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Presymmetry*

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Even in the presence of external fields, space-time symmetry implies nontrivial relations between observables at one time, i.e., kinematical relations. Symmetry operations at one time—translations, rotations, and (for Galilei symmetry) velocity shifts—can be performed on observation-producing and on state-producing instruments, regardless of the existence of an external field. Furthermore, it is possible to give an operational definition of every initial state intrinsically, i.e., regardless of the external field. The precise statement of this empirical fact explains, for example, why a particle in an external field has integral or half-integral eigenvalues of the spin, why a Hamiltonian exists even in the presence of a time-dependent external field, and why (for Galilei symmetry) the canonical commutation relations are still valid, although the full space-time symmetry from which these results can be derived has been destroyed. It is pointed out that the rigorous validity of kinematical relations, in spite of strong breaking of the underlying space-time symmetry, is analogous to the rigorous validity of equal-time current commutation rules, in spite of the breaking of the underlying $U(3)$ symmetry.

I. INTRODUCTION

THE exploration of the constraints imposed on quantum mechanics by space-time symmetry has been very successful, both in predicting power and in the elimination of redundant assumptions. The prime example for the first kind of capability is the prediction that spins can be only integral or half-integral. Perhaps the most spectacular, but not highly advertised, success of symmetry theory in decreasing the number of independent assumptions is the derivation of the canonical commutation relations from the theory of the Galilei group.[1,2]

The presence of an external field destroys space-time symmetry, and at first one sees no reason why any of the results of symmetry theory should remain valid. Yet there are strong indications that some of them do.

Consider a single nonrelativistic particle in an external field. Why is it generally assumed that here, too, the spin can have only integral and half-integral eigenvalues, although there is no rotational symmetry? If one believes that Galilei symmetry is the origin of the canonical commutation relations, one asks why these should remain valid in absence of symmetry. The existence of a Hamiltonian in the time-dependent Schrödinger equation is usually derived from time-translation symmetry. But why is there a Hamiltonian also for time-dependent external fields? Comparison with experiment has favored these assumptions from the beginnings of quantum mechanics, but theory has yet to meet the challenge of showing their relation to prime principles.

Assumptions of the kind discussed may be called *kinematic* because they are supposed to hold regardless of the dynamical specification of the external forces by the Hamiltonian. In most applications, kinematic and dynamical assumptions combine to produce an observable result, and one can ask whether kinematic relations by themselves lead to observable results. The most striking example for the affirmative answer is the historical fact that the canonical commutation relations were originally suggested to Heisenberg by the Thomas-Kuhn sum rules and that, conversely, the canonical commutation relations imply these sum rules regardless of the Hamiltonian.[3]

Presymmetry—the survival of some results of space-time symmetry even when this is broken by an external field—should be deducible from prime principles. It will be shown that the accepted principles are in fact sufficient *if they are explicitly stated.* In discussing observations to test space-time symmetry, a number of tacit assumptions are made habitually; and it is the precise statement of some of these tacit assumptions which explains presymmetry. To compare states and observables connected by symmetry, the experimenter performs an active transformation of observation-producing and of state-producing instruments. Before testing space-time symmetry, the experimenter must be able to test the proper operation of the actively transformed instruments. Another tacit assumption is the possibility of characterizing an initial state intrinsically, regardless of external forces. The feasibility of such a testable active transformation of initial states, quite independent of the existence of an external field, is the statement which leads to presymmetry.

II. OBSERVABLES AND STATES

Common sense seems to provide a simple definition of space transformations: For instance, one-particle wave functions in configuration space $\psi(\mathbf{x},t)$ may be carried into "rotated wave functions" $\psi(R\mathbf{x},t)$ by a rotation R: $\mathbf{x}\to R\mathbf{x}$ of the argument vector $\mathbf{x}$. This transformation defines a unitary operator $U(R)$: $\psi\to U(R)\psi$ on Hilbert space, and one should expect that this transformation induces a similar rotation of all vector observables. For instance, if the rotation R is characterized by $x\to y$, $y\to -x$, $z\to z$, one should

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[1] V. Bargmann, Ann. Math. **59**, 1 (1954).
[2] M. Hamermesh, Ann. Phys. (N. Y.) **9**, 518 (1960).
[3] A. W. Sommerfeld, *Wave Mechanics* (Methuen and Company, Ltd., London, 1930), p. 173.

expect that for every vector observable A_x, A_y, A_z

$$[U(R)\psi, A_x U(R)\psi] = (\psi, A_y \psi),$$

or more generally

$$U(R)\mathbf{A}U^{-1}(R) = R\mathbf{A}.$$

However, in presence of an external field, a vector operator such as the acceleration **a** (being proportional to the external force) will not undergo the desired transformation under $U(R)$. One may be able to define a unitary operator V that carries a_x, a_y, a_z into a_y, $-a_x$, a_z, but it will not be the operator U. In the presence of an external magnetic field, not even the velocity vector will undergo the desired rotation through transformation by U. It appears that common sense is misleading in this case, and that there may be either no or many reasonable definitions of a rotation operator. Clearly, one has to return to prime principles.

As an introduction to presymmetry, it is convenient to reconsider the statement of full space-time symmetry in quantum mechanics in a more explicit manner. The physical statement and the mathematical implications of symmetry under the connected subgroups of the Poincaré and Galilei groups are well understood, but some of the implicit assumptions of the physical statement need further clarification. It has been mentioned occasionally[4] that the definition of an active transformation (e.g., a rotation $A \rightarrow RA$) of an observable A already presupposes a certain amount of symmetry. The following discussion will make these implicit assumptions explicit.

Observables are images of procedures by which a system is made to interact with a macroscopic apparatus. Similarly, states are images of procedures by which sample systems are ejected from a macroscopic generator. A large number of samples produced successively by a given state-producing procedure are subjected to observation. The mean value of many observations by a given procedure is the expectation value $\omega_s(A)$ of the observation procedure A with respect to the state-producing procedure s.

The instructions for the procedure consist of two parts: (a) specifications for the construction of the "black box" and (b) specifications for the spatial position, spatial orientation, state of uniform motion, and time of operation of the instrument. An active transformation consists in changing the latter specifications in the same way for all observables or all states or both. *Space-time symmetry requires that such an active transformation of both observables and states leave the expectation values invariant.* This definition implicitly assumes that the same instrument operates in the same way under different circumstances, e.g., in different regions of space—but this assumption may not be valid. For instance, if the active transformation is accomplished by a bodily rotation of an instrument, the angular acceleration during the act of rotation may have ruined the instrument. Clocks may be ruined by being accelerated. Instruments must be properly shielded from external influences: their operation must be independent of an external field. Instead of bodily moving an observation instrument, an active transformation may be accomplished by manufacturing an instrument according to the same instructions in different locations or on different uniformly moving vehicles. It is also more realistic to use different instruments for measuring the same observable in huge fields and in field-free regions. In either case, the instructions for the construction cannot be relied upon for producing the correct results: A given manufacturing procedure, performed in a huge field, may produce a different instrument, and the proper overlapping of measurements by different types of instruments must be verified by tests. A number of requirements, implicitly understood by all experimenters, need explicit formulation.

In quantum mechanics, the set $\mathcal{O}$ of observation procedures is mapped into the algebra $\mathfrak{A}$ of linear operators on Hilbert space. The comparison of different mappings $\Phi_n\colon \mathcal{O}\rightarrow\mathfrak{A}_n$ corresponding to different external forces makes it desirable to single out those features of the mapping that are common to all systems and therefore characterize the observation instruments rather than the diverse dynamical systems. According to the previous definition of observation procedures, two observations performed with the same instrument at different times are represented by distinct points in $\mathcal{O}$. In accordance with conventional quantum mechanics, it will be assumed that every self-adjoint operator in the algebra generated by $\Phi_n\mathcal{O}$ is the image of an observation procedure in $\mathcal{O}$. This assumption is at least dubious for relativistic quantum mechanics.[5] With this reservation in mind, both the Poincaré and the Galilei groups will be considered as possible groups of active transformations.

Observation procedures fall into equivalence classes $\{x_i\}$ such that their images in every mapping Φ_n coincide, i.e., such that $\Phi_n x_i = \Phi_n x_j$ for all n. There are procedures that yield expectation values in the form of real linear combinations of expectation values of other procedures for all systems characterized by $\mathfrak{A}_n$, i.e.,

$$\alpha\Phi_n x_1 + \beta\Phi_n x_2 = \Phi_n x_3, \quad \text{for all } n;\ \ \alpha, \beta \text{ real}. \qquad (2.1)$$

Finally, there are observation procedures whose images are always squares of images of other procedures [in the sense of Eq. (2.4)], i.e.,

$$\Phi_n x_1 = (\Phi_n x_2)^2 \quad \text{for all } n. \qquad (2.2)$$

In general, there are universal polynomial relations of the type

$$P(\Phi_n x_1, \Phi_n x_2, \cdots, \Phi_n x_r) = 0 \quad \text{for all } n, \qquad (2.3)$$

where P is a polynomial in r elements of $\mathfrak{A}_n$.

[4] R. M. F. Houtappel, H. Van Dam, and E. P. Wigner, Rev. Mod. Phys. **37**, 595 (1965).

[5] W. C. Davidon and H. Ekstein, J. Math. Phys. **5**, 1588 (1964).

They define subsets of observation procedures whose results are universally related to each other. Procedures that measure the x and y coordinates of particle positions and all polynomials in these form such a class, but a calorimeter (measuring the energy) and an x-measuring procedure do not belong to the same universal subset, because the energy is independent of the coordinates in some external fields, but not in all fields.

A subset $\mathcal{O}_u$ of $\mathcal{O}$ is universal, if (a) the subalgebras $\{\Phi_n\mathcal{O}_u\}''$ generated by their images $\Phi_n\mathcal{O}_u$ for all mappings Φ_n are isomorphic under all interchanges of indices $n\rightarrow n'$, and (b) if it is closed universally, i.e., if

$$\Phi_n x\in\Phi_n\mathcal{O}_u \quad \text{for all } n, \quad \text{implies } x\in\mathcal{O}_u.$$

The simplest example of a universal subset is generated by one observation procedure x. Its members are all observation procedures whose images are real functions of $\Phi_n x$ for all n. Generally, a subset $\mathcal{O}_{tb}$ is said to be generated by elements $\{x_i\}\in\mathcal{O}_{tb}$ if it includes all those and only those elements x whose images $\Phi_n x$ are self-adjoint members of the algebra generated by $\{\Phi_n x_i\}$ for all n.

Tests for the acceptability of observation procedures for the purpose of symmetry studies will be discussed. Consider the x component j_x of an electric current density and b_x, its square (i.e., a quantity related to power). The measurement of j_x may be made with an ammeter and that of b_x with a bolometer which measures the heat generated by a wire. The statement that $\Phi_n b_x=(\Phi_n j_x)^2$ for all n signifies that the mean values of large numbers of individual measurements j_{xi}, b_{xi} are related through the expression

$$\frac{1}{N}\sum_{i=1}^{N}(j_{xi})^2=\frac{1}{N}\sum_{i=1}^{N}b_{xi} \tag{2.4}$$

for all states and for all n. The observation procedure will be acceptable only if j_y and b_y, the corresponding procedures rotated by 90°, have the same relation

$$\frac{1}{N}\sum_{1}^{N}(j_{yi})^2=\frac{1}{N}\sum_{1}^{N}b_{yi}. \tag{2.5}$$

More generally, if Vx is the rotated procedure x, then the statement that

$$(\Phi_n x)^2=\Phi_n y \quad \text{for all } n \tag{2.6a}$$

implies that

$$(\Phi_n Vx)^2=\Phi_n Vy \quad \text{for all } n. \tag{2.6b}$$

Generally, one of the tests for the acceptability of measurement procedures is that an active transformation preserves squares of images of elements $x_i\in\mathcal{O}_u$ in all $\mathfrak{A}_n$. This requirement is previous to any symmetry statement, and should hold also in the presence of external fields. Next, consider a linear combination of two Cartesian components of the position vector, e.g.,

$$\Phi_n x_1\cos\theta+\Phi_n x_2\sin\theta=\Phi_n x_\theta, \quad \text{for all } n. \tag{2.7}$$

Again, the equal sign means that expectation values with respect to all states are equal. A rotation by 90° will produce the observation procedures

$$x_1\rightarrow x_2, \quad x_2\rightarrow -x_1, \quad x_\theta\rightarrow x_{\theta+\pi/2}, \tag{2.8}$$

and unless

$$\Phi_n x_2\cos\theta-\Phi_n x_1\sin\theta=\Phi_n x_{\theta+\pi/2},$$

the instruments will be discarded as improper. More generally, an active transformation preserves linear combinations of images of elements of $\mathcal{O}_u$ in all $\mathfrak{A}_n$. These statements can be extended to all universal algebraic relations between images of observation procedures, i.e., the statement that

$$P(\Phi_n x_1,\Phi_n x_2\cdots\Phi_n x_r)=0 \quad \text{for all } n, \quad x_1\cdots x_r\in\mathcal{O}_u \tag{2.9a}$$

implies

$$P[\Phi_n V(L)x_1,\Phi_n V(L)x_2\cdots\Phi_n V(L)x_r]=0 \quad \text{for all } n, \tag{2.9b}$$

and all permutations $V(L)$ of $\mathcal{O}$ induced by elements L of the space-time group.

The requirements on active transformations summarized in Eqs. (2.9) imply then that such a permutation restricted to a universal subset $\mathcal{O}_u$ induces an automorphism of the subalgebra $\Phi_n\mathcal{O}_u''$ in $\mathfrak{A}_n$, if $\mathcal{O}_u$ is invariant under the permutation.

Does a permutation $V(L)$ of $\mathcal{O}$ induce an automorphism Q of $\mathfrak{A}$? If L is not a symmetry of the system, the answer is negative. For instance, if $\Phi x=\Phi y$ but $\Phi Vx\neq\Phi Vy$, there is no operator Q on $\mathfrak{A}$ such that $\Phi Vx=Q\Phi x$ for all $x\in\mathcal{O}$.

Another test of the proper operation of instruments refers to the sequence of active transformations. If two space translations are performed consecutively, the order should not affect the operation of the instrument. More generally, if multiplication is identified with successive application of two active transformations, then the group of these transformations must be isomorphic to the Poincaré or Galilei groups.

It is now possible to answer the question raised at the beginning of this section: What is the formal expression of an active transformation (e.g., a rotation) in presence of external fields? For instance, what does tensor observable or tensor-observation procedure mean? If, as usual, the term "observable" is associated in a unique way to an operator, the term "tensor observable" is meaningless, because an operator may be the image of a rotational invariant in $\mathcal{O}$ and also the image of a tensor component in $\mathcal{O}$. For instance, the energy-observation procedure E may be performed by a calorimeter, and a rotation $E\rightarrow V(R)E$ of $\mathcal{O}$ leaves the image invariant by construction of the instrument;

$$\Phi_n E=\Phi_n V(R)E$$

for all n and all R. However, the Hamiltonian may be (for a special field n):

$$\Phi_n E=H=\Phi_n(p^2/2m+x^2)$$

and the observation procedure $(p^2/2m+x^2)$ is not a rotational invariant, but transforms like the sum of tensor components:

$$V(\tfrac{1}{2}\pi)(p^2/2m+x^2)=p^2/2m+y^2$$

and, of course,

$$\Phi_n y^2 \neq \Phi_n x^2.$$

Therefore, it is imperative to consider the space $\mathcal{O}$ of the hardware explicitly. In this language, the term "tensor-observation procedure" can be given a precise meaning: the subset $\{V(R)x\}$ obtained by letting all active rotations $V(R)$ act on an observation procedure $x\in\mathcal{O}$ is a universal subset $\mathcal{O}_u$. It generates a subalgebra $\{\Phi_n V(R)x\}''$ in $\mathfrak{A}$. This subalgebra may be considered as a linear space on which a representation of the group $V(R)$ is given. This representation may be finite-dimensional and it may contain a tensor representation of the rotation group. The elements $x\in\mathcal{O}_u$ whose images are in this subspace are then properly called tensor-observation procedures.

III. PRESYMMETRY

It is essential for the conceptual framework of classical mechanics that initial conditions can be either set or measured independently of (external) forces. If it were otherwise, one could not use test particles to study force fields. For this intrinsic definition of the initial condition, one can use positions and velocities, but not arbitrary complete sets of variables because their definition may involve the external field. For instance, if the energy is used as one of the variables to characterize the initial state, then it may happen that a negative initial value of this variable cannot exist in one external field, while it exists in another field. If one wishes to compare world lines with *the same* initial condition in different fields, only a restricted subset of all observables can be used for the operational definition of the initial state. More precisely, there exists a unique algebra of basic observables $\{A(t)\}$ for every time such that the expectation values of these observables determine every state intrinsically, i.e., regardless of external forces. In this form, the statement is implicitly assumed to be valid in quantum mechanics by those who either perform or analyze observations on symmetry, at least for nonrelativistic phenomena.

What basic sets of observation procedures at one time are usable for the purpose of defining the initial state intrinsically? Clearly, one must postulate that the algebraic relations between their images in the space of observables should be independent of the external field. Hence, the subsets in question are precisely universal subsets $\mathcal{O}_u$ as defined in Sec. II.

Those elements $L\in G_t$ of the space-time group that leave the hypersurface $t=\text{const}$ invariant, will transform a subset $\mathcal{O}_{tb}$ of the basic observation procedures at a time t into another such subset $\mathcal{O}_{tb}'$ for the same time. Since the subset is unique, it must be invariant under the group of active transformations $\{V(L), L\in G_t\}$ so so that $\mathcal{O}_{tb}'=\mathcal{O}_{tb}$. The basic subset must define each state completely, i.e., the positive linear form $\omega(\Phi_n\mathcal{O}_{tb})''$ must have a unique extension to the whole algebra $\mathfrak{A}_n$. Hence, $\Phi_n\mathcal{O}_{tb}$ must be a generating set of the whole algebra of observables $(\Phi_n\mathcal{O})''=\mathfrak{A}_n$.

According to Sec. II, an active transformation $V(L)\mathcal{O}_{tb}(L\in G_t)$ induces an automorphism $Q(L,t)$ of the subalgebra $(\Phi_n\mathcal{O}_{tb})''$; but since this is in fact the whole algebra $\mathfrak{A}_n$, the automorphism is defined everywhere in $\mathfrak{A}_n$. It may be well to consider its physical meaning. The mere existence of a group of automorphisms of $\mathfrak{A}_n$ which is isomorphic to a given group is an empty statement, because one can select a suitable subgroup from the immense group of all automorphisms of $\mathfrak{A}_n$ to realize virtually any group, and that in many different ways. The only relevant automorphisms of $\mathfrak{A}$ are those induced by an operation on the hardware, or, more formally, by a permutation of the set of $\mathcal{O}$ of observation procedures or of a subset of $\mathcal{O}$. The automorphisms $Q(L,t)$ have a restricted physical meaning. For instance, $Q(L,0)$, being an automorphism of $\mathfrak{A}$, will perform a definite operation on the acceleration Φa_x of a particle in an external field, but this operation will have no physical meaning. The reason is that Φa_x is not transformed as the image in $\mathfrak{A}$ of $a_x\in\mathcal{O}$, but as the operator that is assigned by the map $\Phi\mathcal{O}_{tb}$. For example, if the x component a_x of the external force is proportional to x, and its y component vanishes,

$$(\Phi a_x=k\Phi x,\ \Phi a_y=0),$$

then a 90° rotation V induces the transformation $Q(V)k\Phi x=k\Phi y$, rather than $\Phi a_x\rightarrow\Phi a_y=0$.

The group of automorphisms $Q(L,t)$ for a fixed time and for $L\in G_t$ is a homomorphic image of G_t. To see this, consider the two equations

$$\left.\begin{aligned}\Phi V_1x&=Q_1\Phi x\\ \Phi V_2x&=Q_2\Phi x\end{aligned}\right\}\text{for all } x\in\mathcal{O}_{tb},\quad V_{1,2}\in V(G_t).\quad(3.1)$$

Since $\mathcal{O}_{tb}$ is stable with respect to V_1, one can insert $x=V_2y$ into Eq. (3.1) and obtain

$$\Phi V_1(V_2y)=Q_1\Phi V_2y=Q_1Q_2\Phi y\quad\text{for}\quad y\in\mathcal{O}_{tb}.\quad(3.2)$$

The simplest case occurs if the Galilei group is chosen as the space-time group and a single particle is put in an external field. What does the term "single particle" mean? Without external field, it is of course described by an irreducible ray representation of the Galilei group. By extension, we can define a single particle as a system on whose basic algebra $\Phi\mathcal{O}_{tb}''$ the subgroup G_t induces a group of automorphisms $Q(L,t)(L\subset G_t)$ such that these automorphisms are implementable by an irreducible ray representation of G_t.[6] In the case of the

[6] According to an important theorem, there exists for every automorphism Q of $\mathfrak{A}$ a unitary operator U such that $QA=UAU^{-1}$ for every bounded operator A in $\mathfrak{A}$. J. Dixmier, *Les Algèbres d'Opérateurs dans l'Espace Hiberlien* (Gauthier-Villars, Paris, 1957), p. 253.

Galilei group, G_t is the 9-dimensional derivative group[7] whose irreducible ray representations are "the same" as those of the full group (i.e., every irreducible ray representation of G is an extension of an irreducible ray representation of G_t). The well-known results of the ray-representation theory of G can be applied. For the representation of the Lie group of the central extension[7] of G, one has the canonical commutation relations

$$[Q_i,P_j]=i\delta_{ij}, \tag{3.3}$$

where mQ_i are the generators of velocity shifts.[2] For an irreducible representation, the operators Q_i, P_i and the spin operators s_i generate the entire algebra $\mathfrak{A}$. Hence, the inverse images of these operators in $\mathcal{O}_{tb}$ generate this subset in the sense of Sec. II. In this case, the kinematics, i.e., the algebraic relations between the images $\Phi_n\mathcal{O}_{tb}$ of the elements of a basic subset is completely determined by the representation of G_t. These relations are universal, i.e., equally valid for all external fields. For reducible representations of G_t, the canonical commutation relations and the algebra of orbital and spin angular momenta at one time follow from this consideration.

Truly observable consequences of these results are obtained only if specific operational meaning is assigned to the generators. As for the free particles, one interprets Q as a position and P as the momentum. Unfortunately, the operational meaning of the momentum (the canonical momentum) is not as clear as in the case of the free particle, since momentum is neither identical with mass×velocity nor conserved in time. Similarly, the canonical angular momentum

$$J=Q\times P \tag{3.4}$$

is neither conserved in time nor proportional to the direct product of velocity and position. It is necessary to devise operational definitions of these observables to give a physical interpretation to the theoretical results. This will be discussed in a subsequent paper.

It is remarkable that, contrary to classical mechanics, the velocities cannot be used as elements of basic universal sets. In the presence of a magnetic field, the commutation relations between the three velocity operators depend on the magnetic field and are therefore not universal. Therefore, a state defined by sharp numerical values of the velocities exists (improperly) without a magnetic field, but "the same" state does not exist in the presence of a magnetic field, since the velocity components fail to commute.

It is questionable whether the basic assumptions, in particular the existence of observation procedures for measurements at a sharply defined instant, are tenable for the Poincaré group.[5] Even if one assumes that they are, application to the Poincaré group gives only meager results. For the Poincaré group, the subgroup G_t is only the Euclidean group. Its irreducible representations cannot be extended to representations of the whole group, and its Lie algebra does not generate the algebra $\mathfrak{A}$ for any physical system. One can still conclude that momenta and angular momenta are well defined for each time and that their algebra is a representation of the Lie algebra of the Euclidean group. But not even for a single particle can the inverse images in $\mathcal{O}_{tb}$ of these operators generate a basic universal subset $\mathcal{O}_{tb}$.

Let us summarize the results of this section. The empirical statement that it is possible to give an operational definition of any initial state intrinsically (regardless of the external field) has the following formal implication: There exists, for every instant t, a basic universal subset $\mathcal{O}_{tb}$ of the set of observation procedures $\mathcal{O}$ such that

(a) The basic subset $\mathcal{O}_{tb}$ is invariant under the group G_t of those active transformations L that leave the hyperplane $t=$const invariant, i.e.,

$$V(L)\mathcal{O}_{tb}=\mathcal{O}_{tb}\ (L\in G_t).$$

(b) Each image $\Phi_n\mathcal{O}_{tb}\in\mathfrak{A}_n$ generates the whole algebra $\mathfrak{A}_n$.

It follows that there exists a homomorphism $G_t\rightarrow Q(L,t)$ of G_t into a group of automorphisms of $\mathfrak{A}_n$. These automorphisms have a physical meaning only for the images of $\mathcal{O}_{tb}$, i.e., the equation

$$\Phi_n V(L)x=Q(L,t)\Phi_n x \quad (L\in G_t,\ \text{all } n)$$

holds only for $x\in\mathcal{O}_{tb}$.

IV. THE HAMILTONIAN FOR A TIME-DEPENDENT EXTERNAL FORCE

The permutation $V(\tau)$ of $\mathcal{O}$ induced by a time translation τ is the formal expression of a time delay of the instant at which measurements are performed—this delay being understood as relative to a clock that starts a state-producing procedure. This permutation $V(\tau)$ induces an automorphism Q of $\mathfrak{A}$ only in the absence of external fields, but its restriction $V(\tau)\mathcal{O}_{tb}$ to a universal subset of basic observation procedures will be shown to induce an automorphism $Q(\tau,t)$ of $\mathfrak{A}$ always. Indeed, the subset $\mathcal{O}_{tb}$ is transformed into another subset $\mathcal{O}_{t+\tau,b}$, and since $\Phi_nV(\tau)$ is in the class of the mappings $\{\Phi_n\}$, the universal nature of the subset requires that there exist an isomorphism $Q(\tau,t)$ from $\Phi_n\mathcal{O}_{tb}$ to $\Phi_n\mathcal{O}_{t+\tau,b}$ such that

$$\Phi V(\tau)x_t=Q(t,\tau)\Phi x_t, \quad (x_t\in\mathcal{O}_{tb}). \tag{4.1}$$

And since the algebra generated by $\Phi_n\mathcal{O}_{tb}$ constitutes all of $\mathfrak{A}_n$, the isomorphism is in fact an automorphism of $\mathfrak{A}_n$.

However, the group of automorphisms $Q(t,\tau)$ is not necessarily a realization of the one-parameter Abelian time-translation group. The argument in Sec. III that proved the corresponding statement for members L of G_t does not apply because the range of $V(\tau)\mathcal{O}_{tb}$ is not

[7] J.-M. Léby-Leblond, J. Math. Phys. 4, 776 (1963).

$\mathcal{O}_{tb}$, but $\mathcal{O}_{t+\tau,b}$. Nevertheless, a residue of time-translation symmetry remains. The relation

$$V(\tau_2)V(\tau_1)\mathcal{O}_{tb}=V(\tau_1+\tau_2)\mathcal{O}_{tb} \tag{4.2}$$

induces the relation

$$Q(\tau_2, t+\tau_1)Q(\tau_1,t)=Q(\tau_1+\tau_2, t) \tag{4.3}$$

on $\mathfrak{A}$.

These automorphisms can be implemented by unitary operators.[6] With a slight change of notation, we choose the association $Q(x,y)\rightarrow U(x+y, y)$, with $U(y,y)=1$, so that the multiplication table of the unitary operators reads

$$U(t_3,t_2)U(t_2,t_1)=U(t_3,t_1). \tag{4.4}$$

The Schrödinger equation is obtained by

$$i(\partial/\partial t_3)U(t_3,t_1)\lim_{\epsilon\rightarrow 0}(1/\epsilon)[U(t_3+\epsilon, t_2) - U(t_3,t_2)]U(t_2,t_1). \tag{4.5}$$

On the right-hand side, one can set $t_2=t_3=t$, so that

$$i(\partial/\partial t)U(t,t_1)=i\lim_{\epsilon\rightarrow 0}(1/\epsilon)[U(t+\epsilon, t)-1]U(t,t_1). \tag{4.6}$$

The operator

$$V(t)=i\lim_{\epsilon\rightarrow 0}(1/\epsilon)[U(t+\epsilon, t)-1] \tag{4.7}$$

will be shown to be Hermitian. Differentiating the equation

$$U(t)U^{\dagger}(t)=1 \tag{4.8}$$

and inserting Eq. (4.6) gives

$$iVUU^{\dagger}-iUU^{\dagger}V^{\dagger}=0, \tag{4.9}$$

which shows that $V=V^{\dagger}$. The usual form of the Schrödinger equation is obtained by defining

$$\Psi(t)=U(t,t_1)\Phi, \tag{4.10}$$

where Φ is a Heisenberg state. With this substitution, Eq. (4.6) becomes

$$i(\partial\Psi/\partial t)=V(t)\Psi(t). \tag{4.11}$$

This deduction is purely formal. To make it rigorous, one would have to show (1) that the phases of the ray representation of the group $\{Q\}$ can be set equal to zero without loss of generality, (2) that the limit $V(t)$ exists, and (3) that it is self-adjoint.

V. RELATION TO CURRENT ALGEBRAS

Gell-Mann[8] and others have introduced an apparently new concept of symmetry into physics. A set of equal-time commutation rules define a Lie algebra of operators on Hilbert space, but they do not imply the existence of symmetry operators which commute with either the Hamiltonian, the S operator, the Lagrangian, or even with the restriction of any of these operators to a subspace. Nevertheless, rigorous sum rules can be deduced from the assumption that their commutation rules are rigorously satisfied. Actually, this situation is not as novel as it seems to be at first. Quantum mechanics has, from its beginning, assumed a set of equal-time commutation rules quite independently of the dynamics. The canonical commutation relations by themselves do not induce any dynamical symmetry, but they lead to rigorous sum rules. The Thomas-Kuhn sum rules for spectral intensities depend only on the *kinematics*, i.e., the canonical commutation rules. As pointed out in the Introduction, the canonical commutation relations were suggested to Heisenberg first by the empirical sum rule, while the recent discoveries were motivated more indirectly by approximate symmetries and led to sum rules by mathematical deduction.

This analogy may seem rather far-fetched and superficial, because one does not usually think of the nonrelativistic kinematics as reflecting a symmetry, while the current-commutation rules originate in a (badly broken) symmetry. However, the study of the ray representations of the Galilei group has shown that the canonical commutation relations have their origin in the Galilei symmetry of nonrelativistic physics: The momentum is the generator of unitary translation operators, and the product mass $\times$ position is the generator of unitary "acceleration" operators. If the Galilei symmetry is broken by an external field, it is traditional to maintain the canonical equal-time relations. Similarly, for a particle with spin, the Pauli commutation relations between the spin components are traditionally maintained as kinematic equations, even when an external field is present, although they originate in the representation of the rotation group. A systematic investigation of space-time symmetry broken by an external field has shown that a residue of the original symmetry, a presymmetry, remains and necessarily requires the rigorous persistence of the kinematics of the fully symmetric system. Thereby, the analogy becomes strong. On the one hand, Galilei symmetry broken by an external field still leads to equal-time canonical commutation rules, and thereby to the Thomas-Kuhn rigorous sum rules. On the other hand, $U(3)$ symmetry broken by an unknown agent still leaves a residue of presymmetry, reflected in rigorous kinematic commutation relations and consequent rigorous sum rules.

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[8] M. Gell-Mann, Physics 1, 63 (1964).