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Bethe-Salpeter Equation and Goldstone Bosons in Quantum Electrodynamics

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The relevance of the Goldstone theorem to quantum electrodynamics with zero bare mass for the electron is studied. Subject to the necessary approximations, it is concluded that the Goldstone theorem has no physical consequences for this theory.

I. INTRODUCTION

IT has been proposed[1] in the context of a particular nonperturbative approach to quantum electrodynamics that the electron should have zero bare mass. Then, under certain conditions discussed in Ref. 1, the usually divergent self-energy integrals become conditionally convergent integrals, and the Dyson-Schwinger equation for the electron propagator has finite solutions to any order in a perturbation expansion of the Bethe-Salpeter kernel which appears in the equation for the vertex function. With no bare mass, there is no mass scale in the theory; and the self-energy equation has a solution for any value of the physical electron mass. The solution with the actual electron mass is picked out by imposing the boundary condition that the inverse propagator be zero at the physical electron mass.

With no bare mass the field theory underlying the Dyson-Schwinger Green's-function equations is γ_5-invariant; the existence of solutions with nonzero mass implies the "spontaneous" breakdown of this symmetry. In the field-theoretic context this can only occur through a γ_5 degeneracy of the vacuum. Also in the field-theoretic context, there exist general proofs[2] that spontaneous breakdown of invariance under a continuous group of transformations implies the existence of zero-mass states of appropriate quantum numbers. The theorem customarily goes by the name of the Goldstone theorem, and the corresponding zero-mass particles by the name of Goldstone particles. None of these proofs provide any information about the coupling of the Goldstone particles to other particles in the theory. Thus, the Goldstone theorem, by itself, has no observable consequences. Such consequences can only follow when the coupling of the Goldstone particles to the other particles in the theory is determined.[3]

The original suggestion that spontaneous symmetry breakdown might play a role in field theory was by Nambu and Jona-Lasinio[4] in a paper specifically devoted to the question of spontaneous breakdown of γ_5 invariance in a different theory. In that paper the observability of the ensuing massless particles was demonstrated by displaying the corresponding pole in the solution of the Bethe-Salpeter equation for the scattering amplitude. The residue of this pole is identified as the square of the coupling constant. The primary purpose of the present paper is to carry out the analogous calculation[5] for the version of quantum electrodynamics presented in Ref. 1. The result is that there is no pole in the electron-positron scattering amplitude corresponding to a massless pseudoscalar Goldstone particle.

In recent years there has also been considerable interest in the Bethe-Salpeter equation in its own right as a nonperturbative source of information on the analytic properties and high-energy behavior of the relativistic scattering amplitude. With very little additional work we are able to find the contribution of a certain infinite set of diagrams to the high-energy behavior of one of the scalar amplitudes for electron-positron forward scattering.

II. THE BETHE-SALPETER EQUATION

The Bethe-Salpeter equation for the off-mass-shell electron-positron scattering amplitude is

$$
{}_{ab}T_{cd}(p,p',k) = {}_{ab}K_{cd}(p,p',k) + \int (dl)\, {}_{ae}T_{fd}(p,l,k) S_{eg}(l_+) \times {}_{gb}K_{ch}(l,p',k) S_{hf}(l_-)\,. \quad (2.1)
$$

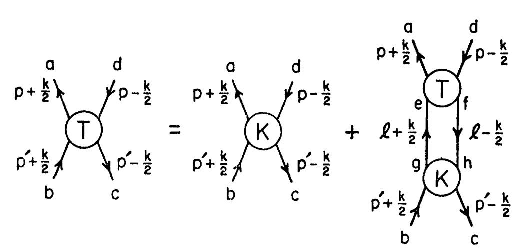

FIG. 1. Bethe-Salpeter equation for electron-position scattering.

[1] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136**, B1111 (1964).

[2] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[3] N. G. Deshpande and S. Bludman, Phys. Rev. **146**, 1186 (1966).

[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[5] In Appendix A, we review briefly that part of the Nambu–Jona-Lasinio model which is of interest in the present context. Since the mathematics involved is much less complicated than in the analogous quantum electrodynamic calculations, it may be useful to read this Appendix before the main body of the paper.

The choice of momentum variables is illustrated in Fig. 1, where the equation is given in diagrammatic form. The spin-matrix indices are written out explicitly. (dl) means $d^4l/(2\pi)^4$ and $l_\pm = l \pm k/2$.

If there is a massless pseudoscalar particle which couples to the electron, a pole will occur in T in the form[6]

$$ {}_{ab}T_{cd}(p,p',k) = \gamma^5{}_{ad}\gamma^5{}_{cb}\frac{g(p_+,p_-)g(p_-',p_+')}{k^2} + \text{terms regular at } k^2=0\,. \quad (2.2)$$

This is illustrated in Fig. 2. The matrix indices of T imply a lengthy expansion in terms of invariant functions and Dirac matrices. We concentrate on a single invariant function defined by the γ_5 projection

$$\gamma^5{}_{da}\,{}_{ab}T_{cd}(p,p',k)\gamma^5{}_{bc} = 12ie^2 f(p,p',k)\,. \quad (2.3)$$

A sufficient condition for no zero-mass pseudoscalar pole to appear in T is that no such pole appear in f.

The off-mass-shell scalar amplitude f is a function of six scalar variables:

$$f(p,p',k) = f(k^2, p\cdot p', p^2, p'^2, p\cdot k, p'\cdot k)\,. \quad (2.4)$$

The mass-shell scalar amplitude is obtained by setting

$$(p \pm \tfrac{1}{2}k)^2 = (p' \pm k)^2 = -m^2\,, \quad (2.5)$$

which requires

$$p\cdot k = p'\cdot k = 0\,, \quad (2.6a)$$

$$p^2 = p'^2 = -m^2 - \tfrac{1}{4}k^2\,. \quad (2.6b)$$

We write the mass-shell invariant amplitude as

$$f(k^2,\, p\cdot p',\, -m^2-\tfrac{1}{4}k^2,\, -m^2-\tfrac{1}{4}k^2,\, 0,\, 0) = f(k^2, p\cdot p')\,. \quad (2.7)$$

Since $-k^2$ is the square of the total center-of-mass energy, the physical region for k^2 is $-4m^2 \geq k^2$. Thus, a massless pseudoscalar particle, if coupled to the electron, appears as a pole at the unphysical value $k^2=0$. We demonstrate the nonexistence of this pole by showing that $f(0,p\cdot p')$ exists. In the Bethe-Salpeter equation we cannot restrict the scattering amplitude to the mass shell, but we can obtain the function $f(0,p\cdot p')$ by solving the Bethe-Salpeter equation for

$$f(0, p\cdot p', p^2, p'^2, 0, 0) = f(p,p') \quad (2.8)$$

and at the end setting $p^2 = p'^2 - = m^2$. Thus, to compute the mass-shell amplitude $f(k^2, p\cdot p')$ at the unphysical value, $k^2=0$; we solve the Bethe-Salpeter equation for $k=0$ and at the end set $p^2=p'^2=-m^2$. In terms of the usual Mandelstam variables,

$$s=-k^2,\quad t=-(p-p')^2\,, \quad (2.9a)$$

$$F(s,t) = f(k^2, p\cdot p')\,, \quad (2.9b)$$

we look at the amplitude for physical values of t, $t<0$; and unphysical value of s, $s=0$.

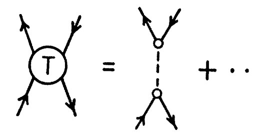

FIG. 2. Pole contribution if pseudoscalar particle is coupled to $e\bar{e}$.

Now we must determine the kernel to be used in the Bethe-Salpeter equation. Since the exact kernel includes an infinite sum of terms, clearly we must use an approximate kernel. The approximation should be one which, when applied to the self-energy problem, leads to a symmetry-breaking solution. In Ref. 1, a systematic approximation scheme is presented for the calculation of the electron propagator for large off-mass-shell momenta. This scheme includes the boundary condition $S^{-1}(p)=0$ when $\gamma p = -m$. The behavior of $S(p)$ near the mass shell is to be determined by ordinary renormalized perturbation theory. Thus, according to Ref. 1, the electron propagator equation has approximate solutions with the properties

$$S(p) \to \frac{1}{\gamma p} + \frac{m}{p^2}\left(\frac{m^2}{p^2}\right)^{\epsilon} + \cdots, \qquad p^2 \to \infty$$

$$\frac{1}{S(p)} \to 0, \qquad \gamma p \to -m$$

$$\epsilon = \frac{3\alpha_0}{4\pi} + \cdots. \quad (2.10)$$

Because of the zero mass of the photon, the point $\gamma p = -m$ is an infinite branch point rather than a simple pole. The combination of the approximation scheme for the far-off-mass-shell behavior with renormalized perturbation theory for the near-mass-shell behavior can be achieved as follows. The first nontrivial approximation to the Dyson-Schwinger equation used in the asymptotic approximation scheme with $m_0 = 0$ is

$$S^{-1}(p) = \gamma p + ie_0^2 \int (dp') D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta\,. \quad (2.11)$$

This is illustrated in Fig. 3. In this approximation we may write

$$S^{-1}(p) = \gamma p + A(p^2)\,, \quad (2.12)$$

with $A(p^2)$ the solution of the integral equation

$$A(p^2) = ie_0^2 \int (dp') D_{\alpha\beta}(p-p')\gamma^\alpha \frac{A(p'^2)}{p'^2 + A^2(p'^2)}\gamma^\beta\,. \quad (2.13)$$

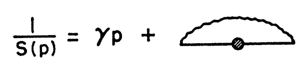

FIG. 3. Approximate Dyson-Schwinger equation for the electron propagator.

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[6] If one takes Eq. (2.2) as an anzatz, it follows that the function $g(p_+,p_-)$ must satisfy a homogeneous vertex equation. M. Baker, K. Johnson, and B. W. Lee [Phys. Rev. **133**, B209 (1964)] have shown that this equation necessarily has a solution when the γ_5-invariant Dyson-Schwinger propagator equation has a γ_5-nonvariant solution. They further point out that this does not imply a pole in the corresponding inhomogeneous equation because of the non-Fredholm nature of the equations.

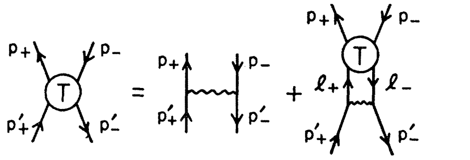

FIG. 4. The ladder-approximation Bethe-Salpeter equation.

Since the approximation (2.11) is only intended for the asymptotic region, (2.13) is used only to determine the asymptotic behavior of $A(p^2)$. Thus, in Ref. 1, the A^2 term in the denominator is dropped and the solution of the resulting linear integral equation yields the asymptotic behavior of (2.10). To get an equation which also has the correct properties near the mass shell, we can replace $A(p^2)$ *in the denominator* by its perturbation expansion

$$A(p^2)=m+\alpha_0 A_1(p^2)+\alpha_0^2 A_2(p^2)+\cdots \quad (p^2 \text{ finite}). \quad (2.14)$$

Thus, to order e_0^2 in the combined perturbative and asymptotic non-perturbative approximation scheme, we have the equation[7]

$$A(p^2)=ie_0^2\int (dp')\,D_{\alpha\beta}(p-p')\gamma^\alpha \frac{A(p'^2)}{p'^2+m^2}\gamma^\beta. \quad (2.15)$$

This equation has as solution a hypergeometric function which has a branch cut from $p^2=-m^2$ to $-\infty$ and asymptotic behavior $(m^2/p^2)^\epsilon$. With the boundary condition $A(-m^2)=m$, this provides an explicit solution with the properties (2.10). Now we may determine the corresponding approximation to the Bethe-Salpeter equation for the electron-positron scattering amplitude. From Fig. 3 it is evident that the corresponding kernel is just the one-photon-exchange diagram.[8]

$$_{ab}K_{cd}(p,p',k)=-ie^2 D_{\alpha\beta}(p-p')\gamma^\alpha{}_{ab}\gamma^\beta{}_{cd}$$
$$=-ie^2\left[g_{\alpha\beta}-\frac{(p-p')_\alpha(p-p')_\beta}{(p-p')^2}\right]\times\frac{1}{(p-p')^2}\gamma^\alpha{}_{ab}\gamma^\beta{}_{cd}. \quad (2.16)$$

Furthermore, we are to make the approximation of replacing $S(p)$ by $(\gamma p+m)^{-1}$ everywhere that the modified behavior of the second term in the large p expansion of $S(p)$ is not required for convergence of an integral. We arrive at the ladder-approximation Bethe-Salpeter equation illustrated in Fig. 4.

Now following the discussion of Eq. (2.8) we set $k=0$ and then carry out the projection indicated in in Eq. (2.3). In Appendix B we show that for $k=0$, the amplitude f decouples from all other invariant amplitudes. Thus, we obtain an integral equation for $f(p,p')$:

$$f(p,p')=\frac{1}{(p-p')^2}-3ie^2\int (dl)\times\frac{1}{(p'-l)^2}\frac{1}{l^2+m^2}f(p,l). \quad (2.17)$$

III. SOLUTION OF THE APPROXIMATE BETHE-SALPETER EQUATION

Although we eventually desire the solution of (2.17) for $p^2=p'^2=-m^2$, we will proceed[9] by solving the equation for $p^2,p'^2>0$ (space-like), and then obtain the desired solution by analytic continuation. For p^2,p'^2 space-like, we can rotate the l_0 integration contour and work in a four-dimensional Euclidean metric. The only angle in the problem is the angle between p and p', so we write

$$f(p,p')=\frac{1}{2\pi^2}\sum_{n=0}^{\infty}(n+1)f_n(p^2,p'^2)U_n(\cos\theta), \quad (3.1)$$

where

$$\cos\theta=\frac{p\cdot p'}{(p^2p'^2)^{1/2}} \quad (3.2)$$

and

$$U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta} \quad (3.3)$$

is a Tchebycheff polynomial of the second kind.[10]

Substituting (3.1) into (2.17), we can carry out the angular integrations and obtain a one-dimensional integral equation for the "partial-wave" amplitude $f_n(x,x')=m^2 f_n(p^2,p'^2)$,

$$f_n(x,x')=\frac{2\pi^2}{n+1}\frac{1}{x_>}\left(\frac{x_<}{x_>}\right)^{n/2}+\frac{\lambda}{n+1}\int_0^{x'}dy\frac{y}{x'(y+1)}\left(\frac{y}{x'}\right)^{n/2}f_n(x,y)+\frac{\lambda}{n+1}\int_{x'}^{\infty}dy\frac{1}{y+1}\left(\frac{x'}{y}\right)^{n/2}f_n(x,y), \quad (3.4)$$

where

$$\lambda=3\alpha/4\pi,\quad x=p^2/m^2,\quad x'=p'^2/m^2.$$

[7] This equation was first proposed by Th. A. J. Maris, V. E. Herscovitz, and G. Jacob, Phys. Rev. Letters **12**, 313 (1964). The extension of this combined approximation scheme to higher orders has been considered by M. Baker and F. H. Lewis (unpublished).

[8] Formally, this follows from Eq. (11) and the definition $K=-\delta\Sigma/\delta S$. Furthermore, we neglect vacuum polarization so we may replace e_0^2 by e^2.

[9] In obtaining the solution of Eq. (2.17), we make use of techniques developed by a number of people for the Bethe-Salpeter equation for scalar particles, e.g., J. D. Bjorken, J. Math. Phys. **5**, 192 (1964) and M. Baker and I. Muzinich, Phys. Rev. **132** 2291 (1963). I am also indebted to Professor Muzinich for a copy of an unpublished manuscript by M. K. Banerjee, L. S. Kisslinger, C. A. Levinson, and I. J. Muzinich dealing with the Bethe-Salpeter equation for the ϕ^4 theory.

[10] We use the definition and properties of $U_n(\cos\theta)$ given in the *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II.

By differentiation we may convert Eq. (3.4) into a differential equation

$$\left[\frac{d^2}{dx^2}+\frac{2}{x}\frac{d}{dx}-\frac{n(n+2)}{4x^2}+\frac{\lambda}{x(x+1)}\right]f_n(x,x')=-2\pi^2\frac{\delta(x-x')}{xx'},\quad (3.5)$$

with the boundary conditions implied by the integral equation

$$x^{n/2+2}f_n(x,x')\to 0 \qquad (x\to 0) \qquad (3.6a)$$

$$\frac{1}{x^{n/2}}f_n(x,x')\to 0 \qquad (x\to\infty). \qquad (3.6b)$$

The details of the reduction of (2.17) to (3.4) and to (3.5) are given in Appendix C. Here we only note that in obtaining (3.5) from (3.4) we have interchanged x and x' at the end for typographical convenience. The solution is symmetric in these variables.

The particular solution of the inhomogeneous equation (3.5) is

$$f_n(x,x')=-2\pi^2 y_1^{(n)}(x_<)y_2^{(n)}(x_>), \qquad (3.7)$$

where $y_1^{(n)}$, $y_2^{(n)}$ are two independent solutions of the homogeneous equation normalized such that the Wronskian $y_1y_2'-y_1'y_2=1/x^2$.

The homogeneous differential equation is simply transformed into the hypergeometric equation; two appropriate independent solutions are

$$y_1^{(n)}(x)=x^{n/2}F(a,\,a-\nu;\,n+2;\,-x), \qquad (3.8a)$$

$$y_2^{(n)}(x)=-\frac{\Gamma(a)\Gamma(a+1)}{\Gamma(n+2)\Gamma(\nu+1)}\times x^{n/2-a}F(a-1-n,\,a;\,\nu+1;\,-1/x), \qquad (3.8b)$$

where

$$a=(n+1+\nu)/2, \qquad \nu=[(n+1)^2-4\lambda]^{1/2}. \qquad (3.9)$$

Some useful properties of these functions are collected in Appendix D.

To construct the complete amplitude $f(p,p')$, we have to do the sum (3.1) which we can rewrite as a contour integral by a Sommerfeld-Watson transformation

$$m^2f(p,p')=\frac{1}{2\pi^2}\sum_n (n+1)f_n(x,x')U_n(\cos\theta)$$

$$=\frac{1}{4\pi^2 i}\int_C\frac{dn}{\sin\pi n}\times(n+1)f_n(x,x')U_n(\cos(\pi-\theta)). \qquad (3.10)$$

The contour C runs just below the real n axis from zero

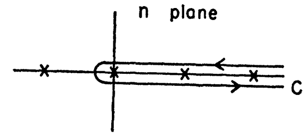

FIG. 5. The Sommerfeld-Watson contour C.

to infinity and back just above it, including the poles of $(\sin\pi n)^{-1}$ at integer values of n (Fig. 5). The amplitude exists if the sum (integral) converges. From the properties of the hypergeometric functions, we have

$$f_n(x,x')\underset{n\to\infty}{\longrightarrow}\frac{2\pi^2}{n+1}\frac{1}{x_>}\left(\frac{x_<}{x_>}\right)^n\sim\frac{1}{n}e^{-n\ln(x_</x_>)};$$

so the integral converges for $x\neq x'$. This demonstrates the existence of $f(p,p')$ in the Euclidean region $p^2,p'^2>0$ and $|\cos\theta|<1$. This already indicates the noncoupling of the Goldstone boson since, if coupled, it should produce a pole in the off-mass-shell amplitude as well. However, it is still worthwhile to check that our solution can be analytically continued to give the mass-shell amplitude.

According to Eqs. (1.6) and (1.9) the desired continuation is to

$$p^2=p'^2=-m^2 \quad \text{for} \quad k^2=0,$$
$$t=-(p-p')^2=2m^2(1-\cos\theta)<0,$$
$$\cos\theta>1.$$

We start by deforming the integration contour. Since $F(\alpha,\beta;\gamma;z)/\Gamma(\gamma)$ is an entire function of α, β, and γ, the product $y_1^{(n)}(x_<)y_2^{(n)}(x_>)$ is an entire function of the parameters a, n, and ν. But ν involves a square root (3.9), which implies branch points in the n plane at $n=-1\pm2\sqrt{\lambda}$. For $\lambda=3\alpha/4\pi<\frac{1}{4}$, we may restrict the branch cut to the left half-plane. Then we can open the contour to enclose the entire right half-plane (Fig. 6). Let

$$n=\xi+i\eta. \qquad (3.11)$$

Then, asymptotically

$$\frac{1}{\sin\pi n}\sim e^{-\pi|\eta|}, \quad U_n(\cos(\pi-\theta))\sim e^{(\pi-\theta)|\eta|},$$

$$f_n(x,x')\sim\frac{1}{n}e^{-\xi\ln(x_</x_>)}. \quad (0<\theta<\pi) \qquad (3.12)$$

For $x\neq x'$, the integral converges everywhere on C' and

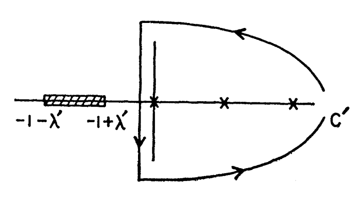

FIG. 6. The contour C' ($\lambda'=2\sqrt{\lambda}$).

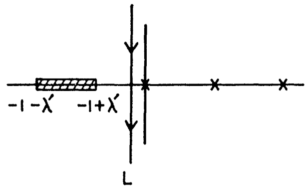

FIG. 7. The contour L.

in fact, there is no contribution from the infinite semicircle. So we may write

$$m^2 f(p,p') = \frac{1}{4\pi^2 i}\int_L \frac{dn}{\sin\pi n} \times (n+1) f_n(x,x') U_n(\cos(\pi-\theta)) . \quad (3.13)$$

The contour L runs parallel to the imaginary n axis and is displaced by an infinitesimal amount into the left half-plane to avoid the pole of $(\sin\pi n)^{-1}$ at $n=0$ (Fig. 7). At this point we may set $x=x'=-1$ and the integral still converges. Evaluating the hypergeometric functions gives

$$f_n(x,x')\big|_{x=x'=-1} = \frac{\pi}{i\lambda}[e^{-i2\pi b}-1] , \quad (3.14)$$

where

$$b=(n+1-\nu)/2 .$$

It remains to continue to $\cos\theta>1$ (fixed negative t). To this end we first give θ an imaginary part and then let the real part of θ go to zero. In this limit the integral diverges. However, this just reflects a singularity at $t=0$ which is associated with the exchange of massless photons. To determine the finite amplitude for fixed negative t, it is necessary to separate and sum explicitly that part of (3.1) which contributes to the singularity at $t=0$. Expanding (3.14) for large n we find

$$(n+1)\frac{\pi}{i\lambda}[e^{-i2\pi b}-1] = -2\pi^2\left[1-\frac{i\pi\lambda}{n+1}+(\lambda-\tfrac{2}{3}\pi^2\lambda^2)+\frac{1}{(n+1)^2}+O\left(\frac{1}{(n+1)^3}\right)\right]$$
$$= -2\pi^2\left[1-\frac{i\pi\lambda}{n+1}+(\lambda-\tfrac{2}{3}\pi^2\lambda^2)\frac{1}{(n+1)(n+2)}+O\left(\frac{1}{n^3}\right)\right]. \quad (3.15)$$

We define a new function $\phi_n(x,x')$ by writing

$$(n+1)f_n(x,x') = 2\pi^2\left[1-\frac{i\pi\lambda}{n+1}+(\lambda-\tfrac{2}{3}\pi^2\lambda^2)\frac{1}{(n+1)(n+2)}\right]\frac{1}{x_>}\left(\frac{x_<}{x_>}\right)^n+\phi_n(x,x') . \quad (3.16)$$

The function $\phi_n(x,x')$ has the property

$$\phi_n(x,x')\big|_{x=x'=-1} \sim 1/n^3 \quad \text{for} \quad n\to\infty . \quad (3.17)$$

From (3.3), we see that for $\theta=0$

$$U_n(1)=n+1 .$$

Thus,

$$\sum_n \phi_n(-1\ \ -1)U_n(\cos\theta)$$

converges for $\theta=0(t=0)$. The entire $t=0$ singularity comes from the terms separated explicitly in (3.16). The sum over n for these terms can be done using the generating function for the Tchebycheff polynomials. The sums are evaluated in Appendix E. The sum over ϕ_n may be replaced by the Sommerfeld-Watson integral and the contour deformed as described above. The result is

$$f(p,p')\big|_{p^2=p'^2=-m^2} = \frac{1}{-m^2}\left\{\frac{1}{2(1-\cos\theta)} - \frac{i\pi\lambda}{2[\cos^2\theta-1]^{1/2}}\ln\left[\frac{[\cos\theta-1]^{1/2}-[\cos\theta+1]^{1/2}}{[\cos\theta-1]^{1/2}+[\cos\theta+1]^{1/2}}\right]+\left(\lambda-\frac{2\pi^2}{3}\lambda^2\right)\left[\frac{1}{2}\left[\frac{\cos\theta-1}{\cos\theta+1}\right]^{1/2}\right.\right.$$
$$\left.\left.\times\ln\left[\frac{[\cos\theta-1]^{1/2}+[\cos\theta+1]^{1/2}}{[\cos\theta-1]^{1/2}-[\cos\theta+1]^{1/2}}\right]-\tfrac{1}{2}\ln 2(1-\cos\theta)\right]\right\}+\frac{1}{4i\pi^2m^2}\int_L\frac{dn}{\sin\pi n}\phi_n(-1,-1)U_n(\cos(\pi-\theta)) . \quad (3.18)$$

All terms exist for $\text{Re}\theta=0$, $\text{Im}\theta\neq 0$ ($\cos>1$). Thus, there is no pole in $F(s,t)$ for $s=0$ and fixed negative t.

There is one other point which should be considered. The general solution of the differential equation (3.5) includes an arbitrary linear combination of the independent solutions of the homogeneous equation, subject to the boundary conditions. Thus, we might add to the particular solution (3.7) terms of the form

$$[c_1^{(n)}y_1^{(n)}(x)+c_2^{(n)}y_2^{(n)}(x)] \times[c_1^{(n)}y_1^{(n)}(x')+c_2^{(n)}y_2^{(n)}(x')] ,$$

where $c_1^{(n)}$ and $c_2^{(n)}$ are arbitrary functions of n. Each of these four extra terms satisfies the boundary conditions (3.6). However, since $f_n(x,x')$ is a function of

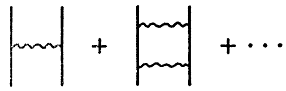

FIG. 8. Sum of ladder diagrams for $e\bar{e}$ scattering.

two variables, the integral equation is more restrictive than the boundary conditions (3.6) which involve only dependence on one variable while the other is fixed. In particular, $y_2^{(n)}(x)y_2^{(n)}(x')$ is not a solution of the homogeneous integral equation since it behaves like $(x')^{-n/2-1}$ for $x'\to 0$, while the integral obtained by substituting this proposed solution behaves like $(x')^{-n/2}$ in this same limit. Thus, $c_2^{(n)}=0$. On the other hand, we have not been able to find any restrictions on $c_1^{(n)}$. Thus, the solution (3.18) obtained by summing the "partial-wave" series with the particular solution (3.7) is not proven to be unique. However, the important point is the existence of a solution in which the Goldstone boson is not coupled to the physical particles. We note that in the Nambu-Jona-Lasinio model in which the Goldstone boson does appear in the scattering amplitude, there is no solution to the Bethe-Salpeter equation for $k=0$ because the pole appears in the particular solution (Appendix A).

IV. HIGH-ENERGY BEHAVIOR IN THE t CHANNEL

We have determined the contribution of all ladder diagrams (Fig. 8) to $F(s,t)$ for $s=0$. Crossing symmetry for $e+\bar{e}\to e+\bar{e}$ implies

$$F(s,t)=F(t,s). \tag{4.1}$$

The approximate $F(s,t)$ computed from just the ladder diagrams does not have this property, but there is another set of "crossed" diagrams (Fig. 9) whose contribution to $F(s,t)$ is obtained from the ladder approximation $F(s,t)$ by simply interchanging s and t. The contribution of these diagrams to physical forward electron-positron scattering may thus be determined from the ladder approximation $F(0,t)$ by making the analytic continuation to $t>4m^2$ rather than $t<0$.

We start with the integral representation (3.10) of the off-mass-shell solution $f(p,p')$ and deform the contour as described in the previous section. The continuation to $t>4m^2$ means $\cos\theta<-1$ which requires $\theta=\pi+i\tau$. The fact that the real part of θ is to be π rather than zero means that there will be no difficulty with the convergence of the final integral and hence no need for any subtraction. The t dependence (θ dependence) of the integrand is all in the Techebycheff polynomial $U_n(\cos(\pi-\theta))$. We write

$$n=\xi+i\eta\,,$$
$$\theta=\pi+i\tau\,;$$

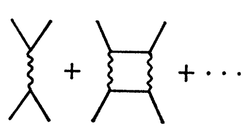

FIG. 9. "Crossed" diagrams.

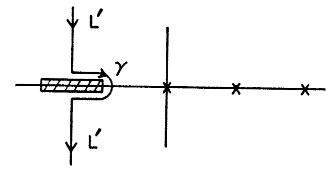

FIG. 10. The contour $L'+\gamma$.

then

$$U_n(\cos(\pi-\theta))=\frac{e^{(1+\xi)\tau}e^{i\tau\eta}-e^{-(1+\xi)\tau}e^{-i\tau\eta}}{e^{\tau}-e^{-\tau}}. \tag{4.2}$$

The behavior for large τ depends on the value of ξ. It is minimized for $\xi=-1$. With this in mind, we start with the integral representation (3.10) for $x,x'>1$ and real θ, and deform the contour to C' (Fig. 6). We now open the contour farther to the left so that it consists of an infinite semicircle closed by a straight line with $\xi=-1$ except for an indentation around the cut on the negative real axis (Fig. 10). Again there is no contribution from the integral along the infinite semicircle so we have

$$m^2f(p,p')=\frac{1}{4\pi^2 i}\int_{L'+\gamma}\frac{dn}{\sin\pi n}\times(n+1)f_n(x,x')U_n(\cos(\pi-\theta)). \tag{4.3}$$

We may now perform the analytic continuation by setting $x=x'=-1$ and $\theta=\pi+i\tau$. According to (4.2), at each point along L' the integrand behaves like $e^{-\tau}$ as $\tau\to\infty$. Along the contour γ encircling the right end of the cut, the integrand behaves like $e^{\xi\tau}$, where ξ ranges from -1 to $-1+2\sqrt{\lambda}$. Thus, this is the dominant term for $\tau\to\infty$. The integral along the contour γ is equal to the line integral from -1 to $-1+2\sqrt{\lambda}$ of the discontinuity of the integrand across the cut. Thus,

$$m^2f(p,p')\underset{t\to\infty}{\sim}\frac{1}{4\pi^2 i}\int_{-1}^{-1+2\sqrt{\lambda}}\frac{d\xi}{\sin\pi\xi}(\xi+1)\frac{\pi}{i\lambda}e^{-i2\pi(\xi+1)}\times\{\exp[-2\pi[4\lambda-(\xi+1)^2]^{1/2}]-\exp[2\pi[4\lambda-(\xi+1)^2]^{1/2}]\}e^{\tau\xi}. \tag{4.4}$$

The integral is difficult to do exactly, but the asymptotic behavior is determined by the upper limit

$$m^2f(p,p')\underset{t\to\infty}{\sim}e^{(-1+2\sqrt{\lambda})\tau}\sim\frac{1}{t^{1-2\sqrt{\lambda}}}. \tag{4.5}$$

ACKNOWLEDGMENTS

I have benefitted from several conversations with Professor M. Baker and Professor I. Muzinich concerning the solutions of Bethe-Salpeter equations.

APPENDIX A: THE NAMBU-JONA-LASINIO MODEL

We outline some of the results of the Nambu-Jona-Lasinio (NJL) paper[4] in a form which makes explicit

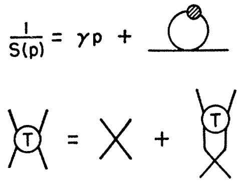

FIG. 11. The approximate self-energy and Bethe-Salpeter equations in the Nambu–Jona-Lasinio model.

the parallel between their calculation and ours. NJL consider fermions interacting through a direct four-fermion interaction. The approximate self-energy equation corresponding to our (2.11) is

$$1/S(p)=\gamma p-i\lambda_0\int(dp')\,\mathrm{Tr}S(p'). \qquad \text{(A1)}$$

NJL look for a solution of the form of Eq. (2.12) with $A=m=$const. Then the self-energy equation reduces to

$$m=\frac{\lambda_0}{4\pi^2}\int dp'^2\,p'^2\frac{m}{p'^2+m^2}. \qquad \text{(A2)}$$

This equation has the trivial solution $m=0$. It also has the desired nontrivial, nonperturbative solution, $m\neq 0$, provided the condition

$$1=\frac{\lambda_0}{4\pi^2}\int dp'^2\frac{p'^2}{p'^2+m^2} \qquad \text{(A3)}$$

is satisfied. The integral is divergent and requires a cutoff to define the model. If this model is considered as an approximation to a Lagrangian theory, the γ_5 invariance of the Lagrangian implies conservation of the axial-vector current. NJL show that the existence of the conserved current implies the existence of massless pseudoscalar particles.[11] The Bethe-Salpeter equation in the approximation corresponding to (1) is the analog of the ladder approximation in a Yukawa-coupling theory,

$$f(p,p')=1+\frac{\lambda_0}{4\pi^2}\int dl^2\frac{l^2}{l^2+m^2}f(p,l). \qquad \text{(A4)}$$

Since the kernel is a constant, the solution is trivial:

$$f(p,p')=\left[1-\frac{\lambda_0}{4\pi^2}\int dl^2\frac{l^2}{l^2+m^2}\right]^{-1}+\text{const}. \qquad \text{(A5)}$$

The extra constant is an arbitrary solution of the homogeneous equation when (A3) is satisfied. However, the particular solution is seen to be infinite, indicating a pole in $f(p,p',k)$ at $k=0$, precisely when (A3) is satisfied, providing for a γ_5-noninvariant solution of (A1).

[11] K. Johnson, Phys. Letters **5**, 253 (1963) has pointed out that formal arguments based on conserved axial-vector currents are unreliable.

APPENDIX B: DECOUPLING OF THE AMPLITUDE $f(p,p')$

We have to take the γ_5, γ_5 projection of

$$\int(dl)\;T(p,l,k)S(l_+)K(l,p',k)S(l_-) \qquad \text{(B1)}$$

with $k=0$. With the spinor indices explicit and substituting Eq. (2.16) for $K(l,p',k)$, the integrand is

$$-ie^2\frac{1}{(l-p')^2}\left[g_{\alpha\beta}-\frac{(l-p')_\alpha(l-p')_\beta}{(l-p')^2}\right]\gamma^5{}_{da\,ae}T_{fd}(p,l,k)$$
$$\times S_{eg}(l_+)\gamma^\alpha{}_{gb}\gamma^5{}_{bc}\gamma^\beta{}_{ch}S_{hf}(l_-)=-3ie^2\frac{1}{(l-p')^2}$$
$$\times\gamma^5{}_{da\,ae}T_{fd}(p,l,k)S_{eg}(l_+)\gamma^5{}_{gh}S_{hf}(l_-). \qquad \text{(B2)}$$

Using $S(p)=(\gamma p+m)^{-1}$, the last three factors are

$$\left[\frac{m-\gamma l_+}{l_+^2+m^2}\gamma^5\frac{m-\gamma l_-}{l_-^2+m^2}\right]_{ef}=\left[\gamma^5\frac{m+\gamma l_+}{l_+^2+m^2}\,\frac{m-\gamma l_-}{l_-^2+m^2}\right]_{ef}$$
$$=\gamma^5{}_{eh}[m^2+l^2-\tfrac{1}{4}k^2+m\gamma k+\tfrac{1}{2}(\gamma l\gamma k-\gamma k\gamma l)]_{hf}. \qquad \text{(B3)}$$

For $k=0$, substitution of (B3) into (B2) gives

$$-3ie^2\frac{1}{(l-p')^2}\gamma^5{}_{da\,ae}T_{fd}(p,l)\gamma^5{}_{ef}\frac{1}{l^2+m^2}.$$

APPENDIX C: SEPARATION OF ANGULAR VARIABLES

After rotation of the l_0 integration contour, Eq. (2.17) may be written

$$f(p,p')=\frac{1}{(p-p')^2}+\frac{3\alpha}{4\pi^3}\int l^3dld\Omega_l\frac{f(p,l)}{(p'-l)^2(l^2+m^2)}. \qquad \text{(C1)}$$

In four-dimensional hyperspherical coordinates

$$d\Omega=\sin^2\theta_1\sin\theta_2d\theta_1d\theta_2d\phi. \qquad \text{(C2)}$$

We introduce the expansions

$$\frac{1}{(p'-l)^2}=\sum_{s=0}^{\infty}\frac{1}{l'_>{}^2}\left(\frac{l'_<}{l'_>}\right)^s U_s(\cos\theta_{p'l}), \qquad \text{(C3a)}$$

$$f(p,l)=\frac{1}{2\pi^2}\sum_{r=0}^{\infty}(r+1)f_r(p^2,l^2)U_r(\cos\theta_{pl}), \qquad \text{(C3b)}$$

where $l_<'(l_>')$ is the lesser (greater) of l,p'. Using the orthogonality relation

$$\int d\Omega_l\,U_r(\cos\theta_{pl})U_s(\cos\theta_{p'l})=2\pi^2\frac{\delta_{rs}}{r+1}U_r(\cos\theta_{pp'}), \qquad \text{(C4)}$$

$$[U_r(\cos\theta_{pp})=U_r(1)=r+1],$$

we compute

$$\frac{1}{n+1}\int d\Omega_p\, U_n(\cos\theta_{pp'})f(p,p')=f_n(p^2,p'^2)$$

$$=\frac{2\pi^2}{n+1}\frac{1}{p_>{}^2}\left(\frac{p_<}{p_>}\right)^n+\frac{3\alpha}{2\pi}\frac{1}{n+1}$$

$$\times\int dl\frac{l^3}{l'_>{}^2(l^2+m^2)}\left(\frac{l'_<}{l'_>}\right)^n f_n(p^2,l^2). \quad \text{(C5)}$$

In terms of the variables x and x', and $\lambda=3\alpha/4\pi$,

$$f_n(x,x')=\frac{2\pi^2}{n+1}\left[\frac{1}{x}\left(\frac{x'}{x}\right)^{n/2}\theta(x-x')+\frac{1}{x'}\left(\frac{x}{x'}\right)^{n/2}\theta(x'-x)\right]$$

$$+\frac{\lambda}{n+1}\int_0^{x'}dy\frac{y}{x'(y+1)}\left(\frac{y}{x'}\right)^{n/2}f_n(x,y)$$

$$+\frac{\lambda}{n+1}\int_{x'}^{\infty}dy\frac{1}{y+1}\left(\frac{x'}{y}\right)^{n/2}f_n(x,y). \quad \text{(C6)}$$

This equation may be written in abbreviated form as

$$f_n(x')=I_n(x')+\frac{1}{x'^{n/2+1}}F_n(x')+x'^{n/2}G_n(x'),$$

where the dependence on x has been suppressed. Differentiation of this equation twice with respect to x' leads to the differential equation (3.5).

APPENDIX D: PROPERTIES OF THE HYPERGEOMETRIC FUNCTIONS

We collect some useful properties of the functions (3.8). Using the integral representation

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}\frac{(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}}dt, \quad \text{(D1)}$$

or the formulas of the Bateman Manuscript Project (Vol. I, Sec. 2.10) we obtain

$$y_1(x)\to x^{n/2},$$

$$y_2(x)\to -\frac{1}{n+1}\frac{1}{x^{n/2+1}}, \qquad \text{for } x\to 0 \quad \text{(D2)}$$

$$y_1(-1)=e^{-in\pi/2}\frac{\Gamma(n+2)}{\Gamma(a+1)\Gamma(b+1)},$$

$$y_2(-1)=e^{i\pi(a-n/2)}\frac{\Gamma(a)}{\Gamma(1-b)\Gamma(n+2)}, \quad \text{(D3)}$$

$$y_1(x)\to\frac{\Gamma(n+2)\Gamma(\nu)}{\Gamma(a)\Gamma(a+1)}x^{(\nu-1)/2},$$

$$y_2(x)\to\frac{-\Gamma(a)\Gamma(a+1)}{\Gamma(n+1)\Gamma(\nu+1)}\frac{1}{x^{(\nu+1)/2}}, \qquad \text{for } x\to\infty \quad \text{(D4)}$$

$$y_1(x)\to x^{n/2},$$

$$y_2(x)\to -\frac{1}{n+1}\frac{1}{x^{n/2+1}}, \qquad \text{for } n\to\infty. \quad \text{(D5)}$$

APPENDIX E: EVALUATION OF SUMS

We evaluate the sums

$$I=\sum_{n=0}^{\infty}U_n(z), \quad \text{(E1a)}$$

$$J=\sum_{n=0}^{\infty}\frac{1}{n+1}U_n(z), \quad \text{(E1b)}$$

$$K=\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}U_n(z), \quad \text{(E1c)}$$

$$(-1<z=\cos\theta<1).$$

The generating function for the Tchebycheff polynomials of the second kind is

$$I(\xi,z)=\frac{1}{1-2\xi z+\xi^2}=\sum_{n=0}^{\infty}U_n(z)\xi^n, \quad |\xi|<1. \quad \text{(E2)}$$

The left-hand side provides the analytic continuation of the right-hand side for $|\xi|\geq 1$. Specializing to $\xi=1$ gives

$$I=1/2(1-z). \quad \text{(E3)}$$

Next we write

$$J(\xi,z)=\sum_{n=0}^{\infty}U_n(z)\frac{\xi^{n+1}}{n+1}=\int_0^{\xi}\frac{d\xi'}{1-2\xi'z+\xi'^2}. \quad \text{(E4)}$$

The integral is

$$\frac{1}{2[z^2-1]^{1/2}}\ln\left[\frac{(\xi-z-[z^2-1]^{1/2})(-z+[z^2-1]^{1/2})}{(\xi-z-[z^2-1]^{1/2})(-z-[z^2-1]^{1/2})}\right].$$

Setting $\xi=1$, we have

$$J=\frac{1}{2[z^2-1]^{1/2}}\ln\left[\frac{[z-1]^{1/2}-[z+1]^{1/2}}{[z-1]^{1/2}+[z+1]^{1/2}}\right]. \quad \text{(E5)}$$

Finally,

$$K(\xi,z)=\sum_{n=0}^{\infty}U_n(z)\frac{\xi^{n+2}}{(n+1)(n+2)}=\int_0^{\xi}d\xi'\,J(\xi',z). \quad \text{(E6)}$$

Setting $\xi=1$ and carrying out the integral, we find after some combination of terms

$$K=\frac{1}{2}\left[\frac{z-1}{z+1}\right]^{1/2}\ln\left[\frac{[z-1]^{1/2}+[z+1]^{1/2}}{[z-1]^{1/2}-[z+1]^{1/2}}\right]-\tfrac{1}{2}\ln 2(1-z). \quad \text{(E7)}$$