$$s=(q_1+q_2)^2,$$
$$t=(q_1-q_3)^2,$$
$$u=(q_1-q_4)^2, \quad (20)$$
$$s+t+u=\sum_{i=1}^{4} q_i^2.$$

In terms of $A$, $B$, and $C$, our consistency condition in Eq. (17) becomes

$$A(s=\mu^2,\, t=0,\, u=\mu^2;\, q_1^2=0,\, q_2^2=\mu^2,\, q_3^2=0,\, q_4^2=\mu^2)=0,$$
$$B(\mu^2,0,\mu^2;0,\mu^2,0,\mu^2)=\mu^2/c_\pi^2, \quad (21)$$
$$C(\mu^2,0,\mu^2;0,\mu^2,0,\mu^2)=0.$$

The Adler-Weisberger sum rule for $\pi\pi$ scattering also has two external-pion momenta taken to zero. However, it essentially gives a consistency condition on the derivative of the odd $\pi\pi$ amplitude at $\nu=0$.

One could easily repeat our calculation to get Adler's consistency condition[2] for the $\pi\pi$ amplitude with $q_1\to 0$ and $q_2$, $q_3$, $q_4$ all on the mass shell. We do not do this here since the Adler consistency condition is a special case of the general consistency condition to be derived in the next section. Adler's consistency condition gives

$$A(\mu^2,\mu^2,\mu^2;0,\mu^2,\mu^2,\mu_2)=B=C=0. \quad (22)$$

To go from Eq. (21) and Eq. (22) to a statement about physical quantities such as scattering lengths, one has to go through extrapolations which at first sight would seem quite dangerous. Weinberg's method of extrapolation consisted of expanding $A$, $B$, and $C$ in powers of $s$, $t$, $u$, and $q_i^2$ and keeping terms only up to first order in these variables. Crossing symmetry and Bose statistics require the off-mass-shell amplitude to have an expansion of the form

$$A=a+b(t+u)+cs+O(s^2,st,\cdots,q_i^2q_j^2,\cdots,q_i^4,\cdots),$$
$$B=a+b(s+u)+ct+\cdots, \quad (23)$$
$$C=a+b(s+t)+cu+\cdots.$$

The main point here is that in Eq. (23) there could be no first-order terms in the $q_i^2$ variables.

In this approximation one can use Eq. (21) and Eq. (22) to determine $a$, $b$, and $c$. From Eq. (21) we get two equations:

$$a+\mu^2b+\mu^2c=0,$$
$$a+2\mu^2b=\mu^2/c_\pi^2; \quad (24)$$

and from Adler's consistency condition, Eq. (22), we have

$$a+2\mu^2b+\mu^2c=0. \quad (25)$$

The solution of Eqs. (24) and (25) is

$$a=\mu^2/c_\pi^2;\quad b=0;\quad c=-1/c_\pi^2, \quad (26)$$

where $c_\pi=M_Ng^A/G_{\pi NN}$. This is the same as the result obtained by Weinberg,[1] where in his notation $c_\pi=F_\pi/2$. If one uses Eq. (23) to give the amplitude at threshold, one gets the scattering lengths given in Ref. 1.

However, there are several troubles with the expansion in Eq. (23). First, it is known to be divergent at threshold. Weinberg gets around this difficulty by assuming that the unitarity branch point is a weak singularity which allows him to use Eq. (23) at least as an asymptotic expansion up to and somewhat beyond threshold. Since he gets small scattering lengths in the end, this shows that his argument is self-consistent, but does not prove that the scattering lengths are indeed small.

The strong consistency condition which we obtain in the next section enables us to estimate the coefficients of the power-series expansion up to second order in $s$, $t$, $u$, and $q_i^2$. The remarkable result is that all the second-order coefficients are not only small but also negligible within our approximation.

## III. A GENERAL CONSISTENCY CONDITION ON THE PION-PION AMPLITUDE

In this section we extend our method to get a general consistency condition on the $\pi\pi$ amplitude which gives restrictions not only at one point in the six-dimensional space of the $\pi\pi$-scattering off-shell variables, but in a three-dimensional region.

We write for the off-shell $\pi\pi$ amplitude the following reduction formula:

$$-i(2\pi)^4\delta(q_1+q_2-q_3-q_4)M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)a_\pi{}^3c_\pi\mu^2$$
$$=(\prod_{i=1}^{4}(\mu^2-q_i^2))\int d^4x_1\cdots d^4x_4$$
$$\times\exp(-iq_1\cdot x_1-iq_2\cdot x_2+iq_3\cdot x_3+iq_4\cdot x_4)$$
$$\times\langle 0|T(\partial_\mu A_\alpha{}^\mu(x_1)v_\beta(x_2)v_\gamma(x_3)v_\delta(x_4))|0\rangle. \quad (27)$$

Again in the limit where all $q_i^2\to\mu^2$, $M$ as defined above gives the exact $\pi\pi$ amplitude.

If we integrate Eq. (27) by parts we get the identity

$$-i(2\pi)^4\delta(q_1+q_2-q_3-q_4)M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)a_\pi{}^3c_\pi{}^2\mu$$
$$=iq_{1\mu}(\prod_{i=1}^{4}(\mu^2-q_i^2))\int d^4x_1\cdots d^4x_4$$
$$\times\exp(-iq_1\cdot x_1-iq_2\cdot x_2+iq_3\cdot x_3+iq_4\cdot x_4)$$
$$\times\langle 0|T(A_\alpha{}^\mu(x_1)x_\beta(x_2)v_\gamma(x_3)v_\delta(x_4))|0\rangle$$
$$-(\prod_{i=1}^{4}(\mu^2-q_i^2))\int d^4x_1\cdots d^4x_4\delta(x_1{}^0-x_2{}^0)$$
$$\times\exp(-iq_1\cdot x_1-iq_2\cdot x_2+iq_3\cdot x_3+iq_4\cdot x_4)$$
$$\times\langle 0|T([A_\alpha{}^0(x_1),v_\beta(x_2)]v_\gamma(x_3)v_\delta(x_4))|0\rangle$$
$$-\text{permutations of the last term over the } v\text{'s}. \quad (28)$$

In the limit $q_1\to 0$ the first term in Eq. (28) vanishes. The other three terms, after using the equal-time commutation relation, Eq. (1′), give us three terms proportional to the $\sigma\pi\pi$ vertex.

We obtain

$$-i\lim_{q_1\to 0}M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)a_\pi{}^3c_\pi$$
$$=i(\mu^2-q_2^2)a_\pi{}^2f^\sigma(q_3^2,q_4^2;(q_3+q_4)^2)\delta_{\alpha\beta}\delta_{\gamma\delta}$$
$$+i(\mu^2-q_3^2)a_\pi{}^2f^\sigma(q_2^2,q_4^2;(q_2-q_4)^2)\delta_{\alpha\gamma}\delta_{\beta\delta}$$
$$+i(\mu^2-q_4^2)a_\pi{}^2f^\sigma(q_2^2,q_3^2;(q_2-q_3)^2)\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (29)$$

where to obtain Eq. (29) we have used the identity

$$a_\pi{}^2f^\sigma(q^2,k^2;(q-k)^2)\delta_{\alpha\beta}$$
$$=-(\mu^2-k^2)(\mu^2-q^2)\int d^4xd^4y\,e^{-iq\cdot x}e^{+ik\cdot y}$$
$$\times\langle 0|T(\sigma(0)v_\alpha(x)v_\beta(y))|0\rangle. \quad (30)$$

This follows from applying the reduction formula directly to Eq. (9).[8]

[8] With both pions on the mass shell it does not matter whether we use the definition (30) or (10) for $f^\sigma$. In principle, when we go off the mass shell, $f^\sigma$ defined with $\partial_\mu A_\alpha{}^\mu$ as an interpolating field could be different than $f^\sigma$ defined by the $v_\alpha$'s. However, we have assumed that both $v_\alpha$ and $\partial_\mu A_\alpha{}^\mu$ are good smooth interpolating fields and we only use (30) for $0\leq q^2, k^2\leq\mu^2$. So as long as we do not go too far off the mass shell, $f^\sigma$ as defined in (30) and as defined in (10) are within our approximations the same. This problem, of course, would not have arisen if we had used $\partial_\mu A_\alpha{}^\mu$ instead of the $v_\alpha$'s all through.

In the limit as $q_1{}^\mu\to 0$ we have the following relations between the six variables of $\pi\pi$ scattering:

$$q_2=q_3+q_4; \quad (31)$$

and hence when $q_1{}^\mu\equiv 0$,

$$s=(q_3+q_4)^2=q_2{}^2,$$
$$t=(q_2-q_4)^2=q_3{}^2, \quad (32)$$
$$u=(q_2-q_3)^2=q_4{}^2.$$

Thus Eq. (20) becomes

$$\lim_{q_1\to 0}M(q_4\delta,q_3\gamma;q_2\beta,q_1\alpha)=-\frac{1}{c_\pi}(\mu^2-s)\frac{f^\sigma(t,u;s)}{a_\pi}\delta_{\alpha\beta}\delta_{\gamma\delta}$$
$$-\frac{1}{c_\pi}(\mu^2-t)\frac{f^\sigma(s,u;t)}{a_\pi}\delta_{\alpha\gamma}\delta_{\beta\gamma}$$
$$-\frac{1}{c_\pi}(\mu^2-u)\frac{f^\sigma(s,t;u)}{a_\pi}\delta_{\alpha\delta}\delta_{\beta\gamma}. \quad (33)$$

We now use Eq. (12) to eliminate $a_\pi$ from Eq. (33) and get a relation between the off-shell $\pi\pi$ amplitudes and the $\sigma\pi\pi$ vertex. In terms of the amplitudes $A$, $B$, and $C$ we now have

$$A(s,t,u;q_1^2=0,q_2^2=s,q_3^2=t,q_4^2=u)$$
$$=\frac{1}{c_\pi{}^2}(\mu^2-s)\frac{f^\sigma(t,u;s)}{f^\sigma(\mu^2,0;\mu^2)},$$
$$B(s,t,u;q_1^2=0,q_2^2=s,q_3^2=t,q_4^2=u)$$
$$=\frac{1}{c_\pi{}^2}(\mu^2-t)\frac{f^\sigma(s,u;t)}{f^\sigma(\mu^2,0;\mu^2)}, \quad (34)$$
$$C(s,t,u;q_1^2=0,q_2^2=s,q_3^2=t,q_4^2=u)$$
$$=\frac{1}{c_\pi{}^2}(\mu^2-u)\frac{f^\sigma(s,t;u)}{f^\sigma(\mu^2,0;\mu^2)}.$$

The functions $f^\sigma$ are by definition symmetric in the first two variables, i.e., the external-pion masses, so Eq. (34) is manifestly crossing symmetric. What we have succeeded in doing so far is to show that when $q_1{}^\mu\equiv 0$, then if one sets the other three external-mass variables equal to $s$, $t$, $u$, respectively, one gets a relation between the off-shell amplitude and a ratio of the $\sigma\pi\pi$ vertex at two different points. Thus the problem reduces to a study of how fast $f^\sigma$ varies in all three of its variables.

We restrict ourselves to the domain $0\leq s,t,u\leq\mu^2$ and show that in this region

$$f^\sigma(t,u;s)/f^\sigma(\mu^2,0;\mu^2)\cong 1;\quad 0\leq s,t,u\leq\mu^2. \quad (35)$$

The fact that $f^\sigma(x,y;z)$ is slowly varying in the first two variables is actually part of the PCAC assumption (or the assumption that $v$ is a smooth interpolating field) as long as $x$ and $y$ do not vary much from their on-mass-shell value, $x=y=\mu^2$. This can be justified by pion-pole-dominance arguments similar to those used by Weisberger.[7] For example, if we let $g^\sigma$ be given by

$$g^\sigma(\mu^2,k^2;(q-k)^2)\frac{\delta_{\alpha\beta}}{(2\pi)^{3/2}(2q^0)^{1/2}} =i\int e^{ik\cdot x}d^4x\langle 0|T(\sigma(0)\phi_\alpha(x))|\pi_\beta(q)\rangle\,,\quad q^2=\mu^2\,, \tag{36}$$

then as a function of $k^2$, for fixed $(q-k)^2$, the function $g^\sigma$ has a pole at $k^2=\mu^2$ and the residue of that pole is just $f^\sigma(\mu^2,\mu^2;s)$, where $s\equiv(q-k)^2$. The PCAC assumption tells us that for $0<k^2<\mu^2$, and $s$ fixed and small, the pion-pole term dominates over contributions from other singularities in the $k^2$ plane. We get

$$g^\sigma(\mu^2,k^2;s)\cong f^\sigma(\mu^2,\mu^2;s)/(k^2-\mu^2)\,,\quad 0\le k^2\le\mu^2. \tag{37}$$

But comparing Eq. (36) with Eq. (10), we get

$$(\mu^2-x)g^\sigma(\mu^2,x;s)=f^\sigma(\mu^2,x;s)\,,$$

and hence

$$f^\sigma(\mu^2,x;s)\cong f^\sigma(\mu^2,\mu^2;s)\,;\quad 0\le x\le\mu^2. \tag{38}$$

Extrapolation in the other pion-mass variable can be handled in the same way. To a good approximation we can therefore write

$$f^\sigma(x,y;s)\cong f^\sigma(\mu^2,\mu^2;s)\,,\quad 0\le x,\,y\le\mu^2. \tag{39}$$

The behavior of $f^\sigma$ in the third variable, the one corresponding to the square of the $\sigma$ four-momentum, could in principle be much more dangerous. Indeed one would argue that a strong $\pi\pi$ $S$-wave, $I=0$, interaction could give the vertex $f^\sigma(\mu^2,\mu^2;s)$ a large derivative in $s$ at $s=0$. Fortunately, dispersion theory gives us a fairly reliable way of estimating the effect of rescattering on a vertex. The Omnes formula for $f^\sigma$ would give us

$$\frac{f^\sigma(\mu^2,\mu^2;s)}{f^\sigma(\mu^2,\mu^2;0)}=\exp\left[\frac{s}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta_0{}^0(s')}{s'(s'-s)}ds'\right]\,, \tag{40}$$

where $\delta_0{}^0$ is the $S$-wave, $I=0$, $\pi\pi$ phase shift. The slope of $f^\sigma(\mu^2,\mu^2;s)$ at $s=0$ could be large either because of a large scattering length or because of a low mass resonance in the $l=0$, $I=0$ channel. Let us first estimate the effect of a scattering length on the slope. Starting with $[(s-4\mu^2)/s]^{1/2}\cot\delta_0{}^0=1/a_0\mu$, we use the expression $\delta_0{}^0(s)\cong a_0\mu[(s-4\mu^2)/s]^{1/2}$ in Eq. (40) and obtain for the derivative of $f^\sigma$,

$$\frac{1}{f^\sigma(\mu^2,\mu^2,0)}\frac{df^\sigma}{ds}(\mu^2,\mu^2;s)\bigg|_{s=0} \cong\frac{a_0\mu}{\pi}\int_{4\mu^2}^{\infty}\frac{(s'-4\mu^2)^{1/2}}{s'^{5/2}}ds'\cong\frac{a_0}{6\pi\mu}. \tag{41}$$

Thus the ratio in Eq. (35) is approximately given by

$$\frac{f^\sigma(t,u;s)}{f^\sigma(\mu^2,0;\mu^2)}\cong\frac{f^\sigma(\mu^2,\mu^2;s)}{f^\sigma(\mu^2,\mu^2;\mu^2)}\cong 1+\frac{a_0}{6\pi\mu}(s-\mu^2)\,;\quad 0\le s\le\mu^2. \tag{42}$$

We note that the form we have used for $\delta_0{}^0$ in Eq. (41) does not vanish as $s\to\infty$, as it would have if we had included an effective range. This makes our correction term in Eq. (42) larger than it actually is. Nevertheless, we easily see that even if $a_0$ is as large as $\mu^{-1}$, the correction term in Eq. (42) is at most $1/6\pi=0.05$, as $s$ varies in the interval $0\le s\le\mu^2$. In the next section we shall keep the second term on the right in Eq. (42) in our calculation of the scattering lengths and show that it only changes Weinberg's result by a few percent. Even including these corrections our final result for $a_0$ is still $a_0\cong 0.20\mu^{-1}$. For the region $0\le s,\,t,\,u\le\mu^2$ one can thus safely neglect the second term in Eq. (42).

If there exists an actual $\sigma$ resonance, in the $l=0$, $I=0$ channel, then the correction to Eq. (35) will be of the form

$$\frac{f^\sigma(\mu^2,\mu^2;s)}{f^\sigma(\mu^2,\mu^2;0)}\cong 1+(s-\mu^2)O\left(\frac{\mu^2}{m_\sigma{}^2}\right);\quad 0\le s,\,t,\,u\le\mu^2. \tag{43}$$

There seems to be no evidence for a narrow ($\Gamma<100$ MeV) $\sigma$ particle with mass lower than 600 MeV.[9] Thus we can also neglect the correction term in Eq. (43). The only possibility left is for a very broad $\pi\pi$ resonance in the region below 600 MeV. But the effect of such a broad resonance ($\Gamma>200$ MeV) on the slope of $f^\sigma$ at $s=0$ will be very similar to that of a large scattering length which we have already shown does not affect our results appreciably.

The consistency condition in Eq. (34) can now be written as

$$\begin{aligned} A(s,t,u;q_1{}^2=0,q_2{}^2=s,q_3{}^2=t,q_4{}^2=u)&\cong c_\pi{}^{-2}(\mu^2-s)\,;\quad 0\le s,\,t,\,u\le\mu^2;\\ B(s,t,u;q_1{}^2=0,q_2{}^2=s,q_3{}^2=t,q_4{}^2=u)&\cong c_\pi{}^{-2}(\mu^2-t)\,,\\ C(s,t,u;q_1{}^2=0,q_2{}^2=s,q_3{}^2=t,q_4{}^2=u)&\cong c_\pi{}^{-2}(\mu^2-u)\,. \end{aligned} \tag{44}$$

As we have mentioned earlier, these consistency con-

[9] M. Deutschmann *et al.*, Phys. Letters **12**, 356 (1964); see also V. Hagopian, W. Selove, J. Alitti, J. P. Baton, and M. Neveu-René, Phys. Rev. **145**, 1128 (1966).

ditions are much stronger than the usual ones which hold only for one point; these hold for any $s$, $t$, $u$ that lie in the cube $0 \leq s, t, u \leq \mu^2$, if the masses are restricted as in (44).

## IV. THE POWER-SERIES EXPANSION OF THE PION-PION AMPLITUDE

We use the consistency condition (44) to estimate the $\pi\pi$ amplitude up to second order in the variables $s$, $t$, $u$, and $q_i^2$.

We expand $A$, $B$, and $C$ in a power series of the variables $s$, $t$, $u$, $q_i^2$, where $u = \sum q_i^2 - s - t$. To second order in these variables, crossing symmetry and Bose statistics require the expansion to take the form

$$A(s,t,u;q_1^2,q_2^2,q_3^2,q_4^2) = a + b(t+u) + cs + d(t+u)^2 + etu + fs^2 + g(t+u)s + h\sum_{i>j} q_i^2 q_j^2; \quad (45)$$

and $B$ and $C$ are obtained by exchanging $s$ and $t$ in (45) or $s$ and $u$, respectively. No terms linear in the $q_i^2$ variables appear. Also terms of the form $q_i^2 s$, $q_i^2 t$, etc., can after using crossing and Bose symmetry be reduced to forms already in (45). Before applying (44), we note that there is one remark we can make in general about the coefficients $a$, $b$, $c$, $\cdots$, $g$, $h$. There is no *a priori* reason to assume that any of the second-order coefficients are small except for $h$. For if $h$ is not small, then the amplitude will vary strongly with the external-pion masses; a situation which is in contradiction to the PCAC philosophy.[10] For example, if this were the case and $h$ were large, then the Adler-Weisberger sum rule for $\pi\pi$ scattering would be practically useless even if we were someday able to measure the $\pi\pi$ total cross sections exactly.

Let us use (44) to determine the coefficients $a$, $b$, $c$, $\cdots$, $h$. We restrict ourselves to the domain $0 \leq s, t, u \leq \mu^2$. Comparing (45) with $q_1^2 = 0$, $q_2^2 = s$, $q_3^2 = t$, $q_4^2 = u$, with (44) we get

$$a + b(t+u) + cs + d(t+u)^2 + etu + fs^2 + g(t+u)s + h(st+tu+su) = c_\pi^{-2}(\mu^2 - s); \quad 0 \leq s, t, u \leq \mu^2. \quad (46)$$

This gives us

$$a = \mu^2/c_\pi^2, \quad b = 0, \quad c = -1/c_\pi^2, \quad d = f = 0; \quad (47)$$

and

$$h = -e = -g.$$

Note that $a$, $b$, and $c$ still have the same value obtained by expanding only up to first order. Only one constant is left undetermined in the second-order terms; and that one is $h$ which, as we mentioned earlier, we expect to be small.

In order to estimate the scattering lengths $a_0$ and $a_2$, we need to assume that the expansion in (45) is at least numerically good up to $s = 4\mu^2$. In extending $s \rightarrow 4\mu^2$, we shall keep track of the correction terms in (42) in order to make sure that they do not make important contributions.

If we keep the correction terms from (42) in the consistency condition (44), then instead of (47) we obtain for the coefficients

$$a \cong (\mu^2/c_\pi^2)(1 - a_0\mu/6\pi), \quad b \cong 0, \quad c \cong -(1/c_\pi^2)(1 - a_0\mu/3\pi), \quad f \cong -(1/c_\pi^2)(a_0/6\pi\mu), \quad (48)$$

and

$$d \cong 0; \quad e \cong g \cong -h.$$

We have mentioned earlier that $h$ must be small compared to the dominant lower order terms. This indeed has to be so if we are to be consistent with the approximation used in (37) and (38). For example, let us consider in detail the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ amplitude, $F$, given by

$$F = A + B + C = \tfrac{1}{3}A^{I=0} + \tfrac{2}{3}A^{I=2}. \quad (49)$$

Let us fix our attention on the symmetry point $s = u$, and $t = 0$. Then by using the arguments of Ref. 7, considering dispersion relations in the external-mass variables $q_1^2$ and $q_3^2$, and assuming dominance by the double pion pole we get

$$F(s=u, t=0; 0, \mu^2, 0, \mu^2) \cong F(s=u, t=0; \mu^2, \mu^2, \mu^2, \mu^2). \quad (50)$$

A similar result was written for the even $\pi N$ amplitude in Ref. 3. The argument is very similar to that used in (36)–(39) above and one can refer to Ref. 7 for details. What we have done here is to keep $t = 0$ fixed, and $s = u$, ($\nu = 0$, fixed) and extrapolate two external-mass variables, $q_1^2$ and $q_3^2$, from $\mu^2$ to zero. On the other hand, we can compute both sides of (50) from our expansion (45). For $F$, using the coefficients in (47), we have

$$F(s,t,u;q_1^2,q_2^2,q_3^2,q_4^2) = 3a + 4\mu^2 c - 3h(st+tu+su) + 3h\sum_{i>j} q_i^2 q_j^2, \quad (51)$$

where $u = \sum_i q_i^2 - s - t$ always. We now use (51) to calculate the difference:

$$F(s=u, t=0; \mu^2, \mu^2, \mu^2, \mu^2) - F(s=u, t=0; 0, \mu^2, 0, \mu^2) \cong 6h\mu^4. \quad (52)$$

Thus to the extent that (50) is a good ext apolation,

[10] In all this paper, we are assuming PCAC is a good approximation. We want to estimate the relative magnitude of the first- and second-order coefficients under that assumption.

we conclude that $6h\mu^4$ must be small when compared to the dominant term in (51) which is $(3a+4\mu^2c)=-\mu^2/c_\pi^2$. Here

$$\mu^2/c_\pi^2=\mu^2G_{\pi NN}^2/M_N^2g_A^2\cong(8/9)\pi.$$

At the end of this section we write a sum rule for $h$ and discuss its magnitude further; however, it is clear that to be consistent with our approximations on $f^\sigma$ earlier we must neglect $h$. We can now compute the scattering lengths.

The $S$-wave scattering lengths are related to our expansion coefficients by

$$a_0\cong-(1/32\pi\mu)[5a+12\mu^2c+48f\mu^4+30h\mu^4],$$
$$a_2\cong-(1/32\pi\mu)[2a+12h\mu^4]. \quad (53)$$

We have kept both terms proportional to $f$ and $h$ in (53). Following our estimate of $6h\mu^4$ when compared with $\mu^2/c_\pi^2$, we see that $30h\mu^4$ is also negligible when compared with $(5a+12\mu^2c)\cong-7\mu^2/c_\pi^2$. Even if $6h\mu^4$ were as large as 20% of $\mu^2/c_\mu^2$, keeping the term $30h\mu^4$ in (53) would only change Weinberg's result by 13% and raise the scattering length at most to $a_0\cong0.23\mu^{-1}$.

We thus have, setting $h\approx0$,

$$a_0\cong-(1/32\pi\mu)[5a+12\mu^2c+48f\mu^4]. \quad (53')$$

Substituting the values (48) for $a$, $c$, and $f$ we get an equation for $a_0$ which we can solve and obtain

$$a_0\cong\frac{7}{4}\left(\frac{1}{8\pi}\frac{\mu}{c_\pi^2}\right)\Big/\left[1-\frac{29\mu^2}{192\pi^2c_\pi^2}\right]. \quad (54)$$

The numerator of this last expression is exactly Weinberg's result. The quantity $(29\mu^2/192\pi^2c_\pi^2)$ is about 0.04. Therefore keeping the correction terms in (48) will only change the result by 4%. We get

$$a_0\cong0.2\mu^{-1}. \quad (55)$$

This means that the ratio in (42) is indeed close to unity and the quantity $a_0\mu/6\pi$ is of the order of 1%. The coefficients are therefore given by (47) to a good approximation.

In a similar way the corrections terms do not affect $a_2$ in any appreciable way and one still gets

$$a_2\cong-\tfrac{1}{2}(\mu/8\pi c_\pi^2)\cong0.06\mu^{-1}. \quad (56)$$

In conclusion we write a dispersion relation for the forward $\pi^0\pi^0\to\pi^0\pi^0$ amplitude $F$ and show how it can be used to give a sum rule for $h$. It is more convenient to use the laboratory energy $\nu$ instead of $s$ as a variable, where

$$s=2\mu^2+2\nu\mu. \quad (57)$$

For $t=0$ the expansion for $F(\nu)$ for physical masses, keeping $h$, is

$$F(\nu)\cong-(\mu^2/c_\pi^2)+18h\mu^4-3h(4\mu^2-s)s$$
$$\cong-(\mu^2/c_\pi^2)+12h\mu^2(\nu^2+\tfrac{1}{2}\mu^2),\quad |\nu|<\mu. \quad (58)$$

This expansion is good, even convergent, for $|\nu|<\mu$. We note that at the points $\nu=\pm i\mu/\sqrt{2}$, $F(\nu)$ is through (58) given by $-\mu^2/c_\pi^2$ and not dependent on $h$. We can therefore write a twice-subtracted dispersion relation for $F(\nu)$ and if we choose the subtraction points to be $\nu=\pm i\mu/\sqrt{2}$, the subtractions do not depend on $h$. We get

$$F(\nu)=-\frac{\mu^2}{c_\pi^2}+\frac{2(\nu^2+\frac{1}{2}\mu^2)}{\pi}\int_\mu^\infty\frac{\mathrm{Im}F(\nu')\nu'}{(\nu'^2+\frac{1}{2}\mu^2)(\nu'^2-\nu^2)}d\nu'. \quad (59)$$

The expansion (58) is certainly good at $\nu=0$, and it gives

$$F(0)\cong-\mu^2/c_\pi^2+6h\mu^4. \quad (60)$$

We see that $6h\mu^4$ is just the difference between $F(\nu=0)$ and $F(\nu=i\mu/\sqrt{2})$. Our assumption is that this is small compared to the value of $F$ at either of these two points. Comparing (60) with (59), we get a sum rule for $h$:

$$6h\mu^4=\frac{\mu^2}{\pi}\int_\mu^\infty\frac{\mathrm{Im}F(\nu')}{\nu'(\nu'^2+\frac{1}{2}\mu^2)}d\nu'. \quad (61)$$

We recall that $F$ is the physical forward fully symmetric amplitude and only $I=0$ or $I=2$ contribute to Im$F$.

The first thing we learn from (61) is that $h$ is negative; Im$F$ in our normalization is negative. The contribution of resonances like the $f^0$ to $h$ through (61) will certainly be negligible for our purposes; so will that of any high-mass (i.e., greater than 500 MeV) resonance. If a low-energy narrow resonance exists, say in the $l=0$, $I=0$ channel, it could change our result appreciably, but it is hard to see how it can increase the scattering length up to more than $a_0=0.3\mu^{-1}$ at worst. Such a resonance would make the Weisberger extrapolation quite bad for $\pi\pi$ scattering, and it has of course not been established experimentally.[9] Many of the theoretical arguments for its existence, like the analyses of $K_{l4}$ decay[11] and $\tau$ decay,[12] have lately been rendered unnecessary. The only remaining question is the saturation of the Adler-Weisberger $\pi\pi$ sum rule.[13] That sum rule has one less power of $\nu$ in the denominator than in (61), and it could easily be saturated with, in addition to known resonances, an $l=0$, $I=0$ resonance of mass>600 MeV. It does not necessarily force us to predict a low-lying resonance. There is one contribution to (61) which might be dangerous and whose effect we can approximately check; namely, the contribution from Im$F(\nu')$ near threshold that is related to the $l=0$, $I=0$ scattering length. This will give a contribution that is proportional to $a_0^2$ from the low-energy part in (61) and that when substituted in (53) will change the functional form of our resulting equation for $a_0$. To make sure that this will not appreciably change our results, we divide the integrationr ange in (61) into two parts, $a\leq\nu\leq6\mu$ and $6\mu\leq\nu\leq\infty$. In the first interval we

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approximate Im$F$ by the contribution from the $l=0$, $I=0$ channel and use $\delta_0^0 \simeq [(s-4)/s]^{1/2} a_0 \mu$ and get

$$6h\mu^4 \cong -2a_0^2\mu^2 + \mu^2/\pi \int_{6\mu}^{\infty} \frac{\mathrm{Im}F(\nu')}{\nu'(\nu'^2+\frac{1}{2}\mu^2)} d\nu' . \quad (62)$$

If we ignore the second term in (62) and assume it to be a fraction of $\mu^2/c_\pi^2 \cong 8\pi/9$, we obtain on substituting (62) into (53)

$$a_0 = -(1/32\pi\mu)[-7(\mu^2/c_\pi^2) - 10a_0^2\mu^2] . \quad (63)$$

This last equation has two roots for $a_0$. One will, to within 2%, give us back the same answer as before, $a_0 \cong 0.2\mu^{-1}$. The other root is ridiculously large, $a_0 \cong 10\mu^{-1}$, and clearly unphysical. The latter root will also give a very large value for $a_2$.


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# $\pi N$ Polarization and Regge Poles*


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We show that the recent high-energy $\pi^- p$ polarization data from CERN are explained in a natural way by the three–Regge-pole model. The prediction of this model for $\pi^+ p$ polarization differs greatly from that for $\pi^- p$ polarization in the region where $|t| < 0.6$ (GeV/$c$)$^2$. In particular, in this region, the $\pi^+ p$ polarization has an opposite sign and comparable magnitude to that for $\pi^- p$.


THIS paper shows that recent high-energy $\pi^- p$ polarization data from CERN[1] are explained in a natural way by the three–Regge-pole model.[2] The prediction of the model for $\pi^+ p$ polarization has an opposite sign and comparable magnitude to that for $\pi^- p$.

Elastic $\pi N$ scattering at small momentum transfer is dominated, in this model, by three Regge poles in the crossed channel. Thus it is a more complicated problem than the charge-exchange reactions, with only one or two poles, for which the Regge hypothesis has had great success.[2–7] However, this complication is largely compensated by the greater variety of data available.

The data we use are total cross sections,[8] differential cross sections for elastic[9,10] and charge-exchange[11,12] scattering, Coulomb interference measurements of the phase of the forward elastic amplitude,[13] and $\pi^- p$ elastic polarization.[1] These data are from 5.9 GeV/$c$ upward, and with squared momentum transfer $|t| < 1$ (GeV/$c$)$^2$. For $d\sigma/dt$ data, we worked with a representative subset of 141 elastic points in the interval $-1 < t < -0.1$ and

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