Subtractions in Dispersion Relations*

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Assuming the unitarity bound in the physical region on a partial-wave amplitude for an elastic scattering process, and assuming further that the amplitude satisfies a dispersion relation, it is shown that the necessity for subtractions in order to make the integral over the left-hand cut converge implies an oscillatory behavior of the discontinuity across the left-hand cut, the violence of the oscillations increasing with increasing number of subtractions. A theorem given by Sugawara and Kanazawa is provided with a rigorous proof and is applied to pion-nucleon and kaon-nucleon dispersion relations.

I. INTRODUCTION

This work arose from a study of the implications of the unitarity bound, in the physical region, on a partial-wave amplitude for an elastic scattering process, for the behavior of that amplitude when continued into the complex s plane (s being the square of the total energy in the center-of-mass system). In particular, assuming that the amplitude satisfies a dispersion relation, is it possible to make any statements about the number of subtractions required or about the behavior of the discontinuity of the amplitude across the left-hand cut? Jin and Martin[1] have shown, using also the threshold behavior of the amplitude, that, independently of the number of subtractions, there is a minimum fluctuation of the sign of the discontinuity across the left-hand cut which increases with increasing angular momentum. In Sec. II of this paper we prove a quite different type of result, one which relates the necessity for subtractions in order to make the integral over the left-hand cut converge, to the existence of an oscillatory behavior of the discontinuity across the far left-hand cut, the violence of the oscillations increasing with increasing number of subtractions.

This led to an attempt to provide a rigorous proof of a theorem given by Sugawara and Kanazawa.[2] Such a proof is given in Sec. III, with sufficient conditions clearly stated. Finally, in Sec. IV we give an application of this theorem to a case of practical interest, the pion-nucleon and kaon-nucleon dispersion relations.

II. PARTIAL-WAVE DISPERSION RELATIONS

Let $f(x)$ be a partial-wave amplitude for an elastic-scattering process, x being the square of the total energy of the two particles in the center-of-mass system. (We adopt standard mathematical notation, since we shall state the main result of this section as a mathematical theorem.) It follows from the unitarity of the S matrix that $f(x)$ may be represented in the form

$$f(x)=[\eta(x)e^{2i\delta(x)}-1]/2iq(x)$$

for $x>x_0$, where $x_0=(m_1+m_2)^2$, m_1 and m_2 being the masses of the two particles. Here $0\leqslant\eta(x)\leqslant 1$, $\delta(x)$ is real and $q(x)$ is the magnitude of the momentum of either particle in the center-of-mass system.

We now define the real-valued function $F(x)$ by

$$F(x)=\mathrm{Re}f(x)-\frac{1}{\pi}P\int_{x_0}^{\infty}\frac{\mathrm{Im}f(x')dx'}{x'-x} \tag{1}$$

for $x>x_0$. We suppose that $\mathrm{Im}f(x)$ satisfies a Lipschitz condition for each $x>x_0$; this will ensure that the principal-value integral exists. The integral clearly converges at infinity, since $0\leqslant\mathrm{Im}f(x)\leqslant 1/q(x)$ and $q(x)\sim\frac{1}{2}x^{1/2}$ as $x\to\infty$. Now suppose further that

$$|\mathrm{Im}f(x+h)-\mathrm{Im}f(x)|<K|h|^{\alpha},\quad (0<K,\ 0<\alpha<1),$$

uniformly for all sufficiently large x and for $|h|\leqslant 1$, say. This condition includes the possibility of cusps in $\mathrm{Im}f(x)$ at energies where inelastic thresholds open up. It then follows[3] that the principal-value integral in (1) approaches 0 as $x\to\infty$. Since $|\mathrm{Re}f(x)|\leqslant 1/2q(x)$, we have $F(x)\to 0$ as $x\to\infty$.

The Mandelstam hypothesis[4] enables us to assert that $f(x)$ is the boundary value, as x approaches from above the segment $x>x_0$ of the real axis, of an analytic function $f(z)$ whose other singularities in the z plane are deducible from the postulated singularity structure of the S matrix. For strong-interaction processes of interest, these singularities consist of poles and cuts in the finite z plane and a cut extending to $-\infty$ along the negative real axis. Singular points of $f(z)$ occur in complex-conjugate pairs and $f(z^*)=f^*(z)$ when z is not a singular point. In writing dispersion relations below, for convenience we omit terms arising from the poles and cuts in the finite z plane, since these do not affect our argument. (These omitted singularities may include

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[1] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[2] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961).

[3] W. S. Woolcock (to be published). Under the slightly stronger condition that

$$|(x+h)^{1/2}\,\mathrm{Im}f(x+h)-x^{1/2}\,\mathrm{Im}f(x)|<K|h|^{\alpha},\quad(0<K,\ 0<\alpha<1),$$

uniformly for all sufficiently large x and for $|h|\leqslant 1$, say, it may be shown that

$$\left|P\int_{x_0}^{\infty}\frac{\mathrm{Im}f(x')dx'}{x'-x}\right|\leqslant C(\ln x)x^{-1/2},$$

where C is a constant. This stronger result does not lead to an improvement of the theorem we are going to prove.

[4] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 (1959); **115**, 1752 (1959).

a cut extending below x_0 if inelastic channels are open below the elastic threshold.) We simply take $f(z)$ to have the two cuts $(-\infty, -x_0]$ and $[x_0, \infty)$, with $x_0>0$. For each x such that $|x|>x_0$, we assume that $f(z)$ approaches a limit [which we shall denote by $f(x)$] as $z\to x$ from above the real axis. We shall use this wording in the statement of theorem 1, but its precise meaning needs to be made clear. Given an arbitrary $\epsilon>0$, we mean that it is possible to find a $\delta>0$ such that $|f(z)-f(x)|<\epsilon$ for all z for which $|z-x|<\delta$ *and* $\mathrm{Im}z>0$. Note that $f(x)$ is a continuous function of x for each x with $|x|>x_0$, and that the discontinuity across either cut is $2i\,\mathrm{Im}f(x)$.

In order to write a dispersion relation for $f(z)$ it is sufficient to assume that there exists a non-negative integer N such that

$$f(z)/z^N\to 0,\quad \text{uniformly as } |z|\to\infty \text{ for } |\arg z|\leqslant\pi.$$

Applying Cauchy's integral theorem to the function $f(z)/z^N$ and assuming that $f(z)(z\mp x_0)\to 0$ as $z\to\pm x_0$, uniformly for $|\arg(z\mp x_0)|\leqslant\pi$, we see that, if z is not a singular point and $R>\max\{x_0,|z|\}$, then

$$f(z)/z^N=\sum_{j=0}^{N-1}\frac{a_j}{z^{N-j}}+\frac{1}{\pi}\int_{x_0}^{R}\frac{\mathrm{Im}f(x')dx'}{x'^N(x'-z)}+\frac{1}{\pi}\int_{-R}^{-x_0}\frac{\mathrm{Im}f(x')dx'}{x'^N(x'-z)}+\frac{1}{2\pi i}\oint\frac{f(z')dz'}{z'^N(z'-z)}.$$

The constants a_j are real; the last integral is taken round the circle $|z|=R$. Now as $R\to\infty$, the last integral approaches 0. But the first integral converges as $R\to\infty$, and so, therefore, does the second. Multiplying by z^N, we have

$$f(z)=\sum_{j=0}^{N-1}a_jz^j+\frac{z^N}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'^N(x'-z)}+\frac{z^N}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^N(x'-z)}.\quad (2)$$

We use an arrow always to indicate that the limit of a Riemann integral is being taken.

We now use the following identity[5] to alter the form of the dispersion relation (2):

$$z/x(x-z)=1/(x-z)-1/x.$$

Suppose that n is the smallest non-negative integer for which

$$\int_{\to-\infty}\frac{\mathrm{Im}f(x)dx}{x^{n+1}}$$

exists; clearly $n\leqslant N$. Successive applications of this identity then give the dispersion relation

$$f(z)=\sum_{j=0}^{N-1}A_jz^j+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'-z}+\frac{z^n}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-z)},\quad (3)$$

where the real constants A_j will in general be different from the constants a_j. Since $\mathrm{Im}f(x)$ satisfies a Lipschitz condition for each $x>x_0$, it follows that

$$\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{(x'-x)-iy}\to\frac{1}{\pi}P\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'-x}+i\,\mathrm{Im}f(x)$$

as $y\to 0$ from above. Therefore, using Eq. (3), we see that the function $F(x)$ defined in Eq. (1) has the representation

$$F(x)=\sum_{j=0}^{N-1}A_jx^j+\frac{x^n}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-x)}.\quad (4)$$

To see the consequences of the fact that $F(x)\to 0$ as $z\to\infty$, we write Eq. (4) in the form

$$\frac{1}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-x)}=-\sum_{k=(n-N+1)}^{n}\frac{A_{n-k}}{x^k}+F(x)/x^n.\quad (5)$$

By a standard theorem,[6] the integral on the left side of (5) approaches 0 as $x\to\infty$. Since the second term on the right side also approaches 0, it is clear that the sum in the first term on the right side must begin with the term A_{n-1}/x; all the constants $A_n, \cdots, A_{N-1}$ must vanish. Hence Eq. (3) becomes

$$f(z)=\sum_{j=0}^{n-1}A_jz^j+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'-z}+\frac{z^n}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x-z)}.\quad (6)$$

This means that if n is the number of subtractions required in order to make the integral over the left-hand cut converge, then the degree of the polynomial in the representation (6) of $f(z)$ is $(n-1)$.

To obtain the most interesting result, note that Eq. (5) may now be written

$$\frac{1}{\pi}\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-x)}=-\sum_{k=1}^{n}\frac{A_{n-k}}{x^k}+F(x)/x^n.$$

This means that, if $n\geqslant 1$, we have

$$\int_{\to-\infty}^{\to -x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-x)}\sim-\pi A_{n-1}/x\quad\text{as}\quad x\to\infty.$$

[5] The argument given here was also used by J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (N.Y.) **17**, 1 (1962) and by J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[6] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, New Jersey, 1941), Chap. VIII.

Further, since n has been chosen to be as small as possible, we know that, when $n\geqslant 1$, $\int_{\to-\infty}[\mathrm{Im}f(x)dx/x^n]$ does not exist. We now appeal to the following theorem of Hardy and Littlewood[7]:

Let $g(x)$ belong to $L(0,X)$ for every $X>0$, and let the integral

$$f(x)=\int_0^{\to\infty}\frac{g(x')dx'}{x'+x}$$

converge. If a constant $K>0$ exists, such that either (i) $g(x)>-Kx^{-1}$ $(0\leq x<\infty)$, or (ii) $g(x)<Kx^{-1}$ $(0\leq x<\infty)$, then $f(x)\sim Ax^{-1}$ $(x\to\infty)$ implies that

$$\int_0^{\to\infty}g(x)dx=A.$$

It is clear that, for the theorem to hold, it is sufficient that either of the conditions (i) and (ii) hold for all $x\geqslant X$, where X is some positive number. The application of the theorem is straightforward; neither of the conditions (i) and (ii) can hold for $\mathrm{Im}f(x)/x^n$ when $n\geqslant 1$. It is therefore *impossible* to find constants $K>0$, $X\geqslant x_0$, such that either $\mathrm{Im}f(x)>-K(-x)^{n-1}$ or $\mathrm{Im}f(x)<K(-x)^{n-1}$ for all $x\leqslant -X$. Thus, if $n\geqslant 1$, $\mathrm{Im}f(x)$ must have an oscillatory behavior as $x\to-\infty$, and this oscillatory behavior becomes more violent as n increases.

We conclude this section with a full statement of the results we have proved.

Theorem 1. Let $f(z)$ be an analytic function, regular in the whole z plane cut along the real axis from x_0 to ∞ and from $-x_0$ to $-\infty$ $(x_0>0)$. Suppose that, when z is not a singular point, $f(z^*)=f^*(z)$. [For this it is necessary and sufficient that $f(x)$ is real for $-x_0<x<x_0$.] For each x such that $|x|>x_0$, let $f(z)$ approach a limit [which we denote by $f(x)$] as $z\to x$ from above the real axis.

Suppose that (a) $\mathrm{Re}f(x)\to 0$ as $x\to\infty$, (b)

$$\int^{\to\infty}\frac{\mathrm{Im}f(x)dx}{x}$$

exists and $\mathrm{Im}f(x)\ln x\to 0$ as $x\to\infty$, (c) $\mathrm{Im}f(x)$ satisfies a Lipschitz condition for each $x>x_0$, and (d)

$$|\mathrm{Im}f(x+h)-\mathrm{Im}f(x)|<K|h|^{\alpha},\quad (0<K,\ 0<\alpha<1),$$

uniformly for all sufficiently large x and for $|h|\leqslant 1$, say.

Suppose that there exists an integer $N\geqslant 0$ such that $f(z)/z^N\to 0$, uniformly as $|z|\to\infty$ for $|\arg z|\leqslant\pi$, and that $f(z)(z\mp x_0)\to 0$ as $z\to\pm x_0$, uniformly for $|\arg(z\mp x_0)|\leq\pi$. Further, let n be the smallest nonnegative integer for which

$$\int_{\to-\infty}\frac{\mathrm{Im}f(x)dx}{x^{n+1}}$$

exists.

Then $f(z)$ has the representation

$$f(z)=\sum_{j=0}^{n-1}A_jz^j+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'-z}+\frac{z^n}{\pi}\int_{\to-\infty}^{\to-x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-z)},$$

where the A_j are real constants.

Further, if $n>0$, it is impossible to find constants $K>0$, $X\geqslant x_0$, such that either $\mathrm{Im}f(x)>-K(-x)^{n-1}$ or $\mathrm{Im}f(x)<K(-x)^{n-1}$ for all $x\leqslant -X$.

III. ANOTHER THEOREM

We now give a rigorous proof of the theorem given by Sugawara and Kanazawa,[2] stating clearly a set of sufficient conditions. We begin by assuming that the conditions of the first paragraph of theorem 1 are satisfied by $f(z)$. Again we omit for convenience poles and cuts in the finite z plane. Likewise, the condition that $f(z^*)=f^*(z)$ when z is not a singular point is convenient but not necessary; it could be dropped at the expense of a more tedious statement of the theorem.

Conditions (a) and (b) are replaced by $\mathrm{Re}f(z)\to c_1$, as $x\to\infty$; $\mathrm{Im}f(x)\to c_2\neq 0$ as $x\to\infty$;

$$\int^{\to\infty}\frac{dx}{x}[\mathrm{Im}f(x)-c_2]$$

exists and $[\mathrm{Im}f(x)-c_2]\ln x\to 0$ as $x\to\infty$. The rest of the conditions remain as in theorem 1.

We can now write for $f(z)$ a representation analogous to that given in Eq. (3), namely,

$$f(z)=\sum_{j=0}^{N-1}A_jz^j+\frac{z}{\pi}\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'(x'-z)}+\frac{z^n}{\pi}\int_{\to-\infty}^{\to-x_0}\frac{\mathrm{Im}f(x')dx'}{x'^n(x'-z)},\quad (7)$$

where the A_j are real constants. When z approaches the cut $[x_0,\infty)$ from above, we replace the left side of (7) by $\mathrm{Re}f(x)$ and the first integral on the right side by a principal-value integral. Now

$$\frac{x}{\pi}P\int_{\to x_0}^{\to\infty}\frac{\mathrm{Im}f(x')dx'}{x'(x'-x)}=\frac{P}{\pi}\int_{\to x_0}^{\to\infty}\frac{[\mathrm{Im}f(x')-c_2]dx'}{x'-x}-\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{[\mathrm{Im}f(x')-c_2]dx'}{x'}+\frac{c_2}{\pi}P\int_{\to x_0}^{\to\infty}dx'\left(\frac{1}{x'-x}-\frac{1}{x'}\right).$$

The conditions imposed above ensure that the first

[7] G. H. Hardy and J. E. Littlewood, Proc. London Math. Soc. **30**, 23 (1930).

integral on the right side approaches 0 as $x\to\infty$, so that

$$\frac{x}{\pi}P\int_{\to x_0}^{\to\infty}\frac{\operatorname{Im}f(x')dx'}{x'(x'-x)}=-\frac{c_2}{\pi}\ln x+c+o(1)\quad\text{for large } x,$$

where

$$c=\frac{c_2}{\pi}\ln x_0-\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{[\operatorname{Im}f(x')-c_2]dx'}{x'}.$$

It follows from Eq. (7) that

$$\frac{1}{\pi}\int_{\to-\infty}^{\to-x_0}\frac{\operatorname{Im}f(x')dx'}{x'^n(x'-x)}=\frac{c_2\ln x}{\pi x^n}-\sum_{k=(n-N+1)}^{n-1}\frac{A_{n-k}}{x^k}-\frac{A_0'}{x^n}+o\left(\frac{1}{x^n}\right)\quad(8)$$

for large x, with $A_0'=A_0+c-c_1$. Again we know that the left side approaches 0 as $x\to\infty$. Since the first term on the right side cannot be cancelled by the other terms as $x\to\infty$, we must have $n\geqslant 1$. It follows, as in Sec. II, that the constants $A_n,\cdots,A_{N-1}$ must vanish. Further, if $n>1$, the left side of (8) $\sim -A_{n-1}/x$ as $x\to\infty$, so that, applying the Hardy-Littlewood theorem, we have: (i) If $n>1$, it is impossible to find constants $K>0$, $X\geqslant x_0$ such that either $\operatorname{Im}f(x)>-K(-x)^{n-1}$ or $\operatorname{Im}f(x)<K(-x)^{n-1}$ for all $x\leqslant -X$; (ii) If there exist constants $K>0$, $X\geqslant x_0$ such that either $\operatorname{Im}f(x)>Kx$ or $\operatorname{Im}f(x)<-Kx$ for all $x\leqslant -X$, then $n=1$. We assume that condition (ii) is satisfied and proceed.

Equation (7) now takes the form

$$f(z)=A_0+\frac{z}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'}{x'}\left(\frac{\operatorname{Im}f(x')}{x'-z}+\frac{\operatorname{Im}f(-x')}{x'+z}\right),$$

which may be rewritten as

$$f(z)=A+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'[\operatorname{Im}f(x')-c_2]}{x'-z}+\frac{z}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'[\operatorname{Im}f(-x')-c_2]}{x'(x'+z)}+\frac{c_2}{\pi}\int_{\to x_0}^{\to\infty}dx'\left(\frac{1}{x'-z}-\frac{1}{x'+z}\right),\quad(9)$$

where

$$A=A_0-\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'}{x'}[\operatorname{Im}f(x')-c_2].$$

Hence, as $x\to\infty$,

$$\frac{x}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'[\operatorname{Im}f(-x')-c_2]}{x'(x'+x)}\to(c_1-A).$$

Provided that there exist constants $K>0$, $X\geqslant x_0$ such that either $\operatorname{Im}f(x)>-K$ or $\operatorname{Im}f(x)<K$ for all $x\leqslant -X$, it follows from the Hardy-Littlewood theorem that

$$\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'}{x'}[\operatorname{Im}f(-x')-c_2]=c_1-A,$$

so that Eq. (9) becomes

$$f(z)=c_1+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'[\operatorname{Im}f(x')-c_2]}{x'-z}-\frac{1}{\pi}\int_{\to x_0}^{\to\infty}\frac{dx'[\operatorname{Im}f(-x')-c_2]}{x'+z}+\frac{c_2}{\pi}\int_{\to x_0}^{\to\infty}dx'\left(\frac{1}{x'-z}-\frac{1}{x'+z}\right)\quad(10)$$

$$=c_1+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}dx'\left(\frac{\operatorname{Im}f(x')}{x'-z}-\frac{\operatorname{Im}f(-x')}{x'+z}\right).$$

Finally, we note that, by a simple extension of the theorem quoted in Ref. 6, we may assert the following.[8] The first integral on the right side of Eq. (10) approaches 0 as $|z|\to\infty$ in any direction for which $\arg z\neq 0$, uniformly for $\delta\leqslant\arg z\leqslant 2\pi-\delta$, where $\delta>0$. Similarly, the second integral approaches 0 as $|z|\to\infty$ in any direction for which $\arg z\neq\pi$, uniformly for $|\arg z|\leqslant\pi-\delta$. Further, by a simple computation, we see that the third integral $\to ic_2$ as $|z|\to\infty$ in any direction for which $0<\arg z<\pi$, uniformly for $\delta\leqslant\arg z\leqslant\pi-\delta$, and $\to -ic_2$ as $|z|\to\infty$ in any direction for which $-\pi<\arg z<0$, uniformly for $-\pi+\delta\leqslant\arg z\leqslant-\delta$. Hence $f(z)\to(c_1+ic_2)$ as $|z|\to\infty$ in any direction for which $0<\arg z<\pi$, uniformly for $\delta\leqslant\arg z\leqslant\pi-\delta$, and $f(z)\to(c_1-ic_2)$ as $|z|\to\infty$ in any direction for which $-\pi<\arg z<0$, uniformly for $-\pi+\delta\leqslant\arg z\leqslant-\delta$ $(\delta>0)$.

We now write down a statement of the theorem we have proved.

Theorem 2. The conditions are exactly as in theorem 1, except that (a) and (b) are replaced by $\operatorname{Re}f(x)\to c_1$ as $x\to\infty$; $\operatorname{Im}f(x)\to c_2\neq 0$ as $x\to\infty$;

$$\int^{\to\infty}\frac{dx}{x}[\operatorname{Im}f(x)-c_2]$$

exists and $[\operatorname{Im}f(x)-c_2]\ln x\to 0$ as $x\to\infty$. c_1 and c_2 are constants. We also impose the extra condition that there exist constants $K>0$, $X\geqslant x_0$ such that either $\operatorname{Im}f(x)<K$ or $\operatorname{Im}f(x)>-K$ for all $x\leqslant -X$. Then $f(z)$ has the representation

$$f(z)=c_1+\frac{1}{\pi}\int_{\to x_0}^{\to\infty}dx'\left(\frac{\operatorname{Im}f(x')}{x'-z}-\frac{\operatorname{Im}f(-x')}{x'+z}\right).\quad(11)$$

Further, $f(z)\to(c_1+ic_2)$ as $|z|\to\infty$ in any direction for which $0<\arg z<\pi$, uniformly for $\delta\leqslant\arg z\leqslant\pi-\delta$ and

[8] For a full proof see W. S. Woolcock (to be published).

$f(z) \to (c_1 - ic_2)$ as $|z| \to \infty$ in any direction for which $-\pi < \arg z < 0$, uniformly for $-\pi + \delta \leqslant \arg z \leqslant -\delta$ ($\delta > 0$). If $f(z)$ is taken to be $f(x)$ on the upper side of the cut $[x_0, \infty)$ and $[f(x)]^*$ on the lower side, the uniform convergence may be extended up to $\arg z = 0$, instead of to δ or $-\delta$.[9]

IV. APPLICATION TO $\pi^{\pm}$-p AND $K^{\pm}$-p DISPERSION RELATIONS

The dispersion relations for $\pi^{\pm}$-p and $K^{\pm}$-p elastic scattering provide a straightforward application of theorem 2. Let $f_{\pm}(\omega)$ denote the forward-elastic-scattering amplitudes for $\pi^{\pm}(K^{\pm})$-p scattering in the *laboratory* system, where ω is the total energy of the $\pi(K)$ in the lab system. Denote the analytic functions of which $f_{\pm}(\omega)$ are the boundary values by $f_{\pm}(z)$. We shall assume that the only singularities of $f_{\pm}(z)$ are the cuts $[\mu, \infty)$ and $(-\infty, -\mu]$, where μ is the $\pi(K)$ mass. The cuts extending below the physical threshold for π^-p and K^-p scattering and the single-particle poles provide extra terms in the dispersion relations which we do not want to write down here.

In writing dispersion relations for $f_{\pm}(z)$ we make use of the *crossing relations*

$$\mathrm{Re} f_{\pm}(-\omega) = \mathrm{Re} f_{\mp}(\omega),$$
$$\mathrm{Im} f_{\pm}(-\omega) = -\mathrm{Im} f_{\mp}(\omega),$$

and the *optical theorem*

$$\mathrm{Im} f_{\pm}(\omega) = \frac{q(\omega)}{4\pi}\sigma_{\pm}(\omega),$$

where $q(\omega) = (\omega^2 - 1)^{1/2}$ is the momentum of the $\pi(K)$ in the lab system and $\sigma_{\pm}(\omega)$ are the total cross sections for $\pi^{\pm}(K^{\pm})$-p. We denote $\mathrm{Re} f_{\pm}(\omega)$ by $D_{\pm}(\omega)$.

We intend to write a dispersion relation for $f_+(z)/z$. The required conditions are

(a) $D_+(\omega)/\omega \to c$, a constant, as $\omega \to \infty$,

(b) $\sigma_+(\omega) \to \sigma_+(\infty)$, a constant, as $\omega \to \infty$,

$$\int^{\to\infty} \frac{d\omega}{\omega}[\sigma_+(\omega) - \sigma_+(\infty)]$$

exists and $[\sigma_+(\omega) - \sigma_+(\infty)] \ln\omega \to 0$ as $\omega \to \infty$,

(c) $\sigma_+(\omega)$ satisfies a Lipschitz condition for each $\omega > \mu$,

(d) $|\sigma_+(\omega + h) - \sigma_+(\omega)| < K|h|^{\alpha}$, ($0 < K$, $0 < \alpha < 1$), *uniformly* for all sufficiently large ω and for $|h| \leqslant 1$, say, and

(e) there exists an integer $N > 1$ such that

$$f_+(z)/z^N \to 0,$$

uniformly as $|z| \to \infty$ for $|\arg z| \leqslant \pi$.

[9] This is true under the uniform Lipschitz condition on $\mathrm{Im} f(x)$ on the right-hand cut. See Ref. 3.

The conditions at $\omega = \pm\mu$ and the condition on $\mathrm{Im} f_+(-\omega)$ are obviously satisfied.

Applying theorem 2, we see that $f_+(z)/z$ has the representation

$$\frac{f_+(z)}{z} = c + \frac{C}{z} + \frac{1}{4\pi^2}\int_{\mu}^{\to\infty} d\omega' \frac{q(\omega')}{\omega'}\left(\frac{\sigma_+(\omega')}{\omega' - z} - \frac{\sigma_-(\omega')}{\omega' + z}\right), \quad (12)$$

where $C = f_+(0)$. Adding the extra condition that $\sigma_-(\omega)$ satisfies a Lipschitz condition for each $\omega > \mu$, letting z approach the cut $[\mu, \infty)$ and the cut $(-\infty, -\mu]$ from above, and using the crossing relation for $\mathrm{Re} f_+(-\omega)$ in the latter case, we obtain the dispersion relations

$$D_{\pm}(\omega) = \pm c\omega + C + \frac{\omega}{4\pi^2} P \int_{\mu}^{\to\infty} d\omega' \times \frac{q(\omega')}{\omega'}\left(\frac{\sigma_{\pm}(\omega')}{\omega' - \omega} - \frac{\sigma_{\mp}(\omega')}{\omega' + \omega}\right). \quad (13)$$

These relations are in a very convenient form for numerical calculations.[10] Empirically it is found that an acceptable statistical fit is obtained when c is taken to be 0. This may be taken as indirect evidence that $D_{\pm}(\omega)/\omega \to 0$ as $\omega \to \infty$.

It is worth noting that, under the conditions assumed, the integrals

$$\int^{\to\infty} \frac{d\omega}{\omega}[\sigma_-(\omega) - \sigma_+(\infty)] \quad \text{and} \quad \int^{\to\infty} \frac{d\omega}{\omega}[\sigma_+(\omega) - \sigma_-(\omega)]$$

exist, but it is not necessarily true that $\sigma_-(\omega) \to \sigma_+(\infty)$ or that $\sigma_+(\omega) - \sigma_-(\omega) \to 0$ as $\omega \to \infty$. If, however, we assume in addition that $\sigma_-(\omega) \to \sigma_-(\infty)$, a constant, as $\omega \to \infty$, it follows from the existence of the integral

$$\int^{\to\infty} \frac{d\omega}{\omega}[\sigma_+(\omega) - \sigma_-(\omega)]$$

that $\sigma_+(\infty) = \sigma_-(\infty)$ (the famous Pomeranchuk theorem[11]). If we go further and assume that conditions (b), (c) and (d) hold with $\sigma_+(\omega)$ replaced by $\sigma_-(\omega)$, it follows that $D_-(\omega)/\omega \to -c$ as $\omega \to \infty$.

It will have been observed that there is an asymmetry in the conditions required in order to establish Eq. (13). In making our assumptions we could have interchanged the subscripts + and − throughout. This asymmetry seems to be necessary if one tries to establish relations of the type (13), with just two constants, by imposing the physically reasonable conditions $D_{\pm}(\omega)/\omega \to c_{\pm}$ and $\sigma_{\pm}(\omega) \to \sigma_{\pm}(\infty)$ as $\omega \to \infty$. All these conditions together

[10] For the $\pi^{\pm}$-p forward relations, see V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965), and for the $K^{\pm}$-p relations, see R. Perrin and W. S. Woolcock (to be published).

[11] This result was first suggested by I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **34**, 725 (1958) [English transl.: Soviet Phys.—JETP **7**, 499 (1958)].

are more than sufficient; only one pair is required. On the other hand, if one attempts to keep the conditions symmetrical by writing dispersion relations for $f_+(z)\pm f_-(z)$, then the above conditions on D and σ are not the natural ones to impose.

We conclude by remarking that theorem 2 can also be applied to the fixed momentum-transfer dispersion relations for the invariant amplitudes $A_\pm(\nu,t)$, $B_\pm(\nu,t)$ for $\pi^\pm(K^\pm)$-p scattering. The invariant t is minus the square of the four-momentum transfer and the invariant $\nu=\omega+t/4M$, where M is the proton mass. The crossing properties of these amplitudes are

$$\begin{aligned}\mathrm{Re}A_\pm(-\nu,t)&=\mathrm{Re}A_\mp(\nu,t),\\ \mathrm{Re}B_\pm(-\nu,t)&=-\mathrm{Re}B_\mp(\nu,t),\\ \mathrm{Im}A_\pm(-\nu,t)&=-\mathrm{Im}A_\mp(\nu,t),\\ \mathrm{Im}B_\pm(-\nu,t)&=\mathrm{Im}B_\mp(\nu,t).\end{aligned}$$

The cuts in the amplitudes are $[\nu_0,\infty)$ and $(-\infty,\nu_0]$, where $\nu_0=\mu+t/4M$. Under the conditions

$$\begin{aligned}\mathrm{Re}A_+(\nu,t)/\nu&\to 0, & \mathrm{Im}A_+(\nu,t)/\nu&\to \text{constant},\\ \mathrm{Re}B_+(\nu,t)&\to 0, & \mathrm{Im}B_+(\nu,t)&\to \text{constant},\end{aligned}$$

as $\nu\to\infty$, for $-t(\geqslant 0)$ small (or the same conditions with the subscript $+$ replaced by $-$), together with conditions analogous to the other conditions for the forward-dispersion relations which it is unnecessary to write down, we have the dispersion relations

$$\mathrm{Re}A_\pm(\nu,t)=A(t)+\frac{\nu}{\pi}P\int_{\nu_0}^{\to\infty}\frac{d\nu'}{\nu'}\times\left(\frac{\mathrm{Im}A_\pm(\nu',t)}{\nu'-\nu}-\frac{\mathrm{Im}A_\mp(\nu',t)}{\nu'+\nu}\right),$$

$$\mathrm{Re}B_\pm(\nu,t)=\frac{1}{\pi}P\int_{\nu_0}^{\to\infty}d\nu'\left(\frac{\mathrm{Im}B_\pm(\nu',t)}{\nu'-\nu}-\frac{\mathrm{Im}B_\mp(\nu',t)}{\nu'+\nu}\right).$$

An exhaustive study of the practical applications of these relations will be found in the review paper of Hamilton and Woolcock.[12]

Note added in proof. While the manuscript of this paper was being prepared, a paper by T. Kinoshita on the same subject appeared [Phys. Rev. Letters **16**, 869 (1966)]. He uses a form of partial-wave amplitude which is bounded as $x\to\infty$, and looks for conditions under which $f(z)$ satisfies a once-subtracted dispersion relation. His method, which is based on that of Jin and Martin,[1] requires the following conditions: (a) There exist constants C, $R(>0)$, $\alpha(>0)$ such that $f(z)<\exp[C(\ln|z|)^{2-\alpha}]$ for $|z|>R$. (b) $N(x)$, the number of times that $\mathrm{Im}f(\xi)$ changes its sign in the interval $(x,-x_0)$, where $x<-x_0$, satisfies the inequality $N(x)\leqslant C'(\ln|x|)^{1-\alpha}$ for all x sufficiently negative, where C' is a constant and α is the same as in (a). Our method, which is a quite different one, makes a stronger assumption than (a), namely, that there exists an integer N such that $f(z)/z^N\to 0$, uniformly as $|z|\to\infty$ for $|\arg z|\leqslant\pi$. This in turn makes it possible to weaken assumption (b). It is clear from the discussion leading to theorem 1 that, provided there exist constants $K>0$, $X\geqslant x_0$ such that *either* $\mathrm{Im}f(x)<K$ *or* $\mathrm{Im}f(x)>-K$ for all $x\leqslant -x_0$, a dispersion relation may be written for $f(z)$ without subtraction ($n=0$). The general conclusion is that, in order to write a dispersion relation for $f(z)$ with the minimum number of subtractions possible, it is necessary to assume some uniform bound on $f(z)$ as $|z|\to\infty$, and also to make some assumption about the behavior of $\mathrm{Im}f(x)$ on the left-hand cut. However, there is considerable freedom in the choice of these assumptions.

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[12] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).