groups are given in Figs. 15–18, and Table VII lists the differential cross sections for the ground state, the 4.55-MeV state, and the 10.79-MeV state groups. The latter group has been recently studied by Detraz, *et al.*[8] at 43.7 MeV. Though the angular distributions agree remarkably well in shape, there exists a difference in the absolute cross sections which is hard to attribute to such a small difference in the incident energy.

D. Summary

Table VIII gives a summary of the reactions induced by the bombardment of Be^9 by 46-MeV protons studied in the present experiment. The simultaneous measurement of proton, deuteron, and triton spectra enabled us to accumulate the data with reasonable speed and to determine the absolute cross sections with increased accuracy.

Optical-model and distorted-wave Born-approximation calculations of the present data will be reported in a separate paper.

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Four-Quasiparticle Excitations and Two-Phonon Vibrational States in Spherical Nuclei. Even-Parity States of Even Tin Isotopes

P. L. OTTAVIANI*
Istituto di Fisica dell'Universita di Bologna, Bologna, Italy
AND
M. SAVOIA†
Istituto di Fisica dell'Universita di Bologna, Bologna, Italy; and Centro de Investigacion y de Estudios Avanzados del I.P.N., Mexico
AND
J. SAWICKI‡
Laboratoire Joliot-Curie de Physique Nucleaire, Orsay, France; and Centro de Investigacion y de Estudios Avanzados del I.P.N., Mexico
AND
A. TOMASINI§
Istituto di Fisica dell'Universita di Bologna, Bologna, Italy
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A microscopic theory of the low-lying states of even-even spherical nuclei is developed in which eigenvectors are linear combinations of two- and four-quasiparticle excitations. The quasiparticles are defined by the Bogoliubov-Valatin canonical transformation. The method is called the quasiparticle second Tamm-Dancoff (QSTD) approximation, since no ground-state correlations are included. It is found that the spurious kets due to the particle-number nonconservation must be absolutely projected out of the secular matrices before their diagonalization. Such a procedure is described and applied. Formulas are given for the electromagnetic transition probabilities. The theory is applied to the study of the 2^+, 4^+, and 0^+ states of the even tin isotopes. The single-particle radial wave functions employed are those of a Saxon-Woods potential and of a harmonic-oscillator potential. The two-nucleon residual interaction potential is spin-dependent and of zero range. Satisfactory numerical agreement with the observed 2^+ and 4^+ low-lying levels is obtained with the Saxon-Woods wave functions for a reasonable strength constant of our zero-range force. Appreciable admixtures of the four-quasiparticle creation components are found even in the lowest lying levels. Poor agreement is obtained for the 0^+ states, for which a more refined theory is necessary (rather unreasonable values of the strength constant of the zero-range potential are required to fit the 0^+ data). Generally, markedly worse 2^+ results are obtained if we replace the Saxon-Woods wave functions with harmonic-oscillator wave functions.

1. INTRODUCTION

RECENTLY, microscopic theories have been proposed for two-phonon-type vibrational states of spherical "superconductor" nuclei in a paper by by three of us[1] (hereafter referred to as I), and in papers by Tamura and Udagawa[2] and by Hsu and French.[3] In the formalism of these papers the two-

* Present address: Centro di Calcolo del Comitato Nazionale per l'Energia Nucleare, Bologna, Italy.
† Present address: Istituto di Fisica di Bologna, Bologna, Italy.
‡ Present address: CERN, Geneva, Switzerland.
§ Deceased on June 15, 1965.

[1] M. Savoia, J. Sawicki, and A. Tomasini, Nuovo Cimento **32**, 991 (1964); this reference contains numerous misprints, and we present here several of the equations of I in their corrected form.
[2] T. Tamura and T. Udagawa, Nucl. Phys. **53**, 33 (1964).
[3] L. S. Hsu and J. B. French, Phys. Letters **19**, 135 (1965); cf. also N. Auerbach, *ibid.* **21**, 57 (1966).

and the four-quasiparticle excitations are treated on the same footing. Essentially, it is a generalization of the particle-hole (or random-phase-approximation) theory for closed-shell nuclei and of the two-quasiparticle random phase approximation (RPA) in the case of "superconductor" nuclei. A corresponding generalization in the case of the "normal"-state nuclei is the higher (second) RPA as discussed in a paper by one of us[4] (cf. also Leonardi *et al.*[5]). This microscopic approach is in contrast to the "semiphenomenological" one in which one assumes two-phonon-type states (i.e., the 0^+, 2^+, 4^+ triplets) to be generated simply by appropriate quadratic combinations of one-phonon RPA modes. In order to attempt to justify such a hypothesis, one has in fact to develop a truly microscopic theory. A criticism of the "semiphenomenological" philosophy and some of the relevant literature are discussed in the Introduction of I.

One way to derive the relevant sceular matrix for such "doubles" (two-quasiparticle operators) and "quadruples" (four-quasiparticle operators) is to linearize the Heisenberg equations of motion for the relevant operators at the level at which one assumes for them a closed soluble system of equations. The difficulties of this method and, in particular, the non-Hermiticity of the resulting secular matrices, are discussed in I and in Ref. 2. In the following we shall concentrate on a simple variational approach previously discussed in I which seems to provide a reasonable approximation to a complete four-quasiparticle-operator theory, and in which one works with Hermitian secular matrices. This is equivalent to the quasiparticle second TD (Tamm-Dancoff) approximation (and probably to what is denoted by TD4 and PTD4 in Ref. 3).

In the numerical part of the present paper we present results for even isotopes of tin. The results on tin isotopes are of particular interest because of very recent experimental results[6,7] on these nuclides.

We compare our results with (1) the recent experimental data for the positions of the first and second vibrational levels, and (2) the simple quasiparticle Tamm-Dancoff (QTD) calculations of Arvieu *et al.*[8,9] for the Sn even isotopes.

2. THEORY

We assume a shell-model Hamiltonian corresponding to the complete set of single-particle states of energies $\{E_\alpha{}^0\}$. These energies are assumed to be determined in an approximately self-consistent way in the Hartree-Fock sense, so that we can exclude in the following any self-energy terms of the two-body residual interaction potential $U\equiv V(1,2)(1-P_{12})$, where P_{12} exchanges 1 and 2. Such terms appear in the process of linearization by contractions of the equations of motion in the RPA (or SRPA) method.

We introduce the pairing correlations by the Bogoliubov-Valatin canonical transformation defining the set of the usual quasiparticle creation and absorption operators $\alpha_\gamma{}^\dagger$ and α_γ. As a result of the usual approximate diagonalization, we obtain the quasiparticle energies as

$$E_\alpha=[(E_\alpha{}^0-\lambda)^2+\Delta_\alpha{}^2]^{1/2};$$

λ and the energy gap Δ_α are determined in terms of the pairing force by the usual BCS equations.[10]

As we have to deal with states of definite spin, z projection of spin, and parity, we consider the following fundamental operators in our theory [our $\alpha_\gamma{}^\dagger$ and α_γ are exactly as in Refs. 8 and 10; we use the Condon-Shortley phase conventions throughout]:

$$\mathbf{A}_{JM}{}^\dagger(aa')=\sum_{m_\alpha m_{\alpha'}}(j_a j_{a'};m_\alpha m_{\alpha'}|JM)\alpha_{\alpha'}{}^\dagger\alpha_\alpha{}^\dagger,$$
$$\mathbf{A}_{JM}(aa')=(\mathbf{A}_{JM}{}^\dagger(aa'))^\dagger \tag{1}$$

and

$$\bar{A}_{JM}(aa')=\sum_{m_\alpha m_{\alpha'}}(-)^{j_a-m_\alpha}(j_a j_{a'};-m_\alpha m_{\alpha'}|JM)\times\alpha_{\alpha'}{}^\dagger\alpha_\alpha. \tag{2}$$

The commutators between these operators are given in Appendix A. In our notations a latin subscript (e.g. p) stands for all the quantum numbers of the corresponding Greek subscript (e.g., π) except the z projection m_π of j_p. The isotopic spin is not involved, as we confine ourselves to one-nucleonic charge state of quasiparticles. In practice we shall consider nuclei with major shells of protons closed and the entire core of protons inert (only the unfilled neutron subshells are active).

With the above notations we can write our total Hamiltonian as

$$H=H_{11}+H_{\rm int}, \tag{3}$$

where

$$H_{11}=-\textstyle\sum_a E_a\hat{j}_a\bar{A}_{00}(aa), \tag{4}$$

with $\hat{c}^2=2c+1$, and

$$H_{\rm int}=H_{40}+H_{31}+H_{22}+H_{13}+H_{04}, \tag{5}$$

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[9] R. Arvieu, E. Salusti, and M. Véneroni, Phys. Letters **8**, 334 (1964); R. Arvieu and E. Salusti, Nucl. Phys. **66**, 305 (1965).

[10] See, e.g., M. Baranger, Phys. Rev. **120**, 957 (1960).

with

$$H_{40}=H_{04}^{\dagger}=-\tfrac{1}{8}\sum_{J'M'}(-)^{J'+M'}\sum_{abcd}u_au_bv_cv_dg_{J'}(bacd)\times\mathbf{A}_{J'-M'}^{\dagger}(ba)\mathbf{A}_{J'M'}^{\dagger}(cd), \quad (6)$$

$$H_{31}=H_{13}^{\dagger}=-\tfrac{1}{4}\sum_{J'M'}(-)^{J'+M'}\sum_{abcd}\times\{u_au_bu_cv_dg_{J'}(bacd)+u_av_bv_cv_df_{J'}(bacd)\}\times\mathbf{A}_{J'-M'}^{\dagger}(ba)\bar{A}_{J'M'}(cd), \quad (7)$$

$$H_{22}=\tfrac{1}{8}\sum_{J'M'}\sum_{abcd}\{(u_au_bu_cu_d+v_av_bv_cv_d)g_{J'}(badc)+4u_av_bu_cv_df_{J'}(badc)\}\mathbf{A}_{J'M'}^{\dagger}(ba)\mathbf{A}_{J'M'}(dc). \quad (8)$$

Our symbols $g_{J'}$ and $f_{J'}(abcd)$ are defined in terms of, $\langle\alpha\beta|U|\gamma\delta\rangle$ in Eq. (6) of I[11]; $g_{J'}(abcd)=-4G(abcdJ')$ and $f_{J'}(abcd)=-4F(abcdJ')$ in the notation of Baranger.[10] The u_c and v_c are the usual coefficients of the Bogoliubov-Valatin canonical transformation;

$$u_c^2+v_c^2=1.$$

We now define the "quadruples" as the operators:

$$B_{(J'J'')JM}^{\dagger}(aa'bb')\equiv N_{(J'J'')J}(aa'bb')\times\mathbf{B}_{(J'J'')JM}^{\dagger}(aa'bb'), \quad (9)$$

where N is a normalization factor and

$$\mathbf{B}_{(J'J'')JM}^{\dagger}(aa'bb')\equiv(\mathbf{A}_{J'}^{\dagger}(aa')\otimes\mathbf{A}_{J''}^{\dagger}(bb'))_{JM}\equiv\sum_{M'M''}(J'J'';M'M''|JM)\times\mathbf{A}_{J'M'}^{\dagger}(aa')\mathbf{A}_{J''M''}^{\dagger}(bb'), \quad (10)$$

and $\mathbf{B}\equiv(\mathbf{B}^{\dagger})^{\dagger}$; N and the commutators involving $\mathbf{B}^{\dagger}$, $\mathbf{B}$ are given in Appendix A.

The operators $B^{\dagger}$ of Eq. (9) are not, in general, linearly independent and mutually orthogonal, and cannot, therefore, serve directly as part of a good basis. This fact is connected with the vector coupling removing the m degeneracy. The symmetry or quasiparticle exchange properties of $\mathbf{B}^{\dagger}$ of Eq. (10) are given in Eqs. (16)–(17) of I. The original basis of $A^{\dagger}$ and $B^{\dagger}$ is redundant and would produce spurious states in the secular problem. To eliminate this redundancy we have to orthonormalize this basis of $A^{\dagger}$ and $B^{\dagger}$ by the Schmidt procedure, as described in I for the case of one subshell, and to assure the correct dimension of the basis (and of the secular matrix). In the next step, we eliminate all the spuriousness due to the nucleon-number nonconservation (project it out in the secular problem), and the final dimension corresponds to the exact number of physical states determined in the subspace of the states considered.

Our formalism can be developed from a variational principle for the excited states $J^{\pi}M$:

$$\delta E=0=\delta\left\{\frac{\langle\Psi_{JM}^{E}|H|\Psi_{JM}^{E}\rangle}{\langle\Psi_{JM}^{E}|\Psi_{JM}^{E}\rangle}\right\}, \quad (11)$$

where we assume our variational trial function in the form of a superposition of two- and four-quasiparticle excitations.

$$|\Psi_{JM}^{E}\rangle=\hat{O}_{JM}^{E\dagger}|0\rangle, \quad (12)$$

where

$$\hat{O}_{JM}^{E\dagger}\equiv\sum_{p\le p'}a_J^{E}(pp')A_{JM}^{\dagger}(pp')+\sum_{(\alpha)}\sum_{r\le r'\le s\le s'}\times b_{(\alpha)}^{E}(rr'ss')\mathfrak{B}_{(\alpha)JM}^{\dagger}(rr'ss'). \quad (13)$$

Our basis of a complete set of operators orthonormal in the sense of the quasiparticle vacuum is defined here as

$$A_{JM}^{\dagger}(aa')=n_J(aa')\mathbf{A}_{JM}^{\dagger}(aa'),\qquad [n_J(aa')]^{-2}=1+(-)^J\delta_{aa'}, \quad (14)$$

(where with $a'=a$ only $J=$ even are possible, as is seen from the symmetry properties of the $A^{\dagger}$), and

$$\mathfrak{B}_{(\alpha)JM}^{\dagger}(bb'cc')\equiv\sum_{J'J''}c_{(\alpha)J}^{(J'J'')}(bb'cc')\times\mathbf{B}_{(J'J'')JM}^{\dagger}(bb'cc'), \quad (15)$$

where the (real) coefficients $c_{(\alpha)J}^{(J'J'')}$ are determined from our numerical orthonormalization procedure. The ordering ($\le$) of the single-nucleon levels (subshells) avoids repetitions of the same sets (pp') or $(rr'ss')$. In fact, we have now

$$\langle0|A_{J'M'}(tt')A_{JM}^{\dagger}(ss')|0\rangle=\delta_{J'J}\delta_{M'M}\delta_{st}\delta_{s't'}$$

and

$$\langle0|\mathfrak{B}_{(\beta)J'M'}(aa'bb')\mathfrak{B}_{(\alpha)JM}^{\dagger}(cc'dd')|0\rangle=\delta_{\alpha\beta}\delta_{J'J}\delta_{M'M}\delta_{ac}\delta_{a'c'}\delta_{bd}\delta_{b'd'}.$$

From Eq. (11) we obtain the following system of equations for the coefficients a and b:

$$(E_s+E_{s'}-E)a_J(ss')+\sum_{t\le t'}\mathcal{E}_J(ss'tt')a_J(tt')+\sum_{(\alpha)}\sum_{p\le p'\le r\le r'}\mathfrak{F}_{(\alpha)J}(ss',pp'rr')\times b_{(\alpha)J}(pp'rr')=0, \quad (16a)$$

$$(E_p+E_{p'}+E_r+E_{r'}-E)b_{(\alpha)J}(pp'rr')+\sum_{s\le s'}\mathfrak{F}_{(\alpha)J}(ss',pp'rr')a_J(ss')+\sum_{(\beta)}\sum_{t\le t'\le u\le u'}\mathcal{G}_{(\alpha\beta)J}(pp'rr',tt'uu')\times b_{(\beta)J}(tt'uu')=0, \quad (16b)$$

[11] $\langle\alpha\beta|U|\gamma\delta\rangle=\tfrac{1}{2}\sum_{J'M'}g_{J'}(abcd)(j_aj_b;m_am_\beta|J'M')\times(j_cj_d;m_\gamma m_\delta|J'M')=\tfrac{1}{2}\sum_{J''M''}f_{J''}(acdb)(j_aj_c;m_a-m_\gamma|J''M'')\times(j_dj_b;m_\delta-m_\beta|J''M'')s_\gamma s_\beta,$

where $s_\alpha=(-)^{j_\alpha-m_\alpha}$. H_{31} can be written also as

$$H_{31}=-\tfrac{1}{4}\sum_{J'M'}(-)^{J'+M'}\sum g_{J'}(bacd)(u_au_bu_cv_d-v_av_bv_cu_d)\times\mathbf{A}_{J'M'}^{\dagger}(ba)\bar{A}_{J'-M'}(cd).$$

where

$$\mathcal{F}_{(\alpha)J}(ss',pp'rr')\equiv\sum_{(J'J'')}c_{(\alpha)J}{}^{(J'J'')}(pp'rr')\times F_{(J'J'')J}(ss',pp'rr') \quad (17)$$

and

$$\mathcal{G}_{(\alpha\beta)J}(pp'rr',tt'uu')\equiv\sum_{J'J''}\sum_{I'II'}c_{(\beta)J}{}^{(J'J'')}(pp'rr')\times G_{(J'J'')J(I'I'')}(pp'rr',tt'uu')c_{(\alpha)J}{}^{(I'I'')}(tt'uu'). \quad (18)$$

The coefficients $\mathcal{E}$, F, and G contain all the information on the interaction $V(1,2)$ and are given explicitly in Appendix B in terms of f and g of Eq. (6) of I.

If we write the Heisenberg equations of motion for the operators $\{O_s^\dagger\}=\{A_s^\dagger,B_s^\dagger\}$ in their commutator form $EO_s^\dagger=[H,O_s^\dagger]$, and linearize them by applying them to the quasiparticle vacuum $|0\rangle$ and by contractions of the appropriate $\alpha_i\alpha_j^\dagger$ (replaced by δ_{ij}), we obtain equations equivalent to Eqs. (16a)–(16b). The corresponding (Hermitian and real) secular matrix is identical to that determined by Eqs. (16a)–(16b).

For a given nuclide we determine first its ground-state chemical potential λ, the energy gaps Δ_s from all the single-particle energies $\{E_s^0\}$ to be considered. We then assume the parameters (λ,Δ_s,E_s) to be valid for our excited states. Given a realistic $V(1,2)$ and the corresponding g_J we could, in principle, solve the coupled system of equations for λ, Δ_s, and $\{E_s\}$ in a self-consistent way in the sense of the method of Hartree and Bogoliubov. This would be a formidable task in itself and not essential for our problem. Instead, we employ the usual pairing-force results as, e.g., in Ref. 8. This means neglecting the "blocking" effects throughout.

For a given family of states characterized by spin-parity J^π and a given "residual" interaction potential $V(1,2)$, we calculate the coefficients $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ and thus, finally, our (Hermitian) secular matrix. By diagonalizing it, we find the relevant energies E and the corresponding eigenvectors $\{a_J{}^E,b_{(\alpha)J}{}^E\}$. The latter are orthonormalized as

$$\sum_{p\le p'}a_J{}^{E'*}(pp')a_J{}^E(pp')+\sum_{(\alpha)}\sum_{r\le r'\le s\le s'}b_{(\alpha)J}{}^{E'*}(rr'ss')\times b_{(\alpha)J}{}^E(rr'ss')=\delta_{EE'}. \quad (19)$$

One of the difficulties of the present approach is the true particle- (nucleon-) number nonconservation and the related spuriousness effects. This particle-number violation due to the Bogoliubov-Valatin quasiparticle transformation is a difficulty even in the usual simple RPA method. The changes in the effective final occupation factors for our higher excited states, relative to the first excited states of the RPA, should most probably mean significantly greater modifications of the ground state λ and Δ_s than would be the case for the lowest excited (RPA) states. Instead, as in the standard RPA calculations, we preserve here in all our excited states the ground-state values of our parameters λ and Δ_s. The spuriousness due to the particle-number nonconservation should be, in general, more dangerous in our case than what it is for the 0^+ states in the usual RPA. One can first examine this spuriousness by calculating the fluctuations of the expectation values of the nucleon-number operator $\hat{N}$ about the correct number N_0 for each of the obtained eigenvectors $|\psi_{JM}{}^E\rangle$. This fluctuation is given by

$$\delta_J{}^E\equiv\langle\psi_{JM}{}^E|\hat{N}|\Psi_{JM}{}^E\rangle-N_0,\quad N_0=\textstyle\sum_s \hat{\jmath}_s{}^2v_s{}^2, \quad (20)$$

where (with a and b real)

$$\langle\Psi_{JM}{}^E|\hat{N}|\Psi_{JM}{}^E\rangle=N_0+\sum_{s\le s'}(u_s{}^2-v_s{}^2+u_{s'}{}^2-v_{s'}{}^2)|a_J{}^E(ss')|^2$$
$$+\sum_{(\alpha)}\sum_{p\le p'\le r\le r'}(u_p{}^2-v_p{}^2+u_{p'}{}^2-v_{p'}{}^2+u_r{}^2-v_r{}^2+u_{r'}{}^2-v_{r'}{}^2)|b_{(\alpha)J}{}^E(pp'rr')|^2$$
$$-4\sum_{(\alpha)}\sum_{J'J''}\sum_{p\le p'\le r\le r'}[\hat{\jmath}_p u_p v_p a_J{}^E(rr')n_J{}^{-1}(rr')\delta_{pp'}\delta_{J'0}\delta_{J''J}+\hat{\jmath}_r u_r v_r a_J{}^E(pp')n_J{}^{-1}(pp')\delta_{rr'}\delta_{J''0}\delta_{J'J}-(-)^{J'+J''+J}$$
$$\times\hat{J}'\hat{J}''u_r v_r a_J{}^E(pr')n_J(pr')W(j_pJ'j_{r'}J'';\,j_rJ)\delta_{p'r}(1+(-)^{J'}\delta_{pp'})(1+(-)^{J''}\delta_{rr'})(1+(-)^J\delta_{pp'}\delta_{rr'})]$$
$$\times c_{(\alpha)J}{}^{(J'J'')}(pp'rr')b_{(\alpha)J}{}^E(pp'rr'). \quad (20a)$$

The quantity $\delta_J{}^E/N_0$ and the corresponding fluctuations of $\hat{N}^2$ measure the relative contamination of our state (E,J) by the parasite states.

The most appropriate and convenient method to handle this difficulty appears to be the projecting most of the spurious kets out of the secular matrix before its diagonalization. In fact, in our numerical work we have applied this technique and we present below the results obtained with the projection method along with those where the parasite kets have not been eliminated. A similar projection method, but for modes involving products of an odd number of quasiparticle operators, has been applied by Kuo, Baranger, and Baranger[12] to a shell-model calculation of the odd tin isotopes.

The method requires an explicit construction of the most important spurious kets to be eliminated. The first obvious one of such kets is the well-known 0^+ parasite—the only one such in the usual TD (RPA) calculations

[12] T. T. S. Kuo, E. U. Baranger, and M. Baranger, Nucl. Phys. **79**, 513 (1966).

of 0^+ states for even isotopes:

$$|\Psi_{sp2}\rangle\equiv N_{sp2}(\hat{N}-N_0)|0\rangle=\sum_a \zeta_a A_{00}^\dagger(aa)|0\rangle. \quad (21)$$

where $\zeta_a\equiv-\sqrt{2}N_{sp2}\hat{\jmath}_a u_a v_a$; N_{sp2} is found as $N_{sp2}^2=\frac{1}{2}(\sum_a \hat{\jmath}_a^2 u_a^2 v_a^2)^{-1}$. The important four-quasiparticle spurious kets are then immediately identified as

$$|\Psi_{sp4,JM}(cc')\rangle=N_{sp4,J}(cc')\times(\hat{N}-N_{cc'})A_{JM}^\dagger(cc')|0\rangle, \quad (22)$$

where $N_{cc'}\equiv\sum_b \hat{\jmath}_b^2 v_b^2-v_c^2+u_c^2-v_{c'}^2+u_{c'}^2$. The number of such kets is equal to the number of the pairs (cc') with $c\leq c'$. Equation (22) can be expressed in the form $(J\neq0)$[13]

$$|\Psi_{sp4,JM}(cc')\rangle=-N_{sp4,J}(cc')\sum_b \hat{\jmath}_b u_b v_b \mathbf{B}_{(0J)JM}^\dagger\times(bbcc')|0\rangle. \quad (23)$$

These kets can now be expanded in terms of our orthonormal complete basis $\{\mathcal{B}_i^\dagger\}$ of Eq. (15) as

$$|\Psi_{sp4,JM}(cc')\rangle=\sum_i \eta_{i,J}(cc')\mathcal{B}_{i,JM}^\dagger|0\rangle, \quad (24)$$

where i is a shorthand notation for the set of indices of Eq. (15), (α), $rr'ss'$. It is obvious that there are only very few coefficients $\eta_i\neq0$ for each set $(bbcc')$ of Eq. (23).

We wish now to project our total secular matrix M of Eqs. (16a)–(16b) onto the subspace free of all the $|\Psi_{sp}\rangle$, i.e., a subspace of states which are orthogonal to all the $|\Psi_{sp}\rangle$. This is done simply by solving the secular problem of the matrix:

$$\tilde{M}=(1-P)M(1-P), \quad (25)$$

where we have $P_{ij}\equiv\eta_i\eta_j^*$ for each $|\Psi_{sp}\rangle$. (In the case $J^\pi=0^\dagger$, $|\Psi_{sp2}\rangle$ also is projected out with the corresponding $P_{ij}=\zeta_i\zeta_j^*$.) The $|\Psi_{sp}\rangle$'s are automatically eigenvectors of $\tilde{M}$ and correspond to zero eigenvalues.

Spurious kets of the type $(\hat{N}-N_{cc'dd'})\mathcal{B}_{(\alpha)JM}^\dagger(cc'dd')\times|0\rangle$ contribute mainly in higher orders, and are less important in our case.

A study of $E2$-transition probabilities and the respective branching ratios for our collective states is of great importance. Unfortunately, in the case when only neutron subshells are involved, a completely microscopic calculation of the transition probabilities is not possible, and one has to introduce the neutron effective charge. The contribution of the core protons to our collective states is still an open problem.[14,15] Only when the complete A-particle SRPA problem is solved can we perform truly microscopic calculations of the transition probabilities.

For an electric $E\lambda$ transition, the corresponding one-particle operator can be written as

$$\hat{M}(E\lambda)\equiv e_\lambda'\sum_{\alpha\alpha'}\langle\alpha'|r^\lambda Y_{\lambda\mu}(\theta\phi)|\alpha\rangle c_{\alpha'}^\dagger c_\alpha, \quad (26)$$

where the c's are the true particle operators. After the canonical transformation, $\hat{M}$ for $\lambda\neq0$ can be rewritten in the form:

$$-\hat{M}(E\lambda,\mu)=\sum_{aa'}M_{aa'}(\lambda)\{u_{aa'}[\mathbf{A}_{\lambda\mu}^\dagger(a'a)+(-)^{\lambda-\mu}\times\mathbf{A}_{\lambda-\mu}(a'a)]-2(-)^{j_a+j_{a'}-\lambda}v_{aa'}\bar{A}_{\lambda\mu}(aa')\}, \quad (27)$$

where the explicit form of $M_{aa'}(\lambda)$, $u_{aa'}$, and $v_{aa'}$ is given in Eq. (55) of I.

The most important and the simplest transitions are the ones from our collective states $|\Psi_{JM}\rangle$ to the ground state $|0\rangle$. For the $E\lambda$ matrix element we find in this case the expression:

$$-\langle\Psi_{JM}{}^E|\hat{M}(E\lambda,\mu)|0\rangle=\delta_{\lambda J}\delta_{\mu M}\sum_{s\leq s'}u_{ss'}a_J{}^E(ss')n_J(ss')\times[M_{s's}(\lambda)-(-)^{j_s+j_{s'}-J}M_{ss'}(\lambda)]. \quad (28)$$

For the case of a general transition between two states $|\Psi_{JM}{}^E\rangle$ and $|\Psi_{J'M'}{}^{E'}\rangle$, we find

$$\langle\Psi_{JM}{}^E|\hat{M}(E\lambda,\mu)|\Psi_{J'M'}{}^{E'}\rangle=(J'\lambda:M'\mu|JM)\times\langle\Psi_J{}^E\|\hat{M}(E\lambda)\|\Psi_{J'}{}^{E'}\rangle, \quad (29)$$

where the reduced matrix element $\langle\|\hat{M}\|\rangle$ is given in terms of a's and b's in Appendix C.

3. NUMERICAL CALCULATIONS AND DISCUSSION

We have performed numerical computations for excited states of even isotopes of tin from 116 through 124. The states in question are of $J^\pi=2^+$, 4^+, and 0^+.

For the nucleon-nucleon interaction potential we take a simple zero-range force:

$$V(1,2)=-V_0\delta(\mathbf{r}_1-\mathbf{r}_2)(a_0+a_\sigma\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \quad (30)$$

For this force we calculate the reduced antisymmetrized matrix elements g_J and $f_J(abcd)$ defined in Eq. (6) of I. Using the notations of Arvieu,[8] we can show that in the case of our $V(1,2)$ of Eq. (30) and for identical nucleons (our neutrons),

$$g_J(abcd)\equiv-V_0'2[G_J{}^\delta(abcd)-(-)^{l_a+l_b+l_c+l_d+J}G_J{}^\delta(abdc)]\equiv-V_0'2(-)^{l_a+l_c}H_J{}^\delta(abcd),\quad V_0'\equiv V_0(a_0-3a_\sigma), \quad (31)$$

where $G_J{}^\delta(abcd)$ is defined on p. 87 of Ref. 8 apart from a parity factor to be multiplied by and equal to π_{abcd} here $\pi_{abcd}\equiv\frac{1}{2}(1+(-)^{l_a+l_b+l_c+l_d})$.

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[13] In the case $J=0$ there would appear the ground state itself $|0\rangle$ as an extra component of $|\Psi_{sp4,00}(cc')\rangle$; however, this component cannot give any contribution to our excited states since they are automatically orthogonal to $|0\rangle$. For $J=0$ we must project out also the spurious ket $=N_{sp}'(\hat{N}^2-N_0^2)|0\rangle$.

[14] A recent paper by Campi-Benet and Lombard (Ref. 15) indicates that such a contribution of the core protons is not at all negligible in an RPA calculation of the first excited states of some even-even nuclei.

[15] X. Campi-Benet and R. J. Lombard, Phys. Letters **19**, 502 (1965); Nucl. Phys. **83**, 303 (1966).

In the notation of Ref. 8 we find for f_J:

$$f_J(abcd) = -V_0'2[H_J{}^\delta(abcd) - K_J{}^\delta(abcd)]$$
$$= -V_0'2(-)^{l_a+l_b+J+j_c-j_d} K_J{}^\delta(abdc), \quad (32)$$

where $H_J{}^\delta$ is defined on p. 87 of Ref. 8, and $K_J{}^\delta(abcd) = (-)^{l_a+l_c} G_J{}^\delta(abcd)$. After some geometrical transformations f_J can be put in the explicit form:

$$f_J(abcd) = -V_0'(-)^{j_a+j_b+j_c+j_d} \pi_{abcd} F_0(abcd) \hat{j}_a \hat{j}_b \hat{j}_c \hat{j}_d$$
$$\times \hat{J}^{-2}[(-)^{j_b+j_d+1+l_a+l_b-J}(j_a j_b; \tfrac{1}{2}-\tfrac{1}{2}|J0)(j_c j_d; \tfrac{1}{2}-\tfrac{1}{2}|J0)$$
$$+(-)^{l_b+l_d+1}(j_a j_b; \tfrac{1}{2}\tfrac{1}{2}|J1)(j_c j_d; \tfrac{1}{2}\tfrac{1}{2}|J1)], \quad (33)$$

where

$$F_0(abcd) \equiv \frac{1}{4\pi} \int_0^\infty R_a R_b R_c R_d r^2 dr.$$

The functions f_J and g_J are related by

$$g_J(cabd) = -\sum_{J'} \hat{J}'^2 W(j_a j_b j_c j_d; J'J) f_{J'}(abdc). \quad (34)$$

For the case of the tin isotopes, we have had available to us all the interesting radial integrals $F_0(abcd)$ of a zero-range $V(1,2)$ involving single-particle wave functions of a Saxon-Woods shell-model potential. These Saxon-Woods (SW) wave functions and single-particle energies were computed numerically at Harwell for parameters appropriate to Sn^{120}.[16] The SW potential with a spin-orbit part has the depth parameter $V_{SW} = 44.5$ MeV, the usual spin-orbit parameter $\lambda = 35$, the diffuseness parameter $a = 0.67$, and the radius parameter $r_0 = 1.3$ F.

The five single-particle subshells (levels) we consider are $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$, and $1h_{11/2}$, with the respective binding energies 9.85, 9.33, 7.97, 7.52, and 6.90 MeV. The effect of any variation of our final results with any reasonable modification of the above single-particle energies is generally small. The total number of all the integrals $F_0(abcd)$ involved in our problem of the 2^+, 4^+, and 0^+ states is 70.

The zero-range force parameters are determined as follows: Let us introduce the normalization $a_0 + a_\sigma = 1$; then we consider the following exchange force mixtures: (1) Wigner (Serber), $a_0 = 1$, $a_\sigma = 0$; (2) Rosenfeld, $a_0 = 0.9$, $a_\sigma = 0.1$; (3) Ferrell-Visscher, $a_0 = 0.9085$, $a_\sigma = 0.0915$; (4) Soper, $a_0 = 0.865$, $a_\delta = 0.135$. A reasonable effective depth parameter V_0' should be of the order of several hundred MeV. In practice it will be our only adjustable parameter. The value equivalent, in the sense of the potential volume, to the Gaussian potential of Arvieu[8] would be about 950 MeV·F³. In our computations we vary V_0' between 650 and 950 MeV F³. The meaning of V_0 depends, then, on the value of the factor $(a_0 - 3a_\sigma)$, which is in our respective cases: (1) 1, (2) 0.6, (3) 0.634, (4) 0.46 and, finally, (5) for a mixture considered by Vinh Mau,[17] 0.487. The equivalence in the sense of an average F_0 would imply a V_0' even smaller than 700 MeV·F³.

It is interesting to compare our final results with those obtained with the harmonic-oscillator shell-model wave functions for our value of V_0' and our $\{E_\nu{}^0\}$ kept fixed. The spring constant of the harmonic oscillator can be taken as that of Arvieu[8]: $\alpha = (M\omega_0/\hbar)^{1/2} = 0.454$ F^{-1}. Our first excited levels can be compared with those of Arvieu[8] in the sense that his V_0' should be of about 950 MeV F³ and the "unperturbed" single-particle energies slightly modified (the levels $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$, and $1h_{11/2}$ are fixed in Ref. 8 as 0, 0.4, 2.3, 3.1, and 3.4 MeV, respectively).

In fact, in order to determine our $\{E_\nu\}$, u_ν, and v_ν, we have just employed the respective numerical values of Ref. 8 throughout in the main bulk of our numerical work. In order to justify this assumption, which is somewhat inconsistent with our radial integrals in the case corresponding to Ref. 16, we have performed the following test calculation. We assumed only the gap parameters Δ_s to be those of Ref. 8 (these are generally rather insensitive to the choice of the pairing force within any reasonable limits), we took the Saxon-Woods single-particle energies $\{E_\nu{}^0\}$ of Ref. 16 as explicitly given above, and we computed the corresponding $\{E_\nu\}$, u_ν, and v_ν. The phases of u_ν and v_ν determined as in Ref. 8 ($u_{h_{11/2}} > 0$ and $v_{h_{11/2}} < 0$).[18]

In fact, a complete calculation of the 2^+ and 4^+ states for this model for Sn^{116} shows that our results are indeed relatively insensitive to the choice of $\{E_\nu{}^0\}$.

The dimensions of the secular matrices for the 2^+, 4^+, and 0^+ states are 175×175, 204×204, and 55×55, respectively. There is an excellent approximation for several of the lowest lying states in the cases 4^+ and 2^+, reducing considerably the dimensions of the respective matrices. In our results, tabulated or described below, we give for each J^π the energies of several of the lowest lying levels (1) without and (2) with the projecting out of the spurious kets, and for the radial integrals with (a) the Saxon-Woods wave functions[16] and (b) with the harmonic-oscillator wave functions.[8]

For each case (2) we give the percentage weight of the b components of the corresponding eigenvectors (the four-quasiparticle terms). This indicates the extent to which a state can have a two-phonon character.

The parameters $\{E_\nu\}$, u_ν, and v_ν have the numerical values of Ref. 8.

A. The 2^+ States

In our search of the best value of our parameter V_0' of Eqs. (30)–(31), we find $V_0' \approx 700$ MeV F³. Increasing V_0' by 50 MeV F³ lowers the energies for case (2) in our Table I by less than 14%, and reducing V_0' by the same gives about the same effect in the opposite direc-

[16] E. Bradford (private communication); we are greatly indebted to Dr. Bradford for evaluating for us the radial integrals with the Saxon-Woods wave functions.

[17] N. Vinh Mau, thesis, University of Paris, 1963 (unpublished).

[18] This choice of signs is essential as explained in Ref. 8.

TABLE I. Results for the four lowest lying 2^+ states of the quasiparticle second Tamm-Dancoff (QSTD) calculations for even tin isotopes. The (best) value of the zero-range force parameter is $V_0' \approx 700$ MeV $(F)^3$. The radial matrix elements are those with Saxon-Woods functions[a] (case a); the values of $\{E_\nu\}$, u_ν and v_ν are those of Ref. 8. The spurious kets of Eqs. (21)–(24) are (1) not projected out or (2) projected out. The percentage of the b components of the respective eigenvectors is indicated for case 2. The QTD levels (with no b components) are those of Ref. 8 (Table 9b). The observed level energies are quoted from Ref. 7.

$J^\pi=2^+$ A	Level energy (MeV) (case a) (1) No projection	(2) With projection	b components (%) (case 2)	QTD levels of Ref. 8 (MeV)	Observed level energies (MeV)
116	0.40	1.49	19	1.56	1.291
	1.24	2.05	27	2.62	2.108
	1.42	2.28	42	2.90	2.224
	1.56	2.49	23	···	···
118	−0.06	1.29	14	1.36	1.229
	0.78	1.95	24	···	···
	1.03	2.04	40	···	···
	1.40	2.38	78	···	···
120	−0.41	1.18	12	1.27	1.166
	0.42	1.81	23	···	···
	0.77	2.09	40	···	···
	1.29	2.36	73	···	···
122	−0.66	1.13	10	1.23	1.142
	0.15	1.76	20	···	···
	0.59	2.17	60	···	···
	1.24	2.33	56	···	···
124	−0.85	1.10	8	1.21	1.132
	−0.06	1.74	17	···	···
	0.47	2.19	92	···	···
	1.22	2.39	25	···	···

[a] See Ref. 16.

tion (increasing the energies). The dependence of the lowest lying energies on V_0' is almost linear in the critical region of our V_0' (attractive potential). With a quite reasonable value of V_0' (700 MeV F^3) we obtain a satisfactory agreement with the lowest lying 2^+ states. The only second 2^+ state observed to date is that of the $A=116$ isotope at 2.108 MeV[7]; for this we find 2.05 MeV with our model. The general trend of the variation of the energy of the first 2^+ state with A is also consistent with our model.

From the first column of Table I (case 1), we see that the spurious kets of Eqs. (21)–(24) must be absolutely projected out in the sense of Eq. (25) in order for any reasonable results to be obtained at all.

Much less satisfactory results are obtained in the case when harmonic-oscillator radial wave functions are employed (with the parameter α of Ref. 8) in the place of our Saxon-Woods radial wave functions in all the radial integrals. If the same zero-range force is employed with $V_0'=700$ MeV F^3 we find the following first four 2^+ level energies of the $A=116$ isotope: (1) without projection: 0.32, 0.69, 1.37, and 1.44 MeV and (2) with projection: 1.35, 1.69, 2.28, and 2.43 MeV. The spacing of the first two levels is much too small. This situation persists also for the other isotopes ($A \geqslant 118$).

From the third column of our Table I, we see the percentage weight of the four-quasiparticle components in each of the eigenvectors (case 2).

Generally (for all the values of A), the eigenvectors of the first 2^+ states are dominated by the following components: $A^\dagger(11/2,11/2)$, $A^\dagger(\frac{1}{2},\frac{3}{2})$, and $A^\dagger(\frac{3}{2},\frac{3}{2})$ (for $A=116$ the respective a components are: −0.660, −0.493, and 0.304). Similarly, the most important components remain the same for the second and the third 2^+ states; only the b components become increasingly important, in particular $b_{(1)2}(\frac{1}{2},\frac{1}{2},11/2,11/2)$, $b_{(2)2}(\frac{3}{2},\frac{3}{2},11/2,11/2)$, and $b_{(2)2}(\frac{1}{2},\frac{3}{2},11/2,11/2)$ where (1) stands for $(J'J'')=(02)$ and (2) for $(J'J'')=(20)$ in our case.

As is seen from Table I the QTD approximation ($b\equiv 0$) is definitely invalid for the third and fourth 2^+ states.

It is impossible to give here numerical tables of all the eigenvectors obtained; such tables or any part of them are available from the authors upon request.

B. The 4^+ States

The numerical results for the 4^+ case are obtained for the same models as those of the preceding 2^+ case. The remarkable feature of these is that the same value of $V_0'=700$ MeV F^3 gives here again a satisfactory agreement with the data. For the $A=116$ case, the first four 4^+ states have been observed[7] as given in Table II. All the important general features of the 2^+ results described above persist in the 4^+ case.

The first four 4^+ levels of Sn^{116} observed[7] at 2.391, 2.531, 2.803, and 3.047 MeV are rather well reproduced by our levels at 2.15, 2.65, 2.95, and 3.01 MeV calculated with the Saxon-Woods radial wave functions and with the same parameter values as our 2^+ levels. The dependence of our 4^+ results on V_0' is about the same as that of our 2^+ results described above.

Here again our model is consistent with the A variation of the first 4^+ state. The projecting out of the spurious kets is necessary, as in the 2^+ case.

The corresponding results with the harmonic-oscillator wave functions of the same α parameter as in the 2^+ case[8] and with the same $V_0'=700$ MeV·F^3 are in the 4^+ case even better. For the first four 4^+ levels of Sn^{116} we find (1) without projection: 0.92, 1.57, 1.82, and 2.04 MeV and (2) with projection: 2.14, 2.49, 2.92, and 3.18 MeV.

The percentage weight of the b components in our 4^+ eigenvectors is generally high. For the first two 4^+ states it is higher than for the first 2^+ states, which is easy to understand. On the other hand, the first 4^+ level definitely cannot be described as a two-phonon-type state. In fact, it is dominated by the $A^\dagger(11/2,11/2)$ component ($a(11/2,11/2)=0.890$ in Sn^{116}). The most important four-quasiparticle component is $b_{(04)4}(\frac{7}{2},\frac{7}{2},11/2,11/2)$. For Sn^{116} the second, the third, and the fourth 4^+ levels are dominated by our components: $a(\frac{1}{2},\frac{7}{2})$ and $b_{(40)4}(\frac{1}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2})$; $a(\frac{3}{2},\frac{7}{2})$ and $b_{(40)4}(\frac{3}{2},\frac{7}{2},11/2,11/2)$;

TABLE II. Results for the four lowest lying 4^+ states of the QSTD calculations for even tin isotopes. The (best) value of the zero-range force parameter is the same as in Table I: $V_0' \approx 700$ MeV F^3. The model, all the parameters employed and the meaning of all the results presented are as in Table I. The QTD levels (no b components) are those of Ref. 8 (Table 9a). The observed level energies are quoted from Ref. 7.

$J^\pi=4^+$ A	Level energy (MeV) (case a) (1) No projection	(2) With projection	b components (%) (case 2)	QTD levels Ref. 8 (MeV)	Observed level energies (MeV)
116	1.07	2.15	20	2.44	2.391
	1.74	2.61	23	3.16	2.531
	1.90	2.95	58	3.49	2.803
	2.19	3.01	65	3.75	3.047
118	0.59	1.87	17	2.30	2.278
	1.74	2.81	96	...	...
	2.01	3.12	20	...	...
	2.03	3.37	16	...	...
120	0.27	1.78	15	2.27	2.183
	1.78	2.76	98	...	...
	1.92	3.39	96	...	...
	1.96	3.41	21	...	...
122	0.02	1.74	12	2.25	...
	1.83	2.71	99	...	...
	1.87	3.33	98	...	...
	1.93	3.65	64	...	...
124	−0.16	1.75	10	2.22	...
	1.79	2.65	99	...	...
	1.84	3.41	99	...	...
	1.94	3.70	88	...	...

and $a(\frac{7}{2},\frac{7}{2})$, $b_{(04)4}(\frac{1}{2},\frac{1}{2},11/2,11/2)$, and $b_{(40)4}(\frac{7}{2},\frac{7}{2},11/2,11/2)$, respectively.

We can conclude that, although the simple QTD (and QRPA) approximation ($b\equiv 0$) are definitely too crude as the four-quasiparticle components are quite appreciable, a naive two-phonon description of the 4^+ states cannot be justified either.

C. The 0^+ States

Unfortunately, no satisfactory results could have been obtained with our model for the 0^+ states. In order to fit the first 0^+ level of Sn^{116} observed[7] at 1.762 MeV, we have to reduce our V_0' to the value ≈ 350 MeV F^3, which appears unreasonable (the Saxon-Woods wave functions employed and all the other parameters are the same as before, spurious kets projected out).

The difficulty with the 0^+ states may be due to one or to several reasons. The effective reduced matrix elements are generally large in this case and capable of giving relatively strong couplings between various different configurations. It is possible that in this connection the role played by the core nucleons (cf. Ref. 15) left out in our treatment should be particularly important in the 0^+ case. It is also possible that six- and more-quasiparticle excitations are particularly important in this case (with such modes the higher order spurious kets, mentioned in the text but not projected out by us, are to be eliminated). Finally, it is possible that the ground-state correlations in the sense of the QSRPA (the "de-excitation" or the "backward-going" graphs) are particularly strong in the 0^+ case [see, however, Note added in proof]. A more refined theory including these points should be applied here, it seems.

D. Final Remarks

Our microscopic model seems to give a satisfactory description of the low-lying 2^+ and 4^+ states and a poor description of the 0^+ states. It is evident that while the simple QTD (QRPA) description of the second and higher 2^+ and 4^+ states[8] cannot be justified, a naive two-phonon description of such states is also hardly possible. The two- and four-quasiparticle excitations have to be treated on the same footing. On the other hand, the question of the numerical stability of our results against the inclusion of six- and more-quasiparticle terms (the convergence of the series of the successive higher QTD approximations) remains unsettled.

The problem of a possibly important role played by the left-out core nucleons discussed in Ref. 15 at the QTD (QRPA) level remains open. Unfortunately, the relevant computational difficulties are prohibitive at the QSTD level.

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Note added in proof. (1) We do not have to project out the spurious states of the center-of-mass motion in our case. In fact, the only single particle (s.p.) state of negative parity is $1h_{11/2}$, and the maximum j of the states of positive parity $=\frac{7}{2}$; consequently, we cannot have any pairs coupled to $J'^\pi=1^-$ present in our $\mathcal{B}_{(a)JM}{}^\dagger$.

(2) We have studied the effects of the ground state (g.s.) correlations by mixing in the BCS (or the q.p. vacuum) $|0\rangle$ itself with our $\mathcal{B}^\dagger|0\rangle$ in the $J^\pi=0^+$ case. For Sn it means working with matrices of 56×56 instead of those of 55×55. The said coupling follows from H_{04} of Eq. (6). Similarly as in Ref. 3 for Ni, we find only a very small shift (lowering) of our Sn g.s. energies due to these g.s. correlations.

All the (seven) described spurious kets are projected out before diagonalization. All the s.p. parameters are as in Sec. 3A,B. For $V_0'=350$ MeV·F^3, $A=116$ and the Woods-Saxon s.p. wave functions we find the lowering of the g.s. energy to be only $=-0.07$ MeV and the first four excited 0^+ levels at 1.71, 2.44, 3.52 and 3.88 MeV; the first excited 0^+ observed[7] lies at 1.762 MeV.

We believe that the discrepancy between our V_0' here and that of Sec. 3A and 3B is due mainly to the particular inadequacy of our zero range force for the 0^+ states rather than to the other reasons discussed in Sec. 3C (cf. also, e.g., the "extra" monopole potential term in Ref. 3). Our results for finite range and realistic forces including also the $B(E2)$- and Q_{2+}-values will be published soon.

One of us (J. S.) is indebted to R. Arvieu, V. Gillet, B. Giraud, M. Jean, G. Ripka, and M. Véneroni for a helpful discussion of these questions.

APPENDIX A

We present here some commutators between the operators defined by Eqs. (1), (2), and (10). We use everywhere the shorthand notation.

$$\delta_J(ab,cd)\equiv\delta_{ac}\delta_{bd}-(-)^{J+j_c+j_d}\delta_{ad}\delta_{bc}.$$

These commutators are necessary to derive the equations of the main text.

$$[\mathbf{A}_{JM}(aa'),\mathbf{A}_{J'M'}{}^\dagger(bb')]=\delta_{JJ'}\delta_{MM'}\delta_J(aa',bb')+\hat{J}\sum_{J''}\hat{J}''(JJ'';M,M'-M|J'M')$$
$$\times\{[\delta_{a'b}W(Jj_aJ'j_{b'};j_{a'}J'')\bar{A}_{J'',M'-M}(ab')]-(-)^{J+j_a+j_{a'}}[a\leftrightarrow a']$$
$$-(-)^{J'+j_b+j_{b'}}[b\leftrightarrow b']+(-)^{J+j_a+j_{a'}+J'+j_b+j_{b'}}[a\leftrightarrow a';b\leftrightarrow b']\},\quad\text{(A1)}$$

$$[\mathbf{A}_{JM}(aa'),\bar{A}_{J'M'}(bb')]=-\hat{J}'\sum_{J''}\hat{J}''(J'J'';M',M-M'|JM)\{[\delta_{ab'}W(Jj_{a'}J'j_b;j_{b'}J'')\mathbf{A}_{J'',M-M'}(ba')]$$
$$-(-)^{J+j_a+j_{a'}}[a\leftrightarrow a']\},\quad\text{(A2)}$$

$$[\mathbf{A}_{JM}(aa'),\mathbf{B}_{(J_1J_2)J'M'}{}^\dagger(bb'cc')]=(J_1J_2;M,M'-M|J'M')\delta_{JJ_1}\delta_J(aa',bb')\mathbf{A}_{J_2,M'-M}{}^\dagger(cc')$$
$$+(J_1J_2;M'-M,M|J'M')\delta_{JJ_2}\delta_J(aa',cc')\mathbf{A}_{J_1,M'-M}{}^\dagger(bb')-\sum_{J''}(J''J;M'-M,M|J'M')$$
$$\times\{[\delta_J(aa',b'c')\langle bc(J'')b'c'(J)J'|bb'(J_1)cc'(J_2)J'\rangle\mathbf{A}_{J'',M'-M}{}^\dagger(bc)]$$
$$-(-)^{J_1+j_b+j_{b'}}[b\leftrightarrow b']-(-)^{J_2+j_c+j_{c'}}[c\leftrightarrow c']+(-)^{J_1+j_b+j_{b'}+J_2+j_c+j_{c'}}[b\leftrightarrow b';c\leftrightarrow c']\}$$
$$+\text{terms containing products of the type }\mathbf{A}^\dagger\bar{A}.\quad\text{(A3)}$$

Here the symbol $\langle\ |\ \rangle$ is defined in terms of a $9-j$ symbol as, e.g., in I.

The terms omitted in the last expression are generally unimportant, and are not involved in the equations of the main text.

$$[\bar{A}_{JM}(aa'),\mathbf{B}_{(J_1J_2)J'M'}{}^\dagger(bb'cc')]=-\hat{J}\hat{J}'\sum_{J''J'''}\hat{J}''(JJ';MM'|J'''';M+M')$$
$$\times\{[\hat{J}_1W(J_2J'''J_1J;J''J')W(J_1j_bJj_{a'};j_{b'}J'')\delta_{ab'}\mathbf{B}_{(J''J_2)J''',M+M'}{}^\dagger(ba'cc')]-(-)^{J_1+j_b+j_{b'}}[b\leftrightarrow b']$$
$$+(-)^{J_1+J_2+J'}[J_1\leftrightarrow J_2;b\leftrightarrow c,b'\leftrightarrow c']+(-)^{J'+j_b+j_{b'}+j_c+j_{c'}}[J_1\leftrightarrow J_2;b\leftrightarrow c',b'\leftrightarrow c]\}.\quad\text{(A4)}$$

We give here only the expression for the main value of the commutator $[\mathbf{B},\mathbf{B}^\dagger]$ in the BCS state because the complete expression is rather complicated and unimportant for our purpose:

$$P_{(J'J'')J(I'I'')}(aa'bb',cc'dd')\equiv\langle0|[\mathbf{B}_{(J'J'')JM}(aa'bb'),\mathbf{B}_{(I'I'')JM}{}^\dagger(cc'dd')]|0\rangle=\delta_{J'I'}\delta_{J''I''}\delta_{J'}(aa',cc')\delta_{J''}(bb',dd')$$
$$+(-)^{J+J'+J''}\delta_{J'I''}\delta_{J''I'}\delta_{J'}(aa',dd')\delta_{J''}(bb',cc')-\{[\delta_{J'}(aa',cd)\delta_{J''}(bb',c'd')\langle cd(J')c'd'(J'')J|cc'(I')dd'(I'')J\rangle]$$
$$-(-)^{I'+j_c+j_{c'}}[c\leftrightarrow c']-(-)^{I''+j_d+j_{d'}}[d\leftrightarrow d']+(-)^{I'+j_c+j_{c'}+I''+j_d+j_{d'}}[c\leftrightarrow c';d\leftrightarrow d']\}.\quad\text{(A5)}$$

In particular, we have for the normalization factor of the $\mathbf{B}^\dagger$ operators:

$$[N_{(J'J'')J}(aa'bb')]^{-2}\equiv P_{(J'J'')J(J'J'')}(aa'bb';aa'bb').\quad\text{(A6)}$$

APPENDIX B

We give below the explicit form of E, F, and G (in terms of g and f) which determine our secular problem.

$$E_J(ss'tt')=\tfrac{1}{8}\{[(u_su_{s'}u_tu_{t'}+v_sv_{s'}v_tv_{t'})g_J(ss'tt')+4v_su_{s'}v_tu_{t'}f_J(ss'tt')]$$
$$-(-)^{j_s+j_{s'}-J}[s\leftrightarrow s']-(-)^{j_t+j_{t'}-J}[t\leftrightarrow t']+(-)^{j_s+j_{s'}+j_t+j_{t'}}[s\leftrightarrow s';t\leftrightarrow t']\},\quad\text{(B1)}$$

and for $\mathcal{E}$ of Eq. (16a) we have: $\mathcal{E}_J(ss'tt')\equiv n_J(ss')n_J(tt')E_J(ss'tt')$;

$$F_{(J'J'')J}(ss';\,pp'rr')=\{\hat{F}_{(J'J'')J}(rs',pp'rr')\delta_{rs}-(-)^{J''+j_r+j_{r'}}\hat{F}_{(J'J'')J}(r's',pp'rr')\delta_{r's}+(-)^{J+J'+J''}$$
$$\times[\hat{F}_{(J''J')J}(ps',rr'pp')\delta_{ps}-(-)^{J'+j_p+j_{p'}}\hat{F}_{(J''J')J}(p's',rr'p'p)\delta_{p's}]\}-(-)^{J+j_s+j_{s'}}\{s\leftrightarrow s'\}\,, \quad \text{(B2)}$$

where

$$\hat{F}_{(J'J'')J}(rs,pp'rr')=-\tfrac{1}{2}n_J(rs)\hat{J}'\hat{J}''W(J'j_sJ''j_r;\,j_{r'}J)\{f_{J'}(p'pr's)(-)^{J'+j_p+j_{p'}}[-u_su_pv_{p'}u_{r'}+v_sv_pu_{p'}v_{r'}]$$
$$+f_{J'}(pp'r's)[u_sv_pu_{p'}u_{r'}-v_su_pv_{p'}v_{r'}]+g_{J'}(pp'r's)[v_sv_pv_{p'}u_{r'}-u_su_pu_{p'}v_{r'}]\}\,; \quad \text{(B3)}$$

$$G_{(J'J'')J(I'I'')}(pp'rr';\,tt'uu')=\delta_{J'I'}\delta_{J''I''}\delta_{J''}(rr'uu')E_{J'}(pp'tt')+(-)^{J+J'+J''}\delta_{J'I''}\delta_{J''I'}\delta_{J''}(rr',tt')E_{J'}(pp'uu')$$
$$+\delta_{J'I'}\delta_{J''I''}\delta_{J'}(pp',tt')E_{J''}(rr'uu')+(-)^{J+J'+J''}\delta_{J'I''}\delta_{J''I'}\delta_{J'}(pp',uu')E_{J''}(rr'tt')-(-)^{J+J'+J''}$$
$$\times\{[E_{J'}(pp't'u')\langle tu(J'')t'u'(J')J\,|\,tt'(I')uu'(I'')J\rangle\delta_{J''}(rr',tu)]-(-)^{I''+j_u+j_{u'}}[u\leftrightarrow u']-(-)^{I'+j_t+j_{t'}}[t\leftrightarrow t']$$
$$+(-)^{I''+j_u+j_{u'}+I'+j_t+j_{t'}}[u\leftrightarrow u';\,t\leftrightarrow t']\}-\{J'\leftrightarrow J'';\,r\leftrightarrow p,\,r'\leftrightarrow p'\}-(-)^{J+I'+I''}$$
$$\times\{[E_{I'}(p'r'tt')\langle pr(I'')p'r'(I')J\,|\,pp'(J')rr'(J'')J\rangle\delta_{J''}(pr,uu')]-(-)^{J''+j_r+j_{r'}}[r\leftrightarrow r']-(-)^{J'+j_p+j_{p'}}[p\leftrightarrow p']$$
$$+(-)^{J''+j_r+j_{r'}+J'+j_p+j_{p'}}[r\leftrightarrow r',\,p\leftrightarrow p']\}-\{I'\leftrightarrow I'';\,t\leftrightarrow u,\,t'\leftrightarrow u'\}+\sum_{\bar{J}}\{[\Omega_{(J'J'')(I'I'')\bar{J}J}(pp'rr',tt'uu')$$
$$\times E_{\bar{J}}(p'r't'u')-(-)^{J''+j_r+j_{r'}}\Omega_{(J'J'')(I'I'')\bar{J}J}(pp'r'r,tt'uu')E_{\bar{J}}(p'rt'u')-(-)^{J'+j_p+j_{p'}}\Omega_{(J'J'')(I'I'')\bar{J}J}(p'prr',tt'uu')$$
$$\times E_{\bar{J}}(pr't'u')+(-)^{J''+j_r+j_{r'}+J'+j_p+j_{p'}}\Omega_{(J'J'')(I'I'')\bar{J}J}(p'pr'r;\,tt'uu')E_{\bar{J}}(prt'u')]-(-)^{I''+j_u+j_{u'}}[u\leftrightarrow u']$$
$$-(-)^{I'+j_t+j_{t'}}[t\leftrightarrow t']+(-)^{I''+j_u+j_{u'}+I'+j_t+j_{t'}}[u\leftrightarrow u',\,t\leftrightarrow t']\}\,, \quad \text{(B4)}$$

where:

$$\Omega_{(J'J'')(I'I'')\bar{J}J}(pp'rr';\,tt'uu')=\sum_{\bar{J}'}\langle pr(\bar{J}')p'r'(\bar{J})J\,|\,pp'(J')rr'(J'')J\rangle\delta_{\bar{J}'}(pr,tu)\langle tu(\bar{J}')t'u'(\bar{J})J\,|\,tt'(I')uu'(I'')J\rangle. \quad \text{(B5)}$$

APPENDIX C

The reduced matrix element of the $E\lambda$ transition operator $\hat{M}(E\lambda)$ for $\lambda\neq 0$ of Eq. (29) can be put in the form:

$$\langle\Psi_J{}^E\|\hat{M}(E\lambda)\|\Psi_{J'}{}^{E'}\rangle=2\hat{\lambda}\hat{J}'\sum_{a\leq a'}\sum_{b\leq b'}a_J{}^{E*}(aa')a_{J'}{}^{E'}(bb')n_J(aa')n_{J'}(bb')\{[\delta_{a'b'}W(J'j_{b'}\lambda j_a;\,j_bJ)v_{ba}M_{ba}(\lambda)]$$
$$-(-)^{J+j_a+j_{a'}}[a\leftrightarrow a']-(-)^{J'+j_b+j_{b'}}[b\leftrightarrow b']+(-)^{J+j_a+j_{a'}+J'+j_b+j_{b'}}[a\leftrightarrow a';\,b\leftrightarrow b']\}$$
$$-\hat{J}'(\hat{J})^{-1}\sum_{\alpha}\sum_{J_1,J_2}\sum_{a\leq a'}\sum_{b\leq b'\leq c\leq c'}a_J{}^{E*}(aa')n_J(aa')b_{(\alpha)J'}{}^{E'}(bb'cc')c_{(\alpha)J'}{}^{(J_1J_2)}(bb'cc')\{\delta_{\lambda J_1}\delta_{JJ_2}\delta_J(aa',cc')u_{bb'}\hat{M}_{bb'}(\lambda)$$
$$+(-)^{J+J'+\lambda}\delta_{\lambda J_2}\delta_{JJ_1}\delta_J(aa',bb')u_{cc'}\hat{M}_{cc'}(\lambda)-[(-)^{J+J'+\lambda}\delta_J(aa',bc)\langle bc(J)b'c'(\lambda)J'\,|\,bb'(J_1)cc'(J_2)J'\rangle u_{b'c'}\hat{M}_{b'c'}(\lambda)]$$
$$+(-)^{J_1+j_b+j_{b'}}[b\leftrightarrow b']+(-)^{J_2+j_c+j_{c'}}[c\leftrightarrow c']-(-)^{J_1+j_b+j_{b'}+J_2+j_c+j_{c'}}[b\leftrightarrow b';\,c\leftrightarrow c']\}$$
$$-(-)^{J+J'+\lambda}\sum_{\beta}\sum_{J_3,J_4}\sum_{a\leq a'}\sum_{b\leq b'\leq c\leq c'}a_{J'}{}^{E'}(aa')n_{J'}(aa')b_{(\beta)J}{}^{E*}(bb'cc')c_{(\beta)J}{}^{(J_3J_4)}(bb'cc')\{\delta_{\lambda J_3}\delta_{J'J_4}\delta_{J'}(aa',cc')u_{bb'}\hat{M}_{bb'}(\lambda)$$
$$+(-)^{J+J'+\lambda}\delta_{\lambda J_4}\delta_{J'J_3}\delta_{J'}(aa',bb')u_{cc'}\hat{M}_{cc'}(\lambda)-[(-)^{J+J'+\lambda}\delta_{J'}(aa',bc)\langle bc(J')b'c'(\lambda)J\,|\,bb'(J_3)cc'(J_4)J\rangle u_{b'c'}\hat{M}_{b'c'}(\lambda)]$$
$$+(-)^{J_3+j_b+j_{b'}}[b\leftrightarrow b']+(-)^{J_4+j_c+j_{c'}}[c\leftrightarrow c']-(-)^{J_3+j_b+j_{b'}+J_4+j_c+j_{c'}}[b\leftrightarrow b';\,c\leftrightarrow c']\}$$
$$-\sum_{\alpha\beta}\sum_{J_1J_2J_3J_4}\sum_{a\leq a'\leq b\leq b'}\sum_{c\leq c'\leq d\leq d'}b_{(\alpha)J}{}^{E}(aa'bb')b_{(\beta)J'}{}^{E'}(cc'dd')c_{(\alpha)J}{}^{(J_3J_4)}(aa'bb')c_{(\beta)J'}{}^{(J_1J_2)}(cc'dd')2\hat{\lambda}_{J'}(-)^{J+J'+\lambda}$$
$$\times\Big(\{[\hat{J}_1\hat{J}_3W(J_2JJ_1\lambda;\,J_3J')\delta_{J_2J_4}\delta_{J_4}(bb',dd')(W(J'j_c\lambda j_{a'};\,j_{c'}J_3)(-)^{j_{c'}+j_{a'}-\lambda}v_{c'a'}M_{c'a'}(\lambda)\delta_{ac}-(-)^{J_3+j_a+j_{a'}}$$
$$\times W(J'j_c\lambda j_a;\,j_{c'}J_3)(-)^{j_{c'}+j_a-\lambda}v_{c'a}M_{c'a}(\lambda)\delta_{a'c})]+(-)^{J_3+J_4+J}[J_3\leftrightarrow J_4;\,a\leftrightarrow b,\,a'\leftrightarrow b']\}-(-)^{J_3+j_c+j_{c'}}\{c\leftrightarrow c'\}$$
$$+(-)^{J_1+J_2+J'}\{J_1\leftrightarrow J_2;\,c\leftrightarrow d,\,c'\leftrightarrow d'\}+(-)^{+J'+j_c+j_{c'}+j_d+j_{d'}}\{J_1\leftrightarrow J_2;\,c\leftrightarrow d',\,c'\leftrightarrow d\}\Big)$$
$$+(\text{terms involving products of 2 nine-}j\text{ symbols which are generally much smaller}). \quad \text{(C1)}$$

Here we have used the notation:

$$\hat{M}_{ba}(\lambda)=M_{ab}(\lambda)-(-)^{\lambda+j_a+j_b}M_{ba}(\lambda)\,. \quad \text{(C2)}$$

$M_{ab}(\lambda)$ and u_{ab}, v_{ab} are defined explicitly in Eq. (55) of I.