Synchrotron Radiation of Neutrinos and Its Astrophysical Significance*†

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The neutrino radiation from a completely relativistic electron gas in the presence of a large magnetic field (the neutrino analog of electromagnetic synchrotron radiation) is computed approximately, under the assumption of a direct electron-neutrino interaction. The radiation is also estimated in the case of a nonrelativistic electron gas. It is shown that the radiation increases strongly with electron energy and magnetic-field strength, and is therefore most likely to have astrophysical significance in the evolution of stars with large electron energies and potentially large magnetic fields, such as white dwarfs. However, computation of the total neutrino luminosity of several model white dwarfs shows that the neutrino luminosity is limited by electron degeneracy to values much less than the photon luminosity. The process is also found to be relatively unimportant in the evolution of neutron stars.

I. INTRODUCTION

It has been shown by a number of authors that radiation of neutrinos can be of great importance during certain stages of stellar evolution. This situation arises because of the extremely large mean free path of a neutrino in matter. The probability of creating a neutrino is minute, but once created, it is almost certain to escape directly from the star, carrying off whatever energy was used in its creation. This is in contrast to electromagnetic radiation, for example, which is copiously created inside a star but must diffuse slowly out through the star and be radiated from the surface. Under some circumstances late in the life of a star, neutrino radiation can be the principal means of energy loss.[1] Calculations or estimates have been made of the neutrino radiation due to plasma oscillations, pair annihilation, bremsstrahlung, photon collisions, and the so-called photoneutrino, photonuclear, and URCA processes.[2] In this paper we consider neutrino pair radiation from electrons accelerated by a large magnetic field, $e\rightarrow e+\nu_e+\nu_e$, under the hypothesis that there exists a direct e-ν_e coupling. This radiation is analogous to the radiation of electromagnetic energy by a magnetically accelerated electron (electromagnetic synchrotron radiation, ESR); hence we call it neutrino synchrotron radiation (NSR).

II. AVAILABLE ELECTRON ENERGIES AND MAGNETIC FIELDS

NSR from relativistic electrons, like ESR, increases strongly with increasing electron energy and magnetic-field strength (see Sec. III 3). Large electron energies are found in stars of very high central temperatures, and in stars in which electron degeneracy is present, such as white dwarfs and neutron stars. For orientation, a temperature of 8×10^7 °K (about that at which helium burning commences in a red giant) corresponds to electron thermal energies of roughly 30 keV, while electrons in a white dwarf have degeneracy energies of about 0.5 to 5 MeV,[3] and the electron Fermi energy in a neutron star is of the order of 100 MeV.[4]

Rather little is known about the magnetic fields inside stars, but it seems possible that some degenerate stars may have very large internal fields. A number of stars on or near the main sequence, particularly A-type stars, have been observed to have surface magnetic fields of the order of 10^3 to 10^4 G.[5] This suggests that there are fields H_0 of the same order of magnitude in the interiors of these stars. Let us assume that H_0 is approximately uniform throughout the star, and consider the contradiction of such a star to a white dwarf of mean density $\bar{\rho}$ from its initial mean density $\bar{\rho}_0$ without substantial loss of mass. The magnetic field lines are nearly "frozen" in the highly ionized material of the star, which is an excellent electrical conductor, and would be squeezed closer together. In the equatorial plane, the number of lines per unit area, and hence the field strength H, would increase as R^{-2}, where R is the stellar radius, or as $\bar{\rho}^{2/3}$. At the end of the contraction, $H\sim H_0(\bar{\rho}/\bar{\rho}_0)^{2/3}$. Now, the most massive white dwarfs cannot have central densities above about 10^{10} g/cm^3 without the star's being unstable owing to inverse β decay,[6] and the "average" white dwarf, with a mass of about 0.6 solar masses[7] (one solar mass $=M_\odot\cong2\times10^{33}$ g), has a central density of a few times 10^6 g/cm^3. The mean densities in these two cases are of the order of

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[1] See, for example, H. Reeves, Astrophys. J. **138**, 79 (1963).
[2] See, for example, J. B. Adams, M. A. Ruderman, and C.-H. Woo, Phys. Rev. **129**, 1383 (1963) and Refs. 3–10 therein; A. Finzi, *ibid.* **137**, B472 (1965); L. Rosenberg, *ibid.* **129**, 2786 (1963); N. Van Hieu and E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **44**, 1003 (1963) [English transl.: Soviet Phys.—JETP **17**, 681 (1963)].
[3] L. Mestel, in *Stellar Structure*, edited by L. H. Aller and D. B. McLaughlin (University of Chicago Press, Chicago, 1965), Chap. 5.
[4] A. Finzi, Phys. Rev. **137**, B472 (1965).
[5] H. W. Babcock, Astrophys. J. Suppl. **3**, No. 30 (1958); H. W. Babcock, in *Stellar Atmospheres*, edited by J. L. Greenstein (University of Chicago Press, Chicago, 1960), Chap. 8.
[6] T. Hamada and E. E. Salpeter, Astrophys. J. **134**, 683 (1961).
[7] V. Weidemann, Z. Astrophys. **57**, 87 (1963).

10^9 and 10^6 g/cm^3, respectively.[8] If we suppose an initial field $H_0 \sim 10^4$ G, and an initial density of the order of 1 g/cm^3, typical of main sequence stars, then we might expect fields in a few of the most massive white dwarfs of as much as 10^{10} G, while fields of order 10^8 G might occur in some average white dwarfs. Note that since the masses of most of the stars which are observed to have large magnetic fields (A-type stars of 2 to 3 $M_\odot$) are more than the maximum possible mass for a white dwarf (the "Chandrasekhar limit," between 1.1 and 1.4 $M_\odot$, depending on composition[9]), substantial mass loss would probably occur in the formation of a magnetic white dwarf, and the estimate above for the expected field is quite crude. It should also be noted that at present there is no secure knowledge of which stars actually become white dwarfs and which do not, and it is quite conceivable that the bulk of the magnetic stars actually evolve to some other final state, such as the neutron star state. All we have established is the possibility that some white dwarfs may have internal magnetic fields in the range 10^8 to 10^{10} G.

In view of the uncertainty in the above estimate of the field which one might expect in white dwarfs, it is worthwhile to examine the limit imposed on the field by the requirement that the total energy of the star, including gravitational, electromagnetic, and thermal or degeneracy components, must be less than zero, so that the star is a bound system. This clearly implies that the total (positive) electromagnetic field energy, $\sim\frac{4}{3}\pi R^3(H^2/8\pi)$, is less than the magnitude of the (negative) gravitational binding energy, $\sim M^2G/R$, where M is the star's mass and g is the gravitational constant. The inequality reduces to

$$H \lesssim 6M^{1/3}g^{1/2}\bar{\rho}^{2/3} \cong 1\times10^8(M/M_\odot)^{1/3}\bar{\rho}^{2/3}\ \mathrm{G}, \quad (1)$$

where $\bar{\rho}=3M/(4\pi R^3)$ is the mean density. Since all white dwarfs have masses of the order of $M_\odot$, this condition reduces approximately to $H \lesssim 1\times10^8\bar{\rho}^{2/3}$ G. for a normal white dwarf of mean density $\bar{\rho}\sim5\times10^5$ g/cm^3, we find an upper limit on H of about 5×10^{11} G; for a very dense white dwarf with $\bar{\rho}\sim5\times10^8$ g/cm^3, we have $H \lesssim 5\times10^{13}$ G. Thus the maximum field which might conceivably be present over a large region of a white dwarf is some three orders of magnitude larger than that which we might expect on the basis of the contraction-without-flux-loss model.

The possibility of such enormous fields raises the question of whether ordinary Maxwell theory is applicable to fields in stellar interiors under all circumstances (so that, for example, the magnetic field energy density is actually given by $H^2/8\pi$). The possibility of virtual pair creation leads to photon-photon scattering and hence to nonlinearity in the field equations for high enough field intensities. The modifications to electromagnetic theory due to this have been investigated by Euler, Heisenberg, and Weisskopf,[10] among others. From their work it is easily found that, *in vacuo*, ordinary Maxwell theory applies for fields less than about 10^{15} G. The presence of matter will undoubtedly change this value somewhat, but we shall take 10^{15} G as the danger line for the application of classical electrodynamics.

Referring to Eq. (1), we see that only for stars of mean density exceeding about 10^{10} g/cm^3 is it possible for fields to arise which exceed the limit of 10^{15} G. Such densities cannot occur in white dwarfs, because of the occurrence of inverse β decay (already mentioned), but neutron stars,[4] if any exist, have densities in the range 10^{14} to 10^{16} g/cm^3 and hence might have fields exceeding the limit 10^{15} G. In this paper, we shall only consider neutron stars briefly and shall not assume fields larger than about 10^{14} G, so that no questions of the applicability of Maxwell's equations arise.

From the above considerations, it appears that the combination of large electron energies and large magnetic fields needed for large amounts of NSR are most likely to be present in dense white dwarfs and neutron stars. The calculations of the NSR rate presented below are therefore carried out for the regimes of electron energy ($E\gtrsim1$ MeV; i.e., fully relativistic electrons) and field strengths ($H\gtrsim10^7$ G) appropriate to these stars. The calculation of the NSR rate is described in Sec. III and applied to various astrophysical situations in Secs. IV–VI. The results obtained are summarized in Sec. VII.

III. NEUTRINO EMISSION FROM A MAGNETICALLY ACCELERATED ELECTRON

Calculation of the emission rate of energy in the form of neutrinos from electrons moving in a large magnetic field is quite analogous to the quantum electrodynamic calculation of the ESR rate.[11] One computes the probability per second for transition of an electron from one Dirac equation eigenstate to another, with the emission of two neutrinos of specified momentum, by first-order perturbation theory, using the weak-interaction Hamiltonian. This probability per second is then integrated over all possible neutrino momenta to find the total probability per second of transition by any neutrino emission between two specified electron eigenstates. The total probability per second is multiplied by the energy carried off by the neutrinos and summed over all the initial electron states available per unit volume and all the final states which can be reached from those initial states, each initial state being weighted by the probability that it is occupied and each final state being

[8] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, 1939), Chap. 11.

[9] Refs. 3 and 5, and E. E. Salpeter, Astrophys. J. **134**, 669 (1961).

[10] H. Euler, Ann. Physik **26**, 398 (1936); W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936); V. Weisskopf, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **14**, No. 6 (1936).

[11] Electromagnetic synchrotron radiation is discussed in considerable detail by A. A. Sokolov, U. S. Atomic Energy Commission Report No. AEC-tr-4322, Sec. 28 (unpublished).

weighted by the probability that it is empty. This gives the energy loss per cm^3 per sec, which may be integrated over the star's volume to find the total neutrino luminosity (assuming negligible reabsortpion of neutrinos). This calculation is summarized below. Details may be found in the author's thesis.[12]

The probability per second that an electron will make a transition between two specified magnetic field eigenstates with emission of a ν, $\bar{\nu}$ pair of specified momenta is

$$\text{prob/sec}=(2\pi/\hbar)\sum|H_{fi}|^2\rho(E_f). \qquad (2)$$

Here $\rho(E_f)$ is the density of final states per unit energy at E_f. $\sum|H_{fi}|^2$ is the square of the matrix element of the weak-interaction Hamiltonian between the initial and final states, summed over final electron spins and averaged over initial spins. The expression for H_{fi} is

$$H_{fi}=\frac{C}{\sqrt{2}}\int d^3x\langle\psi_f{}^\nu|\gamma_4\gamma_\mu(1+\gamma_5)|\psi_i{}^\nu\rangle \times\langle\psi_f{}^e|\gamma_4\gamma_\mu(1+\gamma_5)\psi_i{}^e\rangle\,, \qquad (3)$$

where $C=1.00\times10^{-49}$ erg cm^3.[13] It is necessary to actually compute the electron matrix elements one by one, as in nuclear beta decay, rather than to use the operator methods developed for problems in which all initial and final states may be approximated by plane waves.

It is found that it is possible to write $\sum|H_{fi}|^2$ in the form

$$\sum|H_{fi}|^2=\tfrac{1}{2}C^2\sum(E_\mu{}^*E_\nu)(N_\mu{}^*N_\nu)\,, \qquad (4)$$

where

$$N_\mu=V^{-1}[u^\dagger\gamma_4\gamma_\mu(1+\gamma_5)u]\,,$$
$$E_\mu=\int d^3x\exp[-i(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\mathbf{r}][\psi_f{}^{e\dagger}\gamma_4\gamma_\mu(1+\gamma_5)\psi_i{}^e]\,, \qquad (5)$$

the $\boldsymbol{\kappa}_i$'s are the neutrino wave numbers and the u's are spinors which are independent of $\mathbf{r}$. It is convenient to compute and discuss the quantities $E_{\mu\nu}=E_\mu{}^*E_\nu$ and $N_{\mu\nu}=N_\mu{}^*N_\nu$ separately.

The calculation of the $N_{\mu\nu}$'s from the explicit forms for u given by Sokolov[14] is entirely straightforward. If we write the spherical polar coordinates of $\boldsymbol{\kappa}_i$ as $(\kappa_i,\theta_i,\varphi_i)$ with $i=1$, 2, we get $N_{\mu\nu}$'s which depend only on θ_i and φ_i, of which a typical example is

$$N_{11}=(2/V^2)[1+\sin\theta_1\sin\theta_2\cos(\varphi_1+\varphi_2)-\cos\theta_1\cos\theta_2]. \qquad (6)$$

The evaluation of the $E_{\mu\nu}$'s is somewhat more complicated but no less straightforward. The electron eigenstates in a magnetic field are found to be characterized by the wave-number k of the electron parallel to the magnetic field (which we take to be the z axis) and two quantum numbers $n=0$, 1, 2, $\cdots$ and $s=0$, 1, 2, $\cdots$, both of which refer to the electrons motion normal to the field. Their interpretation is discussed by Sokolov.[11] The quantum number n is analogous to the principal quantum number of atomic electron states; s has no simple interpretation.

If we set $k_0=mc/\hbar$, and $\gamma=eH/2c\hbar$, the relativistic total energy of an electron having quantum numbers (n,k,s) is

$$E=c\hbar K=c\hbar(k_0{}^2+k^2+4\gamma n)^{1/2}. \qquad (7)$$

(Energies, momenta, and masses are converted to wave numbers by multiplication by appropriate powers of $\hbar$ and c, and denoted by various K's, k's, and κ's; K and k refer to the initial electron state and K' and k' to the final electron state.)

A typical expression for an $E_{\mu\nu}$ is

$$E_{11}=I_{ss'}{}^2\delta(k,k'+\kappa_1\cos\theta_1+\kappa_2\cos\theta_2)\times\{2[1+(k_0/K)][1+(k_0/K')]\}^{-1}\times[u_-'u_+I_{n,n'-1}{}^2(x)+u_+'u_-I_{n-1,n'}{}^2(x)-2ww'I_{n,n'-1}(x)I_{n-1,n'}(x)]\,, \qquad (8)$$

where

$$I_{n,p}(x)=(n!p!)^{-1/2}e^{-x/2}x^{(n-p)/2}Q_p{}^{n-p}(x)\,,$$

with

$$Q_p{}^{n-p}(x)=\sum_{j=0}^{p}(-1)^{j+p}\frac{n!p!x^{p-j}}{j!(p-j)!(n-j)!}\,.$$

[$Q_p{}^{n-p}(x)$is the generalized Laguerre polynomial.] The argument x of the $I_{n,p}$'s is $x=\kappa^2\sin^2(\theta/4)$; and other abbreviations are

$$u_\pm=1+(k_0\pm k)/K\pm k_0k/K^2\,,$$
$$u_\pm'=1+(k_0\pm k')/K'\pm k_0k'/K'^2\,,$$
$$w=[1+(k_0/K)](4\gamma n)^{1/2}/K\,,$$

and

$$w'=[1+(k_0/K')](4\gamma n')^{1/2}/K'.$$

The Kronecker delta indicates that along the magnetic field, particle momentum is conserved.

We note in passing that these results apply equally to relativistic and nonrelativistic electrons. Only below do we assume the electrons to be fully relativistic.

We now proceed to compute approximately the total probability per second for the transition of a single electron between an initial electron state having quantum numbers (n,k,s) (and energy $c\hbar K$), and all the final electron states characterized by a particular value n', by integrating Eq. (2) over all allowed neutrino momenta and summing over s'. For any particular one of these transitions, k' is determined from k, $\boldsymbol{\kappa}_1$, and $\boldsymbol{\kappa}_2$ by conservation of z momentum.

It turns out that an enormous degree of simplification is achieved if it is assumed that the radiation from an electron of energy E is roughly independent of k. In the extreme case of an electron moving parallel to the field, and hence unaccelerated, this is clearly wrong, but in an isotropic distribution of electrons in mo-

[12] J. D. Landstreet, thesis, Columbia University, 1965, Chaps. 4–11 (unpublished).

[13] See, for example, M. A. Preston, *Physics of the Nucleus* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Chap. 15.

[14] Reference 11, Sec. 20.

mentum space, most of the electrons of energy E will have a component of momentum transverse to the field only slightly less than that of an electron moving normal to the field, and will thus produce nearly the same radiation. We shall therefore compute, throughout this paper, power radiated as if all the electrons had $k=0$. It is estimated that this leads leads to a value of the neutrino radiation which is too large by a factor of two, which is not a serious error in the light of other approximations which are made below.

The integral we are interested in is

prob/sec (n, $k=0$, $s\rightarrow n'$, all allowed k', all allowed s')

$$=(2\pi/\hbar)\int d\Omega_1\int d\Omega_2\int_0^{K-K'} d\kappa_2 \sum_{s'=0}^{\infty} E_{\mu\nu}N_{\mu\nu} \times\frac{C^2V^2\kappa_1{}^2\kappa_2{}^2}{2(2\pi)^6c\hbar(1+\partial K'/\partial\kappa_1)}. \quad (9)$$

In Eq. (9), κ_1 is determined by conservation of energy for the decay,

$$K=K'+\kappa_1+\kappa_2. \quad (10)$$

It is possible to approximate the behavior of the $E_{\mu\nu}$'s by step functions and to otherwise simplify the integral of Eq. (9) enough to obtain a closed expression which is accurate to within perhaps a factor of 5. One finds that ($\nu=n-n'$)

prob/sec (n, $k=0$, $s\rightarrow n'$, all allowed k', all allowed s')

$$=P(n,n')=1\times10^{-3}C^2\nu^{-5/3}c^{-1}\hbar^{-2}(2\gamma\nu/K)^5. \quad (11)$$

It is also found that the behavior of the matrix elements is such that for almost all transitions

$$\nu=n-n'\lesssim1.5(K/k_0)^3=1.5(E/mc^2)^3\equiv\nu_m, \quad (12)$$

a limit which physically means that the energy spectrum of the NSR has an upper limit above which radiation is strongly damped.[15]

It is next necessary to determine l_ν, the total energy emitted per cubic centimeter per second. This involves multiplying the expression of Eq. (11) by the energy lost in the transition and the probability

$$1-\{\exp[(E'-E_0)/k_bT]+1\}^{-1} =\{\exp[(E_0-E')/k_0T]+1\}^{-1},$$

(where k_b is Boltzman's constant, E_0 is the Fermi energy of the distribution, and T is temperature) that the lower state (of energy E') is empty, and summing over n or ν. Since Eq. (11) is a strongly increasing function of ν up to $\nu_m\gg1$ [see Eq. (12)], we may replace the sum by an integral. We then multiply the resulting expression by the probability $\{\exp[(E-E_0)/k_bT]+1\}^{-1}$ that the upper state (of energy E) is occupied, and integrate over all the electron states in the cubic centimeter. If $\rho_e(E)$ is the density of electron states in energy, which we may approximate to a sufficient degree of accuracy by the free-electron distribution, we have

$$l_\nu=\int_0^\infty\frac{\rho_e(E)dE}{\exp[(E-E_0)/k_bT]+1} \times\int_0^{\nu_m}\frac{P(n,n')d\nu}{\exp[(E_0-E')/k_bT]+1}, \quad (13)$$

where $P(n,n')$ is given by Eq. (11).

It is found in evaluating the integral that two physically distinct cases arise, depending on whether

$$H_8(E_0/mc^2)^2\lesssim5\times10^2T_7 \quad (14)$$

holds or not. (Here we set $H_8=H/10^8$ and $T_7=T/10^7$.) When Eq. (14) holds, the energy spectrum of the NSR is cut off at the high-energy end by the matrix elements of the transition, as ESR is. In this case, the value of l_ν, $l_\nu{}^{\rm I}$, is given (in cgs Gaussian units) by

$$l_\nu{}^{\rm I}=6\times10^{-19}H_8{}^6T_7(E_0/mc^2)^{12}\text{ erg cm}^{-3}\text{ sec}^{-1}. \quad (15)$$

If Eq. (14) does not hold, the NSR energy spectrum is cut off at the upper end not by the matrix elements but by the fact that it is only possible for electrons to make transitions involving energy changes of the order of k_bT or less. This is because only in an energy band of about this width near E_0 are there both empty states for electrons to drop into and occupied states above for the electrons to drop from. In this case, the value of l_ν. which we call $l_\nu{}^{\rm II}$, is given by

$$l_\nu{}^{\rm II}=2\times10^{-4}H_8{}^{2/3}T_7{}^{19/3}(E_0/mc^2)^{4/3}\text{ erg cm}^{-3}\text{ sec}^{-1}. \quad (16)$$

We may convert Eqs. (14), (15), and (16) to a form more appropriate for application to white dwarfs of neutron stars by using the relation[3]

$$E_0/mc^2\cong p_0/mc=(\hbar/mc)(3n_e/8\pi)^{1/3}, \quad (17)$$

where p_0 is the Fermi momentum and n_e is the electron number density in the gas. This relation holds for a relativistic gas. We further note that for a completely pressure-ionized gas of atomic number Z and atomic weight A we have

$$\rho=m_Hn_e(A/Z)\cong2m_Hn_e,$$

where $A/Z\cong2$ in any white dwarf (hydrogen is almost entirely absent because its presence in more than trace amounts would lead to higher luminosities than are observed[3]).

IV. APPLICATION TO WHITE DWARFS

Putting together Eqs. (14)–(17), we find that for relativistic white dwarfs

$$l_\nu{}^{\rm I}=3\times10^{-44}H_8{}^6T_7\rho^4\text{ erg cm}^{-3}\text{ sec}^{-1}, \quad (18)$$

$$l_\nu{}^{\rm II}=4\times10^{-7}T_7{}^{19/3}\rho^{4/9}H_8{}^{2/3}\text{ erg cm}^{-3}\text{ sec}^{-1}, \quad (19)$$

[15] Compare with the ESR spectrum as shown in J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), Fig. 14.11.

where one uses Eq. (18) as long as

$$H_8^{2/3} \lesssim 8\times10^6 T_7 \qquad (20)$$

holds, and Eq. (19) otherwise.

It will be noted that $l_\nu^{\rm I}$ increases very strongly with increasing H and ρ, while $l_\nu^{\rm II}$ is quite insensitive to these parameters. This causes the total NSR flux from a white dwarf to increase very rapidly with increased mass and magnetic field until Eq. (20) becomes an equality, after which further increase in H or ρ has little effect. [The temperature dependence of Eqs. (18) and (19) is here ignored because the interior of a white dwarf is nearly isothermal and is believed to have a temperature within perhaps a factor of two of 1.5×10^7 °K for all observed white dwarfs, so that the temperature cannot be considered a freely variable parameter.]

Equations (18) and (19) have been used to determine the NSR luminosity of a number of model white dwarfs, assuming negligible reabsorption of the emitted neutrinos. Calculations have been made for stars of central densities between 1.6×10^7 g cm^{-3} and 1×10^{10} g cm^{-3} (nearly the maximum possible central density of a white dwarf[3]), central temperatures between 1×10^7 °K and 5.5×10^7 °K, and magnetic-field strengths (assumed uniform throughout the star) of up to 10^{11} G. It is found that for any white dwarf with $T\lesssim5.5\times10^7$ °K and $H\lesssim10^{11}$ G, the NSR luminosity is smaller than the photon luminosity from the surface by a factor of 10^3 or more (even allowing for the uncertainty inherent in this calculation of l_ν), while the NSR luminosity is smaller than the photon luminosity by a factor of more than 10^7 if $H\lesssim10^9$ G. It may therefore be concluded that this process is of essentially no significance in the thermal evolution (i.e., cooling) of a white dwarf, although it might play some role in the more violent stages of evolution which presumably immediately precede the white-dwarf state in stellar evolution.

V. APPLICATION TO NEUTRON STARS

The results embodied in Eqs. (14), (15), and (16) may be applied to neutron stars.[16] A neutron star typically has densities of the order of 10^{15} g cm^{-3} and an electron Fermi energy of the order of 100 MeV, and hence by the discussion of the Introduction may very well have fields of the order of 10^{14} G. It turns out that even at these energies and fields the treatment of the neutrino luminosity given above can be applied. It is easily seen that one must use the luminosity function $l_\nu^{\rm II}$ given by Eq. (16) for magnetic fields greater than 10^{10} G. Estimates have been made of the NSR luminosity of neutron stars of masses between $0.2M_\odot$ and $2M_\odot$ having central temperatures between 10^8 and 10^{10} °K for a magnetic field $H\sim10^{14}$ G. Since Eq. (16) depends so weakly on H, the results apply within an order of magnitude to stars having fields anywhere in the range 10^{12} to 10^{16} G. It has been assumed that negligible reabsorption occurs, since the mean free path of a neutrino in a neutron star is of the order of ten times as large as the radius of the star,[17] as long as the temperature of the star (which limits the energy which can be carried off by a single neutrino to about k_bT) is below about 10^{10} °K. It is found that the NSR luminosity is always less than the dominant radiation process (photon radiation from the surface up to 3 or 4×10^8 °K, URCA[4] or plasmon[18] radiation at higher temperatures) by a factor of 100 or more. However, these estimates are sufficiently crude that the NSR process might actually be comparable in importance to other radiation processes under some circumstances.

VI. EXTENSION TO NONRELATIVISTIC ELECTRONS

For some situations in stellar interiors in which neutrino radiation is important, the electrons are nondegenerate and their energy is not mainly degeneracy energy but thermal energy. In this case, as long as the temperature is well below 6×10^9 °K, the electrons are nonrelativistic. Such a situation occurs in stars burning C^{12}, O^{16}, and Ne^{24}, a state of affairs which may occur near the end of the active life of a massive star.[1] To estimate the NSR rate which is applicable to such cases, it is necessary to redo the calculations of Sec. III in the limit of nonrelativistic electron velocities.

In the theory of ESR is is found that for nonrelativistic electrons, almost all of the radiation comes from transitions having $\nu=1$: that is, from transitions between adjacent energy states. We expect this result to be true in NSR as well. This is supported by an estimate made from the theory of Sec. III of the maximum change in principal quantum number $\nu=n-n'$ which would contribute strongly to the neutrino radiation. We therefore estimate only the radiation from the transition $n-n'=1$ and assume that the total radiation is about the same. We again take $k=0$ and ignore the error that this leads to.

The calculations are very much like those of Sec. III. Using similar approxiations, one finds that the energy emitted per second as NSR by a nonrelativistic electron is

$$P(n)\sim2C^2[(2\pi)^3 15\hbar]^{-1}(2\gamma/k_0)^6$$
$$\sim2\times10^{-46}H_8{}^6 \text{ erg sec}^{-1}.$$

It is remarkable that $P(n)$ is independent of energy (or n) to first approximation. This occurs because the largest of the $E_{\mu\nu}$'s, namely, E_{11} and E_{22}, are independ-

[16] See L. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), Sec. 108 for a brief summary of the properties of these objects, and S. Tsuruta, thesis, Columbia University, 1964 (unpublished) for detailed calculations of model stars.

[17] J. N. Bahcall, Phys. Rev. **136**, B1164 (1964).

[18] J. B. Adams, M. A. Ruderman, and C.-H. Woo, Phys. Rev. **129**, 1383 (1963).

ent of energy in the nonrelativistic limit, and since the energy levels are essentially evenly spaced, the phase space accesible to the neutrino pair is also independent of energy.

For a nondegenerate, nonrelativistic gas, then, we find

$$l_\nu \sim 2\times 10^{-46} H_8{}^6 n_e \text{ erg cm}^{-3}\text{ sec}^{-1}. \quad (21)$$

If the gas is degenerate, only about a fraction $k_bT/(E_0-mc^2)$ of the electrons will participate in the radiation, and the l_ν given by Eq. (21) is multiplied by this factor. Since $E_0-mc^2 \cong p_0{}^2/2m$, using Eq. (17) we find

$$l_\nu \sim 1\times 10^{-28} n_e{}^{1/3} T_7 H_8{}^6 \text{ erg cm}^{-3}\text{ sec}^{-1}. \quad (22)$$

This result assumes that the neutrino energy spectrum is not cut off by degeneracy, which is true as long as

$$T_7 \gtrsim 3\times 10^{-3} H_8.$$

For temperatures of interest, say $T_7 \gtrsim 1$, Eq. (22) holds as long as $H \lesssim 3\times 10^{10}$ G. If larger fields are anticipated, the result may be modified as Eq. (15) is modified to give Eq. (16).

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Perey Effect and the Analytic Properties of the Wave Function*

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The Perey effect is shown to be related to the analytic properties of the wave function, which are different for local and nonlocal potentials.

If a phase shift is fitted by a local or nonlocal potential, the wave functions calculated from the Schrödinger equation using the two potentials differ in a predictable way. If the potential is attractive, the wave function for the local potential is larger in the potential region than the wave function for the nonlocal potential, assuming a normalization for which both have the same asymptotic behavior. If the potential is repulsive, the opposite is true. It is the purpose of this article to point out that this phenomenon, which is called the Perey effect,[1] has a simple explanation in the analytic structure of the two wave functions. The position of the left-hand cut is different in the two cases, being nearer the origin for the nonlocal potential. The Perey effect is a consequence of this.

We restrict ourselves to nonlocal Hermitian potentials which are diagonalizable with eigenvectors $u_i(r)$.[2]

$$\int_0^\infty V(r,r')u_i(r')dr' = \lambda_i u_i(r),$$
$$V(r,r') = \sum \lambda_i u_i(r) u_i{}^*(r').$$

The Schrödinger equation for each partial wave is now immediately soluble for this sum of separable potentials. To understand the proof for the Perey effect, we shall assume that only one of the eigenvalues dominates the sum, and we shall consider for definiteness the S-wave bound-state problem:

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} + \lambda u(r)\int_0^\infty r'^2 R(r')u^*(r')dr' = -\alpha^2 R(r).$$

This may be solved by converting to momentum

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[1] F. G. Perey and D. S. Saxon, Phys. Letters **10**, 107 (1964); F. G. Perey, in *Direct Interactions and Nuclear Reaction Mechanisms*, edited by E. Clementel and C. Villi (Gordon and Breach, Science Publishers, Inc., New York, 1963), p. 125; and N. Austern, Phys. Rev. **137**, B752 (1965).

[2] This is a restriction. For example, for a local potential one would need a continuous set of λ's. However, the class of potentials that are considered is much larger than that for which the Perey effect has been demonstrated up to now. It is not restricted to a short-range nonlocality, the nonlocality can be "infinite," and furthermore it happens to be the natural form which arises from a few resonances in nucleon-nucleon scattering and from Hartree-Fock calculations. The Hermiticity property can be relaxed in favor of symmetry to include optical potentials.