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Static, Axially Symmetric, Interior Solution in General Relativity*

WALTER C. HERNANDEZ, JR.†

Institute for Space Studies, Goddard Space Flight Center, NASA, New York, New York

and

Department of Physics and Astronomy, University of Maryland, College Park, Maryland

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A static, axially symmetric, interior solution of the Einstein field equations which matches on smoothly to one of the Weyl exterior solutions is exhibited. It is obtained essentially by guessing an interior metric and requiring that the calculated stress-energy tensor not be grossly unphysical. Boundary conditions at the surface of the matter are employed which guarantee the existence of, but do not exhibit explicitly, coordinates in which the metric components and their derivatives are continuous. Finally, it is shown that this method can easily be generalized to obtain interior solutions for any Weyl metric which indicates that it has a positive mass source.

I. INTRODUCTION

IN a realistic analysis of many astrophysical problems the lack of spherical symmetry must eventually be considered. For example, stationary rotating bodies and gravitational collapse with angular momentum can, at best, possess axial symmetry. In fact, systems which might serve as sources of gravitational radiation must *not* be spherically symmetric as is evident from Birkhoff's theorem.[1] Of course the study of any of these problems in the framework of general relativity would, in general, be quite complicated. It is reasonable to expect that their solution will involve many new techniques and an improved understanding of many old ones. In the hope of facing simultaneously the demands for only a few of these many needed advances, I have chosen here to study a very simple problem whose main value is that it does lack spherical symmetry.

The problem under consideration here is that of fitting a static, interior solution to one of the Weyl axially symmetric, static, exterior solutions of the Einstein equations. The solution I find serves to verify the standard assumption that the Weyl metrics represent gravitational fields which could be produced by nonspherical solid bodies. In the course of this exercise I found that the Lichnerowicz conditions[2] on the differentiability of the metric (C^1, C^3 by pieces) are too stringent to be applied in practice. (Indeed, they are not imposed in the usual discussions of Schwarzschild's interior solution.) Instead we take advantage of a wider choice of coordinate systems to simplify the computations, but impose boundary conditions which guarantee the existence of some coordinates in which the Lichnerowicz conditions would be satisfied.

As I shall discuss in Sec. II, Weyl and Levi-Civita,[3] in 1919, discovered all the solutions of the axially symmetric Einstein field equations after imposing the simplifying conditions that they describe the static, vacuum states. I shall drop the vacuum requirement, but retain the static condition. If I look at only finite-mass distributions then I am actually searching for *interior* solutions which might serve as sources of the Weyl *exterior* solutions. We must also expect these sources to exhibit some properties of a solid, the usual argument being that any static fluid would also be spherically symmetric. I shall not concern myself with the problem as to whether any nonfluid types of materials can actually exist under very high gravitational stresses. The method I shall use for looking for

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† Presently at the Goddard Institute for Space Studies, New York, New York.

[1] G. D. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, Massachusetts, 1923), p. 253.

[2] A. Lichnerowicz, *Theories Relativistes de la Gravitation et de L'electromagnétisme* (Masson et Cie, Paris, 1955), pp. 5, 6.

[3] H. Weyl, Ann. Physik **54**, 117 (1918); **59**, 185 (1919); T. Levi-Civita, Atti Accad. Naz. Lincei Rend., Classe Sci. Fis., Mat. e Nat. **28**, 101 (1919).

interior solutions is essentially that suggested by Synge[4] which is simply to guess the field $g_{\mu\nu}$ and see whether this guess leads to a $T_{\mu\nu}$ which is grossly unphysical.

It should be emphasized that the main contribution of this work lies not in the physical significance of the interior solution obtained, but in the techniques employed for obtaining this nonspherically symmetric interior metric. In fact, in concluding this paper we show that the above method can immediately be used for obtaining interior solutions for all the Weyl metrics which indicate a source possessing positive mass.

II. AXIAL SYMMETRY AND WEYL'S SOLUTIONS

Any static, axially symmetric metric can be written in the cylindrical-coordinate form as

$$ds^2=\alpha^2(d\rho^2+dz^2)+\beta^2d\phi^2-\gamma^2dt^2, \qquad (1)$$

where the metric coefficients are functions of ρ and z. Weyl and Levi-Civita showed that one could obtain all the solutions of the Einstein field equations for the case of static, axial symmetry in a vacuum by taking the line element (1) in the form

$$ds^2=e^{2\gamma-2\psi}(d\rho^2+dz^2)+\rho^2e^{-2\psi}d\phi^2-e^{2\psi}dt^2, \qquad (2)$$

where ψ and γ satisfy the field equations

$$\psi_{\rho\rho}+\rho^{-1}\psi_\rho+\psi_{zz}=0, \qquad (3)$$

$$\gamma_\rho=\rho(\psi_\rho{}^2-\psi_z{}^2),\quad \gamma_z=2\rho\psi_\rho\psi_z \qquad (4)$$

with subscripts denoting partial derivatives. The solutions of these equations are discussed in detail by Synge.[4] To obtain solutions one chooses for ψ any three-dimensional harmonic function which is independent of ϕ since Eq. (3) is simply Laplace's equation in cylindrical coordinates for a function which is independent of ϕ. It is also required that ψ vanish at infinity at least like $(\rho^2+z^2)^{-1/2}$ so that the space will be flat at infinity. The function γ is obtained by the line integration of Eq. (4) with the boundary condition that $\gamma=0$ on the z axis. I shall pick the most obvious and simple solution[5,6]

$$\psi=-m/R, \qquad (5)$$

where $R^2=\rho^2+z^2$. The integration for γ yields

$$\gamma=-\tfrac{1}{2}m^2\rho^2/R^4. \qquad (6)$$

The attempt now will be to find an interior solution of the Einstein field equations which could serve as a source of this field. Let us see what the requirements on this interior metric will be.

[4] J. L. Synge, *Relativity, The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), pp. 309–317.

[5] J. Chazy, Bull. Soc. Math. France **52**, 17 (1924); H. E. Y. Curzon, Proc. London Math. Soc. **23**, 477 (1924).

[6] This form contains a hint of spherical symmetry, but it has been verified that Eqs. (5) and (6) do not produce the Schwarzschild metric in an unfamiliar coordinate system. For proof, see H. Takeno, Progr. Theoret. Phys. (Kyoto) **8**, 317 (1952).

III. INTERIOR METRIC REQUIREMENTS

The requirements we impose on the interior metric are essentially those discussed by Synge except for the additional demand that the solution yield a "good" Newtonian weak-field limit. We shall be working in a coordinate system where both the interior and the exterior metrics have the form

$$ds^2=g_{rr}dr^2+g_{\theta\theta}d\theta^2+g_{\phi\phi}d\phi^2+g_{tt}dt^2. \qquad (7)$$

The advantage of these coordinates over Weyl's cylindrical coordinates is that the boundary surface may be assumed to have a simple form

$$r=r_1. \qquad (8)$$

Suppose the metric or first fundamental form of the boundary surface s is the same in both interior and exterior coordinates and is given by

$$\mathrm{I}=g_{AB}dx^Adx^B=g_{\theta\theta}d\theta^2+g_{\phi\phi}d\phi^2+g_{tt}dt^2, \qquad (9)$$

where the subscripts (A,B) refer to the coordinates (θ,ϕ,t). In a sufficiently small neighborhood of s we can construct geodesics normal to the surface s such that each point of s uniquely determines a geodesic. Then any point in the neighborhood can be named by a coordinate R, which is just the proper spatial distance of the point from the surface s along one of these geodesics and the three coordinates (θ,ϕ,t) which is the point where this geodesic intersects the surface. Thus the metric in these "normal Gaussian coordinates" is simply

$$ds^2=g_{AB}dx^Adx^B+dR^2. \qquad (10)$$

The second fundamental form of any hypersurface s is given by

$$\mathrm{II}=-[n_{\mu;\nu}dx^\mu dx^\nu]_s\equiv K_{AB}dx^Adx^B, \qquad (11)$$

where n_μ is the unit normal to the surface and the subscript s means that one of the coordinate differentials is to be eliminated using the equation of the surface. For the metric (10) the second fundamental form is simply

$$K_{AB}=\Gamma_{AB}{}^R=-\frac{1}{2}\frac{\partial g_{AB}}{\partial R}. \qquad (12)$$

Thus if we insist that the first and second fundamental forms be the same whether the boundary surface is considered imbedded in the interior or the exterior space, this will guarantee us that there is a coordinate system (namely these normal Gaussian coordinates) where all the $g_{\mu\nu}$ and all the normal derivatives of the $g_{\mu\nu}$ (here simply the derivative with respect to R) are continuous. It also obviously follows that partial derivatives taken parallel to the surface are continuous across the surface and hence that the Lichnerowicz continuity conditions are satisfied in this coordinate system. Hence the requirements we impose on the metric are the following:

(A) The first fundamental form of the boundary surface should be the same whether obtained from the interior or exterior metric.

(B) The second fundamental form should be the same whether the boundary surface is considered imbedded in interior or exterior space-time. Since the Lichnerowicz condition is guaranteed to hold in some coordinate system, the Synge condition,[7] that the normal component of the stress vanish at the surface, is also satisfied. For the case of the metric (9) we find that the K_{AB} are given by

$$K_{AB}=(g_{rr})^{1/2}\Gamma_{AB}{}^{r}=-\tfrac{1}{2}(g_{rr})^{-1/2}\frac{\partial g_{AB}}{\partial r}. \tag{13}$$

So if we choose g_{rr} to be continuous across the boundary then this requirement simply states that the $g_{AB},_r$ must be continuous across the boundary also (but allows a jump in $g_{rr},_r$).

(C) The local energy density should be non-negative, i.e., $T_0{}^0\leq 0$.

(D) All the physical components of the stress should be small enough compared to the local energy density so as to allow an interpretation where the velocity of sound will not be greater than the velocity of light as is demanded by causality. Though we shall not strictly demand that $T_i{}^i$ (not summed) be non-negative as Synge does, since we know this does not hold even in many Newtonian examples of stressed solids, nevertheless it shall turn out to be true for our particular example. This would probably tend to make the solution more stable.

(E) The weak-field (Newtonian) limit should give us good values for the mass density and stresses, i.e., the stress and mass density distributions should describe a body composed of a material with a reasonable equation of state. The idea here is that we know what reasonable equations of state are in the Newtonian limit, but there is little that we can say about what a reasonable equation of state would be for a solid body in the relativistic limit. Actually it also makes the satisfaction of requirements (C) and (D) rather simple.

IV. THE INTERIOR METRIC

I found it much easier to work in the coordinates (r,θ,ϕ,t) given by Eqs. (4), (5), and (7) of Erez and Rosen,[8] rather than the cylindrical coordinates (ρ,z,ϕ,t) discussed earlier. The Erez and Rosen coordinate transformations can be written as

$$z=(r-m)\cos\theta, \tag{14}$$

$$\rho=(r^2-2mr)^{1/2}\sin\theta, \tag{15}$$

so that

$$R^2=r^2-2mr+m^2\cos^2\theta \tag{16}$$

for $r\geq 2m$.

The exterior metric (2), (5), and (6) becomes in these new coordinates

$$ds^2=e^{2\gamma-2\psi}\left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)dr^2+(r^2-2mr+m^2\sin^2\theta)d\theta^2\right]+e^{-2\psi}(r^2-2mr)\sin^2\theta d\phi^2-e^{2\psi}dt^2, \tag{17}$$

with

$$\gamma=-\frac{1}{2}\frac{m^2(r^2-2mr)\sin^2\theta}{(r^2-2mr+m^2\cos^2\theta)^2}, \tag{18}$$

$$\psi=-\frac{m}{(r^2-2mr+m^2\cos^2\theta)^{1/2}}. \tag{19}$$

A crucial step is to notice that if one expands the above metric in powers of m/r and drops all terms in the third order and higher, then one gets

$$ds^2=\left(1+\frac{2m}{r}+\frac{4m^2}{r^2}\right)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)-\left(1-\frac{2m}{r}\right)dt^2 \tag{20}$$

which is identical to the ordinary Schwarzschild metric

$$ds^2=\frac{dr^2}{1-2m/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)-\left(1-\frac{2m}{r}\right)dt^2 \tag{21}$$

expanded up to second order in m/r. So it should be possible to find an interior solution which, to second order, looks spherically symmetric. The spherically symmetric interior solution we shall look at is the Schwarzschild interior solution[9] which describes an incompressible fluid. This is given by

$$ds^2=\frac{dr^2}{1-r^2/B^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)-\left[A-\frac{1}{2}\left(1-\frac{r^2}{B^2}\right)^{1/2}\right]^2dt^2, \tag{22}$$

where

$$A=\frac{3}{2}\left(1-\frac{r_1{}^2}{B^2}\right)^{1/2},\quad \frac{r_1{}^2}{B^2}=\frac{2m}{r_1}<1,\qquad B^2=3/8\pi\rho_{00}. \tag{23}$$

ρ_{00} is the local mass density and $r=r_1$ is the surface. All the metric components and their derivatives are continuous across the boundary except for $g_{rr,r}$, which is consistent with requirements A and B. One could say that the interior metric (22) was obtained from the exterior metric (21) by the substitutions

$$2m/r\rightarrow r^2/B^2 \tag{24}$$

in those metric components which need to be continuous, but need not have a continuous derivative and by

[7] See Ref. 4, p. 317.

[8] G. Erez and N. Rosen, Bull. Res. Council Israel, Sec. F, 8, 47 (1959).

[9] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, England, 1962), p. 245.

the substitution

$$2m/r \to 1-[A-\tfrac{1}{2}(1-r^2/B^2)^{1/2}]^2 \tag{25}$$

in those metric components which need to be continuous and have continuous first derivatives also. It is clear that these substitutions do in fact accomplish this since (24) does not have a continuous first derivative at the surface, but (25) has.

If one uses this same method of substitution on the exterior metric (17), i.e., uses (24) for the g_{rr} metric component and (25) for the $g_{\theta\theta}$, $g_{\phi\phi}$, and g_{tt} metric components, one obtains the "interior" metric

$$ds^2=e^{2\delta}\left(1+\frac{r^4}{4B^4}\frac{\sin^2\theta}{1-r^2/B^2}\right)dr^2 +e^{2\alpha-2\beta}r^2[1+X(r)+\tfrac{1}{4}X^2(r)\sin^2\theta]d\theta^2 +e^{-2\beta}r^2\sin^2\theta[1+X(r)]d\phi^2-e^{2\beta}dt^2, \tag{26}$$

where

$$\delta=-\frac{r^4}{8B^4}\left(1-\frac{r^2}{B^2}\right)\left(1-\frac{r^2}{B^2}+\frac{r^4}{4B^4}\cos^2\theta\right)^{-2}\sin^2\theta +\frac{r^2}{2B^2}\left(1-\frac{r^2}{B^2}+\frac{r^4}{4B^4}\cos^2\theta\right)^{-1/2},$$

$$\begin{aligned}\alpha&=-\tfrac{1}{8}X^2(1+X)[1+X+\tfrac{1}{4}X^2\cos^2\theta]^{-2}\sin^2\theta,\\ \beta&=\tfrac{1}{2}X[1+X+\tfrac{1}{4}X^2\cos^2\theta]^{-1/2},\end{aligned} \tag{27}$$

$$X=\left[\frac{3}{2}\left(1-\frac{r_1^2}{B^2}\right)^{1/2}-\frac{1}{2}\left(1-\frac{r^2}{B^2}\right)^{1/2}\right]^{-2}-1,$$

$$\frac{r_1^2}{R^2}=\frac{2m}{r_1}<1,\quad R^2=\frac{3}{8\pi\rho_{00}}=\text{constant},$$

and $r=r_1$ is the boundary.

It is obvious that again all the metric components and their first derivatives except for $g_{rr,r}$ will be continuous across the boundary, thus satisfying requirements A and B. If we make an expansion of this interior metric in powers of r^2/R^2 up to second order, we get

$$ds^2=\left(1+\frac{r^2}{B^2}+\frac{r^4}{B^4}\right)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) -[A-\tfrac{1}{2}(1-r^2/B^2)^{1/2}]^2dt^2 \tag{28}$$

which is the Schwarzschild interior solution up to second order. Actually the g_{00} component of (28), as it stands, contains higher order terms than second, but we left it in this form for convenience.

Substituting the Schwarzschild interior solution up to second order into the Einstein field equations (with $G=1$, $c=1$)

$$8\pi T_\nu{}^\mu=R_\nu{}^\mu-\tfrac{1}{2}\delta_\nu{}^\mu, \tag{29}$$

we get the ordinary Newtonian expressions for the mass density and pressure of an incompressible sphere of fluid

$$\rho=3m/4\pi r_1^3, \tag{30}$$

$$p=\frac{3}{8\pi}\frac{m^2}{r_1^4}\left(1-\frac{r^2}{r_1^2}\right). \tag{31}$$

It is evident that if we substitute the metric (26) into (29) that we will get the forms

$$\rho=\frac{3m}{4\pi r_1^3}\left(1+\frac{m}{r_1}g_0\right), \tag{32}$$

$$T_\alpha{}^\alpha=\frac{3m^2}{8\pi r_1^4}\left(1-\frac{r^2}{r_1^2}+\frac{m}{r_1}g_\alpha\right), \tag{33}$$

where $T_\alpha{}^\alpha$ is not summed, α refers to r, θ, and ϕ, and g_0, g_α are four functions of order unity. We see that if m/r_1 is not too large then requirements (C) and (D) are satisfied. Actually m/r_1 must be of order unity before these conditions are not met. The exact limit on m/r_1 can only be obtained after the g_0, g_α are obtained explicitly, which we shall not do. The requirement (E) is obviously satisfied by the equations (32) and (33).

V. DISCUSSION OF SOLUTION

By looking closely at the exterior metric we can determine some qualities of the interior metric which acts as a source. If we expand the g_{00} component of the metric (17) in powers of m/r we obtain

$$-g_{00}=1-2\left[\frac{m}{r}-\frac{1}{3}\frac{m^3}{r^3}P_2(\cos\theta)+\cdots\right]. \tag{34}$$

Thus at large distance from the source (weak fields) it appears that this metric describes a source of mass m and quadrupole moment $m^3/3$.[10] Next we notice that the metric of the boundary surface $r=r_1$ and $t=$ const up to order m^3/r^3 is given by

$$ds_2^2=r_1^2\left[1-\frac{2}{3}\frac{m^3}{r_1^3}P_2(\cos\theta)\right](d\theta^2+\sin^2\theta d\phi^2). \tag{35}$$

This can be interpreted as a two-surface in a 3-dimensional Euclidean space given by

$$r=r_1\left[1-\frac{1}{3}\frac{m^3}{r_1^3}P_2(\cos\theta)\right], \tag{36}$$

where r is the ordinary radial distance and again we retained terms only up to third order. This tells us that the approximate amount of mass which contributes to

[10] At first it might appear that the identification of the quadrupole moment as obtained from Eq. (34) was dependent on the coordinate system; it is not, however, in the sense that it is this particular coordinate system which gives a zero quadrupole moment to the spherically symmetric Schwarzschild metric. Also, see Ref. 8.

the quadrupole moment is $m(m^3/r_1^3)$ (if the density is approximately constant). The largest possible quadrupole potential term this leads to is of the order

$$\frac{(mm^3/r_1^3)r_1^2}{r^3}=\frac{m^3}{r^3}\left(\frac{m}{r_1}\right) \tag{37}$$

which is too small by one order, i.e., a factor m/r_1. So we conclude that, in a Newtonian analysis, the deviation from spherical symmetry of the source is not due to a distortion of the surface, but must be a result of variation in the density. The sign of the quadrupole term indicates the density is higher nearer the $\theta=\pi/2$ plane. Actually a variation from constant density which goes like m^2/r_1^2 would be sufficient though not necessary.

From the forms of the expression (32) and (33) we can say that in the case of weak fields the interior metric (26) describes a material which is nearly incompressible, and so represents a material whose equation of state is not very different from many common forms.

VI. OTHER SOLUTIONS

After considering closely the techniques involved in obtaining the above solution, it becomes apparent that this method can immediately be generalized for obtaining sources of other Weyl metrics. Using a notation similar to that of Erez and Rosen,[8] we can write the almost general solution of Eq. (3) as

$$\Psi=\Psi_0+q\Psi_1, \tag{38}$$

where Ψ_0 refers to the spherically symmetric Schwarzschild solution [Eq. (13) of Erez and Rosen], but now Ψ_1 refers, in general, to that part of Ψ which is nonspherically symmetric. The q is simply a constant. The form of Eq. (38) restricts Ψ to describing the gravitational field of a particle which possesses a positive mass and higher mass multipoles (if $q\neq 0$). By consideration of Eq. (4) we can write a similar expression for γ as

$$\gamma=\gamma_0+q\gamma_1+q^2\gamma_2. \tag{39}$$

Assuming we have chosen a particular expression for Ψ and have solved for γ, we can obtain an interior solution by use of the following prescription.

(1) Rewrite this exterior metric using the Erez and Rosen coordinate transformation [their Eq. (7) or our Eqs. (14) and (15)]. See Eq. (1b) of Erez and Rosen.

(2) Use our method of substitution [Eqs. (24) and (25)] on the terms of the metric which do not depend on q (i.e., all terms in the metric except Ψ_1, γ_1, γ_2.

(3) Use any method of substitution on the q-dependent terms which has the necessary continuity conditions at the surface $r=r_1$ and also yields well-behaved metric components. It is evident that the mass density and stress will be given by equations of the form

$$\rho=\rho_0(1+qg_0), \tag{40}$$

$$T_\alpha{}^\alpha=p_0(1+qg_\alpha), \tag{41}$$

where again $T_\alpha{}^\alpha$ is not summed, α refers to r, θ, and ϕ, and g_0, g_α are four functions of order unity. Here ρ_0 and p_0 are the mass density and pressure, respectively, of the Schwarzschild interior solution. Thus if we let $q=0$ our exterior and interior solutions reduce to the Schwarzschild solutions. So by choosing q to be not too large, conditions (C) and (D) are satisfied and the stresses are positive also. Conditions (A) and (B) are satisfied by virtue of our method of substituting. Our condition (E) requires some comment. The original reason for requiring a good weak-field limit was to ensure that the equation of state of the material would not be too unreasonable, especially in the Newtonian limit. In these new solutions which we have just outlined, the Newtonian limit might well possess very large mass multipoles and thus would be difficult to investigate. We do have some assurance, however, that the equation of state is not always grossly unreasonable. If we take the spherically symmetric limit of the above solutions ($q\rightarrow 0$) then the resulting Schwarzschild interior solution describes an incompressible material. Thus in solutions which deviate only a little from spherical symmetry the equation of state deviates only slightly from that of an incompressible material.

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