Electromagnetic Form Factors of H^3 and He^3 with Realistic Potentials

V. K. GUPTA
Centre for Advanced Study in Physics, University of Delhi, Delhi, India

AND

B. S. BHAKAR AND A. N. MITRA
Department of Physics and Astrophysics, University of Delhi, Delhi, India

(Received 22 August 1966)

The charge and magnetic form factors of H^3 and He^3 have been calculated on the lines of Schiff's analysis for the problem. The three-body wave functions used for this purpose are the ones which had earlier been derived in an exact fashion by the authors, using separable potentials involving central as well as tensor forces. These wave functions are all characterized by a small S'-state probability ($\sim 1\%$). The calculations of the form factors and their corresponding radii have been carried out (a) for pure s-wave forces, and (b) for tensor forces, using the potential parameters of both Yamaguchi and Naqvi. It has been found that, whereas the agreement with experiment for pure s-wave forces is poor, the inclusion of tensor forces improves the results considerably, so that they fall short of experimental values by not more than about 10%, which is fully within the scope of hard-core effects. To account for the appreciable difference (~ 0.17 F) between the charge radii of He^3 and H^3, we require a positive value for the slope of neutron charge distribution, which is in agreement with the recent analysis from inelastic electron-deuteron scattering. A reasonable value for this slope, deduced from deuteron-scattering data, however, accounts for only about 0.1 F of this difference in the two radii. The remaining difference of about 0.07 F could probably be ascribed to hard-core effects, electromagnetic violations of charge independence, and effects of exchange moments.

1. INTRODUCTION

THE experiments on elastic electron scattering from H^3 and He^3 by Hofstadter and collaborators[1] opened up a new possibility for probing into the charge structure of the neutron,[2] the estimation of which had hitherto been confined only to deuteron-scattering experiments.[3] While theoretically the deuteron is a simpler structure, scattering from the triton and He^3 provides an independent determination of the neutron form factor, which could be checked against the corresponding deuteron-scattering data.

For such a program to be successful, the first condition is an accurate knowledge of the ground-state wave function of H^3 and He^3. Alternatively, such experiments may themselves throw valuable light on the structure of these nuclei if the neutron form factor is otherwise assumed known. Indeed, such a point of view was advocated by Schiff[4] in a comprehensive analysis of the electromagnetic form factors of H^3 and He^3. This analysis, which is characterized by fairly general formulas for the form factors in terms of certain "body form factors" F_L and F_O, associated with the "like" nucleon and "odd" nucleon, respectively, showed how the percentage of the S state of [2,1] symmetry in the ground-state wave function (called S'), could be estimated from a difference between the observed charge form factors. While the percentage of this S' state in Schiff's earlier analysis was somewhat higher ($\sim 4\%$) than is compatible with data on the Gamow-Teller matrix elements for H^3 decay,[5] with the rate for thermal-neutron capture in deuterium,[6] or with the inelastic scattering of electrons from H^3,[7] it is probably quite sensitive to the assumed (variational) shape of the three-body wave function, and also to the details of the neutron charge form factor. In addition, the effects of Coulomb repulsion in He^3,[8] the possibility of small admixtures of the isobaric $T=\frac{3}{2}$ state,[9] and the uncertainties on the exchange-moment contributions[10] could further obscure the determination of the S' state. Indeed, with so many effects on hand, an "experimental determination" of the ground-state wave function from electron-scattering data, may well have lost its earlier appeal.[11]

We would like to present here an alternative approach to the form-factor problem based on an accurate theoretical determination of the triton wave function by solving the three-body Schrödinger equation in terms of two-body potentials, instead of assuming a variational form for this quantity. As is now well-known, such an approach is possible with the help of separable potentials which allow an exact determination of the

[1] H. Collard, R. Hofstadter, A. Johansson, R. Parks, M. Ryneveld, A. Walker, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. Letters **11**, 132 (1963).
[2] L. I. Schiff, H. Collard, R. Hofstadter, A. Johansson, and M. R. Yearian, Phys. Rev. Letters **11**, 387 (1963).
[3] R. Hofstadter, C. de Vries, and R. Herman, Phys. Rev. Letters **6**, 290 (1961); R. Hofstadter and R. Herman, *ibid.* **6**, 293 (1961).
[4] L. I. Schiff, Phys. Rev. **133**, B802 (1964).
[5] R. J. Blin-Stoyle, Phys. Rev. Letters **13**, 55 (1964).
[6] T. K. Radha and N. T. Meister, Phys. Rev. **136**, B388 (1964); N. T. Meister, T. K. Radha, and L. I. Schiff, Phys. Rev. Letters **12**, 509 (1964).
[7] T. A. Griffy and R. J. Oakes, Phys. Rev. **135**, B1161 (1964).
[8] R. H. Dalitz and T. W. Thacker, Phys. Rev. Letters **15**, 204 (1965).
[9] T. A. Griffy, Phys. Letters **11**, 155 (1964).
[10] D. A. Kreuger and A. Goldberg, Phys. Rev. **135**, B934 (1964); A. Q. Sarker, Phys. Rev. Letters **13**, 375 (1964); Nuovo Cimento **36**, 392 (1965); **36**, 410 (1965).
[11] See, e.g., H. Collard, R. Hofstadter, E. B. Hughes, A Johansson, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. **138**, B57 (1965).

three-body wave function.[12] The only limitation lies in the choice of the potentials. For example, if the N-N potential is approximated by merely the two effective S-wave terms of different strengths (for the singlet and triplet forces, respectively), it gives a rather poor approximation to the wave function. On the other hand, the inclusion of the tensor force in the $T=0$ state significantly improves the wave function, as judged by the results on the triton binding energy, as well as the percentage probabilities of various states.[13] For further improvement one also needs the hard-core effects, symbolized by the change in sign of the $^1S_0$ phase shift around 200 MeV. Unfortunately, the combined effect of the tensor force as well as the hard core on the triton wave function is not as yet available to us because of rather formidable computational difficulties associated with the appearance of four coupled integral equations (which must be solved consistently with the requirement of reasonably small mesh sizes which are essential for computational accuracy). The best we have at this stage is a wave function which takes account of a central plus a tensor force of the Yamaguchi form in the triplet state and a central S-wave force in the singlet state.[13,14] Such a combination yields an S' state of the order of 0.8–1.0%, which seems to be in general agreement with the data on inelastic-electron scattering on H^3 and He^3,[7] as well as thermal neutron capture on deuterium.[6] The D-state probability works out at 3–5%, depending upon the potential parameters chosen, the lower value corresponding to Naqvi's determination. The P-state probabilities are almost completely negligible. These results on P- and D-state probabilities seem to be in general agreement with the analysis of Gibson and Schiff.[15]

These figures on the percentage probabilities which have the advantage of dynamical determination from fairly realistic two-body potentials (without the usual uncertainties accompanying variational treatments), also appear to be quite reasonable from a comparison of contemporary analysis of three-body data.[6–8] If, therefore, these figures are accepted as such, they give a complete determination of the two-body form factors F_0 and F_L. This determination in turn can be incorporated in the general analysis of Ref. 4 to estimate how H^3 and He^3 form factors depend upon other (unknown) factors. For example, the results for F_{H^3} and F_{He^3} could be quite sensitive to the neutron charge form factor (F_n^{ch}) for which the experimental data are still poor.[11] Thus the calculation of F_{H^3} and F_{He^3} with "exact" three-body wave functions could provide a useful probe into F_n^{ch}, or at least serve to bring out the sensitivity to this quantity. This is mainly the point of view that is adopted in this paper for the calculation of F_{H^3} and F_{He^3}.

In Sec. 2, we collect for convenience the basic formulas of Ref. 14 in terms of which the three-body wave functions are defined, both for effective S-wave potentials as well as for the tensor forces. The probabilities P_L for various L states are defined in Sec. 3 and explicit formulas given for their numerical evaluation. In Sec. 4, the charge and magnetic form factors of H^3 and He^3 are expressed in terms of body form factors, on the lines of Schiff's analysis.[4] These body form factors are in turn expressed in terms of the three-body wave function, defined earlier in Sec. 2. Explicit formulas for the S-wave and tensor-force cases are given separately in Sec. 5. A suitable parametrization of the spectator functions which enables the various form-factor integrals to be evaluated by the Feynman method, is described in Sec. 6, together with the results of numerical evaluation of body form factors for several sets of potential parameters considered. The broad procedure used for the evaluation of the integrals is described in the Appendix. Finally, Sec. 7 gives a discussion of the results, with particular reference to the sizes of H^3 and He^3 and the role of the neutron charge form factor in the analysis. A brief comparison with the results of contemporary investigations is also included.

The main conclusions are that while the tensor force appreciably increases the size of the triton, over the results of pure S-wave calculations, it still falls short (by $\lesssim 10\%$) of the experimental determination for this quantity, a gap which could probably be bridged by hard-core effects. The difference between the charge radii of H^3 and He^3 depends rather sensitively on the slope assumed for F_n^{ch}, a positive slope being clearly favored, in conformity with its determination from deuteron-scattering results.

2. STRUCTURE OF THE THREE-BODY WAVE FUNCTION

We collect here the essential features of the three-body wave function obtained with tensor forces given some time ago by one of us.[14] The properly antisymmetrized wave function Ψ is expressed as

$$\Psi=(1/\sqrt{2})(A'\zeta''-A''\zeta'). \tag{2.1}$$

Here (ζ',ζ'') are the two isospin functions which for H^3 are

$$\begin{aligned}\zeta'&=(1/\sqrt{2})u_1(u_2v_3-u_3v_2),\\ \zeta''&=-(1/\sqrt{3})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3)\zeta',\end{aligned} \tag{2.2}$$

and u, v are the states of $\tau_z=\pm\frac{1}{2}$, respectively. For He^3 the corresponding ζ', ζ'' have u and v interchanged. The quantities (A',A'') are the corresponding space-spin functions. We use a separable potential of the type

$$-M\langle\mathbf{p}|V|\mathbf{p}'\rangle=\lambda_{31}g(\mathbf{p})g(\mathbf{p}')P_\sigma{}^+P_\tau{}^- +\lambda_{13}f(p)f(p')P_\sigma{}^-P_\tau{}^+, \tag{2.3}$$

[12] A. N. Mitra, Nucl. Phys. 32, 529 (1962); C. Lovelace, Phys. Rev. 135, B1225 (1964).
[13] B. S. Bhakar and A. N. Mitra, Phys. Rev. Letters 14, 143 (1965).
[14] B. S. Bhakar, Nucl. Phys. 46, 572 (1963).
[15] B. F. Gibson and L. I. Schiff, Phys. Rev. 138, B26 (1965); B. F. Gibson, *ibid.* 139, B1153 (1965).

where $P_\sigma^{\pm}(ij)$ are the triplet- and singlet-spin projection operators and $P_\tau^{\pm}(ij)$ the corresponding isospin operators having the following representation in terms of the permutation operators $(ij)_{\sigma,\tau}$:

$$P_{\sigma,\tau}^{\pm}=\tfrac{1}{2}[1\pm(ij)_{\sigma,\tau}]. \tag{2.4}$$

The function $g(\mathbf{p})$ is in turn taken (a) as a pure S-wave function representing an effective central force, and (b) as a function of Yamaguchi form[16]

$$g(\mathbf{p})=C(p)+8^{-1/2}T(p)S_{12}(\hat{p}), \tag{2.5}$$

for a combination of central and tensor forces. Using these forms of the potentials, and the definitions

$$-\mathbf{P}_k=\mathbf{P}_i+\mathbf{P}_j,\quad 2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j, \tag{2.6}$$

$$D(E)=\tfrac{1}{2}(P_1^2+P_2^2+P_3^2)+\alpha_T^2, \tag{2.7}$$

and

$$\alpha_T^2=ME_B, \tag{2.8}$$

(A',A'') have the following structures:

$$\begin{pmatrix}A'\\A''\end{pmatrix}=D^{-1}(E)\Omega_S\begin{pmatrix}\chi'\\\chi''\end{pmatrix}, \tag{2.9}$$

where

$$\Omega_S=\sum_{k=1}^{3}[g(\mathbf{p}_{ij})P_\sigma^{+}(ij)F_{ij}(\mathbf{P}_k)+f(p_{ij})P_\sigma^{-}(ij)G(P_k)], \tag{2.10}$$

and (χ',χ'') are the two spin-$\frac{1}{2}$ functions of (2,1) symmetry, viz.,

$$\begin{aligned}\chi'&=(1/\sqrt{2})\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2),\\ \chi''&=-(1/\sqrt{3})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)\chi'.\end{aligned} \tag{2.11}$$

For completeness we list the representations of $P_\sigma^{\pm}(ij)$ in the (χ',χ'') basis, viz., Eq. (2.4) and

$$\begin{aligned}{}^{(12)}\sigma,\ {}^{(13)}\sigma&=\begin{pmatrix}1/2 & \pm\sqrt{3}/2\\ \pm\sqrt{3}/2 & -1/2\end{pmatrix};\\ {}^{(23)}\sigma&=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{2.12}$$

For an S-wave triplet force, $F(\mathbf{P}_k)$ is a single scalar function $F(P_k)$, but for a tensor force of the type (2.5), it has the structure[14]

$$F_{ij}(\mathbf{P}_k)=F_1(P_k)+8^{-1/2}F_2(P_k)S_{ij}(\hat{P}_k). \tag{2.13}$$

The coupled integral equations satisfied by the quantities (F,G) for the scalar case and (F_1,F_2,G) for the tensor case are given in Ref. 14.

[16] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); **95**, 1635 (1954).

3. PROBABILITIES OF VARIOUS ORBITAL STATES

The probabilities of various orbital states must be determined in terms of the spatial part of the wave function. Denoting the spatial parts of various symmetries by $(\psi^s,\psi',\psi'',\psi^a)$, these quantities are easily identified from the results of Sec. 2. For the pure S-wave case, these are simply

$$\psi^s=D^{-1}(E)(A_1+A_2+A_3), \tag{3.1}$$

$$\psi'=D^{-1}(E)\tfrac{1}{2}\sqrt{3}(B_3-B_2), \tag{3.2}$$

$$\psi''=D^{-1}(E)(-B_1+\tfrac{1}{2}B_2+\tfrac{1}{2}B_3), \tag{3.3}$$

$$\psi^a=0, \tag{3.4}$$

where (with i, j, $k=1$, 2, 3)

$$A_k=g(\mathbf{p}_{ij})F(P_k)+f(p_{ij})G(P_k), \tag{3.5}$$

$$B_k=g(\mathbf{p}_{ij})F(\mathbf{P}_k)-f(p_{ij})G(P_k). \tag{3.6}$$

There are thus only two types of amplitudes-symmetric (S) and mixed-symmetric (S'). With an over-all normalization to unity, viz.,

$$\langle\psi^s|\psi^s\rangle+\langle\psi'|\psi'\rangle+\langle\psi''|\psi''\rangle=1, \tag{3.7}$$

the two S-state probabilities P_0 and P_0' are simply given by

$$P_0=\langle\psi^s|\psi^s\rangle, \tag{3.8}$$

$$P_0'=2\langle\psi'|\psi'\rangle, \tag{3.9}$$

noting that the two (2,1) states make equal contribution to P_0'.[17]

For the case of tensor forces, the analysis is somewhat more involved because of the presence of several P and D states. Formally, we can, of course, define the quantities A_k and B_k as in Eqs. (3.5) and (3.6), but now the functions $F_{ij}(\mathbf{P})$ and $g_{ij}(\mathbf{p})$ would still involve the spin operators $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$. To identify the various states in this case, we note that after the effects of these additional spin operators have been taken into account, the resultant terms in the wave function can be arranged according to spin-cum-angular structures. Thus the terms associated with

$$\chi' \quad\text{and}\quad \chi''=-(1/\sqrt{3})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)\chi' \tag{3.10}$$

clearly represent the ${}^2S_{1/2}$ contributions, which can be further broken up into the symmetric and mixed-symmetric parts; viz., $\psi_S{}^s$ and (ψ_S',ψ_S''), as for the case of pure S-wave interaction. The terms involving

$$i(\boldsymbol{\sigma}_3\cdot\mathbf{Q})\chi',\quad i(\boldsymbol{\sigma}_1\cdot\mathbf{Q})\chi',\quad (\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_1)\cdot\mathbf{Q}\chi', \tag{3.11}$$

where

$$\mathbf{Q}=\mathbf{p}_{23}\times\mathbf{P}_1=\mathbf{p}_{31}\times\mathbf{P}_2=\mathbf{p}_{12}\times\mathbf{P}_3 \tag{3.12}$$

are the various combinations of ${}^2P_{1/2}$ and ${}^4P_{1/2}$ states.

[17] B. S. Bhakar, Ph.D. thesis, University of Delhi, 1965 (unpublished).

As it is not of much physical interest to classify the P states in detail, it is convenient to lump them together as an effective P-state contribution ψ_P to the wave function. Finally, there are three different ${}^4D_{1/2}$ terms associated with the quartet spin function (in tensor representation)

$$\tfrac{1}{2}[\sigma_{1\mu}\sigma_{3\nu}+\sigma_{1\nu}\sigma_{3\mu}-\tfrac{2}{3}\delta_{\mu\nu}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)]\chi', \tag{3.13}$$

as explained, e.g., in Sachs's book.[18] Again, since it is perhaps unnecessary to classify them in further detail, these will be lumped together under the single head of a D-state contribution ψ_D to the wave function. With an over-all normalization of the wave function to unity, the probabilities P_L of S, S', P, and D states are respectively given by[17]

$$P_0=\langle\psi_{S^s}|\psi_{S^s}\rangle, \tag{3.14}$$

$$P_0'=\langle\psi_{S'}|\psi_{S'}\rangle+\langle\psi_{S''}|\psi_{S''}\rangle, \tag{3.15}$$

$$P_1=\langle\psi_P|\psi_P\rangle, \tag{3.16}$$

$$P_2=\langle\psi_D|\psi_D\rangle, \tag{3.17}$$

where

$$P_0+P_0'+P_1+P_2=1. \tag{3.18}$$

4. THE CHARGE AND MAGNETIC FORM FACTORS

In this section, we closely follow the procedure of Schiff[4] in his corresponding analysis of the form factors. The charge and magnetic form factors are defined as the three-dimensional Fourier transforms of the expectation values of the corresponding density functions in the H^3 and He^3 states. Assuming that the three nucleons contribute additively, and ignoring the contributions from various exchange moments, the density functions are

$$\rho_C=\sum_{i=1}^{3}\rho_C(\mathbf{r},\mathbf{r}_i),\quad \rho_M=\sum_{i=1}^{3}\rho_M(\mathbf{r},\mathbf{r}_i), \tag{4.1}$$

where

$$\rho_C(\mathbf{r},\mathbf{r}_i)=\tfrac{1}{2}(1+\tau_{iz})f_{\mathrm{ch}}{}^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}(1-\tau_{iz})f_{\mathrm{ch}}{}^n(\mathbf{r}-\mathbf{r}_i), \tag{4.2}$$

$$\rho_M(\mathbf{r},\mathbf{r}_i)=\tfrac{1}{2}(1+\tau_{iz})\sigma_{iz}\mu_p f_{\mathrm{mag}}{}^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}(1-\tau_{iz})\sigma_{iz}\mu_n f_{\mathrm{mag}}{}^n(\mathbf{r}-\mathbf{r}_i). \tag{4.3}$$

μ_p and μ_n are the static magnetic moments of the proton and neutron, respectively (in nuclear magneton units), and $\mathbf{r}_i$ is the position coordinate of the ith nucleon. The functions $f(\mathbf{r}-\mathbf{r}_i)$ are the coordinate representations for the various nucleon (charge and magnetic) form factors $F(k)$, normalized, respectively, to

$$F_{\mathrm{ch}}{}^p(0)=1,\quad F_{\mathrm{ch}}{}^n(0)=0,\quad F_{\mathrm{mag}}{}^p(0)=F_{\mathrm{mag}}{}^n(0)=1.$$

[18] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953). See also, L. Cohen and J. B. Willis, Nucl. Phys. **32**, 114 (1962).

We now indicate the broad procedure for the evaluation of the charge form factor of H^3, which is defined as

$$F_{\mathrm{ch}}{}^{\mathrm{H}^3}(k)=\sum_{i=1}^{3}F_i(k), \tag{4.4}$$

where

$$\delta(\mathbf{K})F_i(k)=\int\exp(i\mathbf{k}\cdot\mathbf{r})[\psi_{\mathrm{H}^3}{}^\dagger(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\rho_C(\mathbf{r},\mathbf{r}_i)\times\psi_{\mathrm{H}^3}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)]d\mathbf{r}d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3, \tag{4.5}$$

and the multiplying δ function on the left-hand side representing over-all conservation of momentum, anticipates its appearance on the right-hand side as well, after certain spatial integrations have been carried out. A corresponding expression holds for the He^3 charge form factor with ψ_{H^3} replaced by ψ_{He^3}, except for a factor of 2 on the left-hand side of (4.5) to normalize $F_{\mathrm{ch}}{}^{\mathrm{He}^3}(0)$ to unity.

For the calculation of $F_1(k)$, the transformation $\mathbf{r}-\mathbf{r}_1=\mathbf{z}_1$ reduces it to

$$F_1(k)=F_{\mathrm{ch}}{}^p(k)F_1{}^+(k)+F_{\mathrm{ch}}{}^nF_1{}^-(k), \tag{4.6}$$

where

$$\delta(\mathbf{K})F_1{}^\pm(k)=\int\exp(i\mathbf{k}\cdot\mathbf{r}_1)\psi_{\mathrm{H}^3}{}^\dagger(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\times\frac{1\pm\tau_{1z}}{2}\psi_{\mathrm{H}^3}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3. \tag{4.7}$$

The remaining coordinates in $F_1{}^\pm(k)$ are most easily integrated out through the transformations

$$\begin{gathered}\mathbf{r}_1=\mathbf{R}-\tfrac{2}{3}\boldsymbol{\varrho}_1,\\ \mathbf{r}_2=\mathbf{R}+\tfrac{1}{2}\mathbf{r}_{23}+\tfrac{1}{3}\boldsymbol{\varrho}_1,\quad \mathbf{r}_3=\mathbf{R}-\tfrac{1}{2}\mathbf{r}_{23}+\tfrac{1}{3}\boldsymbol{\varrho}_1,\end{gathered} \tag{4.8}$$

and then expressing ψ_{H^3} in momentum space. Taking due care of the δ function $\delta(\mathbf{K})$ representing over-all momentum conservation, this finally gives

$$F_1{}^\pm(k)=\int\psi_{\mathrm{H}^3}{}^\dagger(\mathbf{p}_{23},\,\mathbf{P}_1+\tfrac{1}{3}\mathbf{k})\times\frac{1\pm\tau_{1z}}{2}\psi_{\mathrm{H}^3}(\mathbf{p}_{23},\,\mathbf{P}_1-\tfrac{1}{3}\mathbf{k})d\mathbf{p}_{23}d\mathbf{p}_1, \tag{4.9}$$

where $\psi_{\mathrm{H}^3}(\mathbf{p}_{23},\mathbf{P}_1)$ is the complete triton wave function,[14] as given in Sec. 2, in the over-all center-of-mass frame $\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0$, *but expressed entirely in terms of the two momentum variables* $(\mathbf{p}_{23},\mathbf{P}_1)$, by virtue of the identities

$$\mathbf{p}_{31}=-(\tfrac{3}{4}\mathbf{P}_1+\tfrac{1}{2}\mathbf{p}_{23}),\quad \mathbf{p}_{12}=(\tfrac{3}{4}\mathbf{P}_1-\tfrac{1}{2}\mathbf{p}_{23}), \tag{4.10}$$

$$\mathbf{P}_2=-\tfrac{1}{2}\mathbf{P}_1+\mathbf{p}_{23},\qquad \mathbf{P}_3=-(\tfrac{1}{2}\mathbf{P}_1+\mathbf{p}_{23}). \tag{4.11}$$

The wave function in (4.9) is normalized according to

$$\int\psi_{\mathrm{H}^3}{}^\dagger(\mathbf{p}_{23},\mathbf{P}_1)\psi_{\mathrm{H}^3}(\mathbf{p}_{23},\mathbf{P}_1)d\mathbf{p}_{23}d\mathbf{P}_1=1. \tag{4.12}$$

Similar definitions hold for $F_2{}^\pm(k)$ and $F_3{}^\pm(k)$, To eliminate the isospin factors, the following 2×2 matrix representations for τ_{iz} in the states (ζ',ζ'') may be employed:

$$\tau_{1z}=(\pm)\begin{pmatrix}1 & 0\\ 0 & -1/3\end{pmatrix},$$
$$\tau_{2z}=(\pm)\begin{pmatrix}0 & 1/\sqrt{3}\\ 1/\sqrt{3} & 2/3\end{pmatrix}, \tag{4.13}$$
$$\tau_{3z}=(\pm)\begin{pmatrix}0 & -1/\sqrt{3}\\ -1/\sqrt{3} & 2/3\end{pmatrix},$$

where the sign $(\pm)$ in front of the matrices are appropriate for the cases of He^3 and H^3, respectively. This leads to the results

$$F_1^+(k)=\tfrac{1}{3}\langle A'|A'\rangle_{(23,1)}, \tag{4.14}$$

$$F_1^-(k)=\tfrac{1}{6}\langle A'|A'\rangle_{(23,1)}+\tfrac{1}{2}\langle A''|A''\rangle_{(23,1)}, \tag{4.15}$$

where (A',A'') are as defined in Sec. 2, but each expressed entirely in terms of $\mathbf{P}_1$ and $\mathbf{p}_{23}$, and the notation $\langle A|A\rangle_{(23,1)}$ is an abbreviation for

$$\sum_{\text{spin}}\int d\mathbf{P}_1 d\mathbf{p}_{23}\langle A|A\rangle. \tag{4.16}$$

Similar expressions are written down for $F_2^\pm(k)$ and $F_3^\pm(k)$, using the cyclic permutations $(\mathbf{p}_{31},\mathbf{P}_2)$ and $(\mathbf{p}_{12},\mathbf{P}_3)$, respectively, of the momentum variables. These expressions finally allow us to obtain the charge form factor for H^3 in the form

$$F_{\text{ch}}{}^{\text{H}^3}=2F_{\text{ch}}{}^nF_L{}^c+F_{\text{ch}}{}^pF_O{}^c, \tag{4.17}$$

where

$$F_L{}^c=\tfrac{1}{12}\langle A'|A'\rangle_{(23,1)}+\tfrac{1}{4}\langle A''|A''\rangle_{(23,1)} +5/24\langle A'|A'\rangle_{(13,2)}+\tfrac{1}{8}\langle A''|A''\rangle_{(13,2)} +5/24\langle A'|A'\rangle_{(12,3)}+\tfrac{1}{8}\langle A''|A''\rangle_{(12,3)}, \tag{4.18}$$

$$F_O{}^c=\tfrac{1}{3}\langle A'|A'\rangle_{(23,1)}+\tfrac{1}{12}\langle A'|A'\rangle_{(13,2)}+\tfrac{1}{4}\langle A''|A''\rangle_{(13,2)} +\tfrac{1}{12}\langle A'|A'\rangle_{(12,3)}+\tfrac{1}{4}\langle A''|A''\rangle_{(12,3)}, \tag{4.19}$$

thus explicitly defining the charge body form factors for H^3 in terms of various elements of the three-body wave function. For He^3, the corresponding result is

$$2F_{\text{ch}}{}^{\text{He}^3}=2F_{\text{ch}}{}^pF_L{}^c+F_{\text{ch}}{}^nF_O{}^c. \tag{4.20}$$

As for the magnetic-moment form factors, the calculations are almost identical, except for the appearance of spin factors σ_{iz}. However, since their matrix elements follow identical rules to those of τ_{iz}, the representation (4.13) will hold with respect to the spins states (χ',χ''), except that the sign $(\pm)$ in front of the matrices is now unnecessary. The results for the magnetic form factors are expressible as

$$\mu_{\text{H}^3}F_{\text{mag}}{}^{\text{H}^3}=\mu_pF_{\text{mag}}{}^pF_O{}^m +\tfrac{2}{3}\mu_nF_{\text{mag}}{}^n[F_O{}^m-F_L{}^m], \tag{4.21}$$

$$\mu_{\text{He}^3}F_{\text{mag}}{}^{\text{He}^3}=\mu_nF_{\text{mag}}{}^nF_O{}^m +\tfrac{2}{3}\mu_pF_{\text{mag}}{}^p[F_O{}^m-F_L{}^m], \tag{4.22}$$

where the magnetic body form factors $F_L{}^m$ and $F_O{}^m$ are given by the explicit formulas

$$F_L{}^m=\frac{1}{12}\langle A'|\sigma_{1z}|A'\rangle_{(23,1)}-\frac{3}{4}\langle A''|\sigma_{1z}|A''\rangle_{(23,1)}-\frac{13}{24}\langle A'|\sigma_{2z}|A'\rangle_{(13,2)}-\frac{1}{8}\langle A''|\sigma_{2z}|A''\rangle_{(13,2)}$$
$$+\frac{5}{4\sqrt{3}}\langle A'|\sigma_{2z}|A''\rangle_{(13,2)}-\frac{13}{24}\langle A'|\sigma_{3z}|A'\rangle_{(12,3)}-\frac{1}{8}\langle A''|\sigma_{3z}|A''\rangle_{(12,3)}-\frac{5}{4\sqrt{3}}\langle A'|\sigma_{3z}|A''\rangle_{(12,3)}, \tag{4.23}$$

$$F_O{}^m=\frac{1}{3}\langle A'|\sigma_{1z}|A'\rangle_{(23,1)}+\frac{1}{12}\langle A'|\sigma_{2z}|A'\rangle_{(13,2)}+\frac{1}{4}\langle A''|\sigma_{2z}|A''\rangle_{(13,2)}+\frac{1}{2\sqrt{3}}\langle A'|\sigma_{2z}|A''\rangle_{(13,2)}$$
$$+\frac{1}{12}\langle A'|\sigma_{3z}|A'\rangle_{(12,3)}+\frac{1}{4}\langle A''|\sigma_{3z}|A''\rangle_{(12,3)}-\frac{1}{2\sqrt{3}}\langle A'|\sigma_{3z}|A''\rangle_{(12,3)}. \tag{4.24}$$

It may be noted that we have four different body form factors, as against two in Schiff's treatment,[4] even for the pure S-wave case, The reason lies simply in our inclusion of the terms involving the squares of the S' amplitude (which Schiff neglects). We recognize, of course, that the S'^2 terms are quite negligible. The only reason for retaining them in our treatment is that their algebraic separation would have been more troublesome. As we shall see, however, their smallness will show up in terms of approximate equality of the quantities $(F_L{}^m,F_L{}^c)$ and $(F_O{}^m,F_O{}^c)$.

5. INTEGRAL FORMULAS FOR THE BODY FORM FACTORS

The body form factors F_O and F_L obtained in the last section are all expressible as linear combinations of several integrals, each involving a product of two distinct pieces of the initial and final wave functions. As Eqs. (2.9) and (2.10) show, the three-body wave function is a sum of three different types of terms, denoted symbolically by

$$\psi_1(\mathbf{p}_{23},\mathbf{P}_1),\quad \psi_2(\mathbf{p}_{31},\mathbf{P}_2),\quad \psi_3(\mathbf{p}_{12},\mathbf{P}_3), \tag{5.1}$$

which label the appearance of various momentum combinations. The evaluation of $F_1(k)$ is most easily achieved in terms of $\mathbf{p}_{23}(\equiv\mathbf{p})$ and $\mathbf{P}_1(\equiv\mathbf{q})$, as shown in Sec. 4, since the other two momentum pairs can also be expressed via (4.10) and (4.11), in terms of $(\mathbf{p},\mathbf{q})$. The four basic integrals are then of the following types:

$$I_1\equiv(1,1)=\int\psi_1^\dagger(\mathbf{p}_{23},\,\mathbf{P}_1+\tfrac{1}{3}\mathbf{k})\phi_1(\mathbf{p}_{23},\,\mathbf{P}_1-\tfrac{1}{3}\mathbf{k})\,, \tag{5.2}$$

$$I_2\equiv(2,2)=\int\psi_2^\dagger(\tfrac{3}{4}\mathbf{P}_1+\tfrac{1}{2}\mathbf{p}_{23}+\tfrac{1}{4}\mathbf{k},\,\tfrac{1}{2}\mathbf{P}_1-\mathbf{p}_{23}+\tfrac{1}{6}\mathbf{k})\times\phi_2(\tfrac{3}{4}\mathbf{P}_1+\tfrac{1}{2}\mathbf{p}_{23}-\tfrac{1}{4}\mathbf{k},\,\tfrac{1}{2}\mathbf{P}_1-\mathbf{p}_{23}-\tfrac{1}{6}\mathbf{k})\,, \tag{5.3}$$

$$I_3\equiv(2,3)=\int\psi_2^\dagger(\tfrac{3}{4}\mathbf{P}_1+\tfrac{1}{2}\mathbf{p}_{23}+\tfrac{1}{4}\mathbf{k},\,\tfrac{1}{2}\mathbf{P}_1-\mathbf{p}_{23}+\tfrac{1}{6}\mathbf{k})\times\phi_3(\tfrac{3}{4}\mathbf{P}_1-\tfrac{1}{2}\mathbf{p}_{23}-\tfrac{1}{4}\mathbf{k},\,\tfrac{1}{2}\mathbf{P}_1+\mathbf{p}_{23}-\tfrac{1}{6}\mathbf{k})\,, \tag{5.4}$$

$$I_4\equiv(1,2)=\int\psi_1^\dagger(\mathbf{p}_{23},\,\mathbf{P}_1+\tfrac{1}{3}\mathbf{k})\times\phi_2(\tfrac{3}{4}\mathbf{P}_1+\tfrac{1}{2}\mathbf{p}_{23}-\tfrac{1}{4}\mathbf{k},\,\tfrac{1}{2}\mathbf{P}_1-\mathbf{p}_{23}-\tfrac{1}{6}\mathbf{k})\,, \tag{5.5}$$

where ψ_i and ϕ_j represent symbolically the different portions of the initial and final wave functions, respectively. It is clear that integrals like (1,3) and (3,3) are trivially expressible in terms of (1,2) and (2,2), respectively. For the quantities $F_2(k)$ and $F_3(k)$, an identical procedure is available with appropriate cyclic permutations of the momentum pair $(\mathbf{p}_{23},\mathbf{P}_1)$.

A further problem arises because each of the initial and final wave functions involves two types of potential factors, viz. $g(\mathbf{p})$ and $f(p)$, with associated form factors $F(\mathbf{P})$ and $G(P)$. This necessitates a further classification of the integrals I_1 to I_4 in (5.2) to (5.5), so as to indicate the precise potential factors involved in each associated pair (ψ_i,ϕ_j) of the components of the wave functions.

In the pure S-wave case, we illustrate this classification by writing these integrals as $I(gf)$, where ψ_i involves $g(p)$ and ϕ_j involves $f(p)$. Thus

$$I_1(gf)=\int d\mathbf{p}d\mathbf{q}g(p)f(p)F(\mathbf{q}+\tfrac{1}{3}\mathbf{k})G(\mathbf{q}-\tfrac{1}{3}\mathbf{k})D^{-1}(p,\,\mathbf{q}+\tfrac{1}{3}\mathbf{k})D^{-1}(p,\,\mathbf{q}-\tfrac{1}{3}\mathbf{k})\,, \tag{5.6}$$

$$I_2(gf)=\int d\mathbf{p}d\mathbf{q}g(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k})f(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}-\tfrac{1}{4}\mathbf{k})F(\mathbf{p}-\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})G(\mathbf{p}-\tfrac{1}{2}\mathbf{q}+\tfrac{1}{6}\mathbf{k})\times D^{-1}(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k},\,\mathbf{p}-\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})D^{-1}(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}-\tfrac{1}{4}\mathbf{k},\,\mathbf{p}-\tfrac{1}{2}\mathbf{q}+\tfrac{1}{6}\mathbf{k})\,, \tag{5.7}$$

$$I_3(gf)=\int d\mathbf{p}d\mathbf{q}g(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k})f(\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k})F(\mathbf{p}-\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})G(\mathbf{p}+\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})\times D^{-1}(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k},\,\mathbf{p}-\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})D^{-1}(\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}\mathbf{q}+\tfrac{1}{4}\mathbf{k},\,\mathbf{p}+\tfrac{1}{2}\mathbf{q}-\tfrac{1}{6}\mathbf{k})\,, \tag{5.8}$$

$$I_4(gf)=\int d\mathbf{p}d\mathbf{q}g(p)f(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}-\tfrac{1}{4}\mathbf{k})F(\mathbf{q}+\tfrac{1}{3}\mathbf{k})G(\mathbf{p}-\tfrac{1}{2}\mathbf{q}+\tfrac{1}{6}\mathbf{k})D^{-1}(p,\,\mathbf{q}+\tfrac{1}{3}\mathbf{k})D^{-1}(\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}-\tfrac{1}{4}\mathbf{k},\,\mathbf{p}-\tfrac{1}{2}\mathbf{q}+\tfrac{1}{6}\mathbf{k})\,, \tag{5.9}$$

where

$$D(p,q)=p^2+\tfrac{3}{4}q^2+\alpha_T^2\,. \tag{5.10}$$

The other combinations like $I(gg)$, $I(ff)$, etc., are easily obtained from the above formulas. This gives, for the body form factors in the S-wave case, the following results:

$$F_L{}^c=\tfrac{3}{8}[5I_1(gg)+3I_1(ff)+7I_2(gg)+9I_2(ff)+I_3(gg)+3I_3(ff)+12I_3(gf)+5I_4(gg)+3I_4(ff)+9I_4(gf)+15I_4(fg)]\,, \tag{5.11}$$

$$F_O{}^c=\tfrac{3}{4}[I_1(gg)+3I_1(ff)+5I_2(gg)+3I_2(ff)+2I_3(gg)+6I_3(gf)+I_4(gg)+3I_4(ff)+9I_4(gf)+3I_4(fg)]\,, \tag{5.12}$$

$$F_L{}^m=\tfrac{3}{4}[3I_1(gg)+I_1(ff)-2I_2(gg)+10I_2(gf)+5I_3(gg)+7I_3(ff)-4I_3(gf)+3I_4(gg)+I_4(ff)+3I_4(gf)+9I_4(fg)]\,, \tag{5.13}$$

$$F_O{}^m=\tfrac{3}{4}[4I_1(ff)+4I_2(gg)+4I_2(gf)+5I_3(gg)+I_3(ff)+2I_3(gf)+4I_4(ff)+12I_4(gf)]\,. \tag{5.14}$$

In the case of tensor forces, the formulas are much more involved. However, certain simplifications are possible, if due regard is paid to the physical magnitudes of the various quantities. Thus the central and tensor terms in the triplet potential $g(\mathbf{p})$, denoted by C and T, respectively, obey the condition $|T|\ll|C|\sim|f|$. Like-

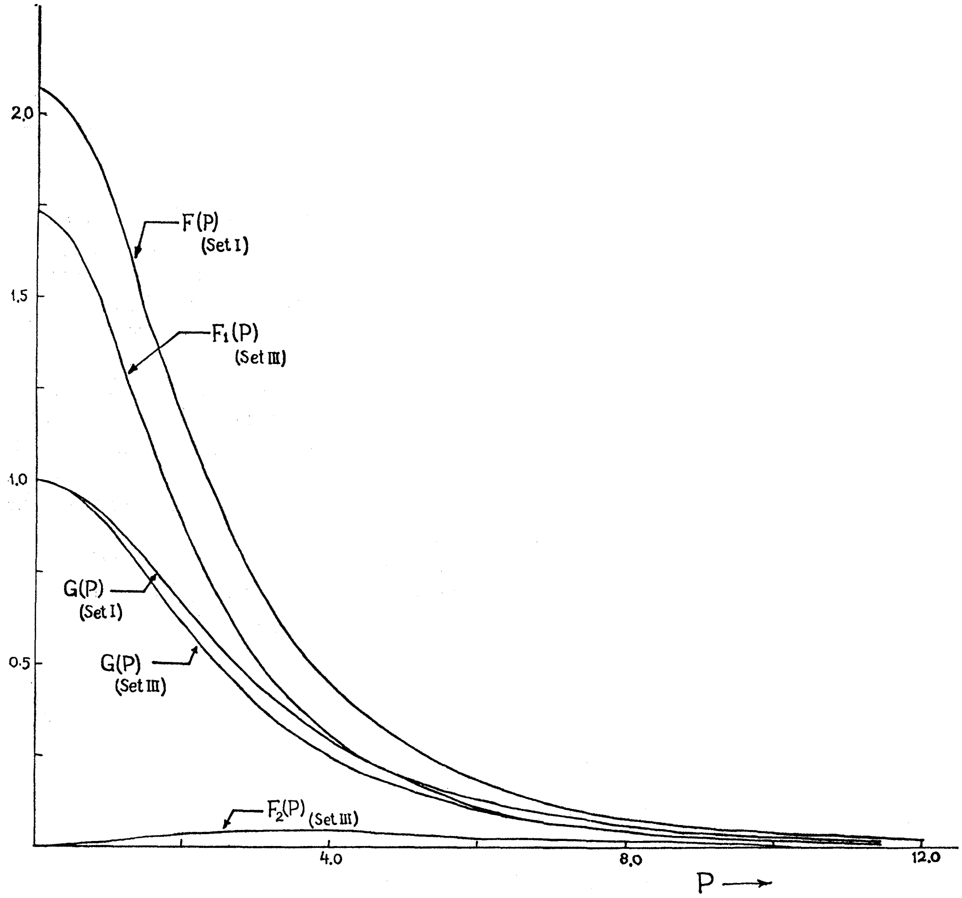

FIG. 1. The spectator functions (F,G) with set I and (F_1,F_2,G) with set III, as functions of momentum P in units of α, the deuteron-binding-energy parameter. The curves are all normalized to $G(0)=1$.

wise, the "central" and "tensor" parts of the triplet spectator function $F(\mathbf{P})$, denoted, respectively, by F_1 and F_2, satisfy the inequality $|F_2|\ll|F_1|\sim|G|$, anticipating the numerical results to be given in the next section. Indeed, the numerical results bring out the following inequalities:

$$y=|F_2|/|F_1|\ll x=|T|/|C|\ll 1. \tag{5.15}$$

These inequalities help in distinguishing between the orders of magnitude of the various terms in the expressions for the body form factors. Thus, while the principal terms in the integrals (5.2)–(5.9) would involve factors like

$$CCF_1F_1,\quad CfF_1G,\quad ffGG, \tag{5.16}$$

the magnitudes of various smaller terms compared with (5.16) are of the following (descending) orders:

$$x,\, y,\, x^2,\, xy,\, x^2y,\, y^2,\, xy^2,\, x^2y^2. \tag{5.17}$$

However, the inequalities (5.15) show that only the terms of orders x, y, x^2, xy need be taken into account, without sacrificing any physical accuracy. With this approximation, the body form factors in this case are formally given by Eqs. (5.11)–(5.14), except for certain modifications in the meaning of various integrals, as indicated below ($\alpha=1, 2, 3, 4$):

$$I_\alpha(ff)\to I_\alpha(ff), \tag{5.18}$$

$$I_\alpha(fg,gf,gg)\to I_\alpha'(fg,gf,gg) \tag{5.19}$$

for the charge form factors, and

$$I_\alpha(fg,gf,gg)\to I_\alpha''(fg,gf,gg) \tag{5.20}$$

for the magnetic-moment form factors. In these modified forms, the principal terms in I_α' and I_α'' are *identical* in structure to the corresponding terms I_α in the S-wave case. However, these terms now contain additional contributions of orders x, y, x^2, xy, which are admissible within our approximation, under all the heads $\alpha=1, 2, 3, 4$. The actual expressions, however, are too lengthy to be reproduced here.

6. NUMERICAL RESULTS FOR FORM FACTORS

The spectator functions F and G were evaluated corresponding to the following shapes of the potentials:

$$f(p)=(p^2+\beta_s^2)^{-1}, \tag{6.1}$$

$$g(p)=(p^2+\beta_t^2)^{-1}, \tag{6.2}$$

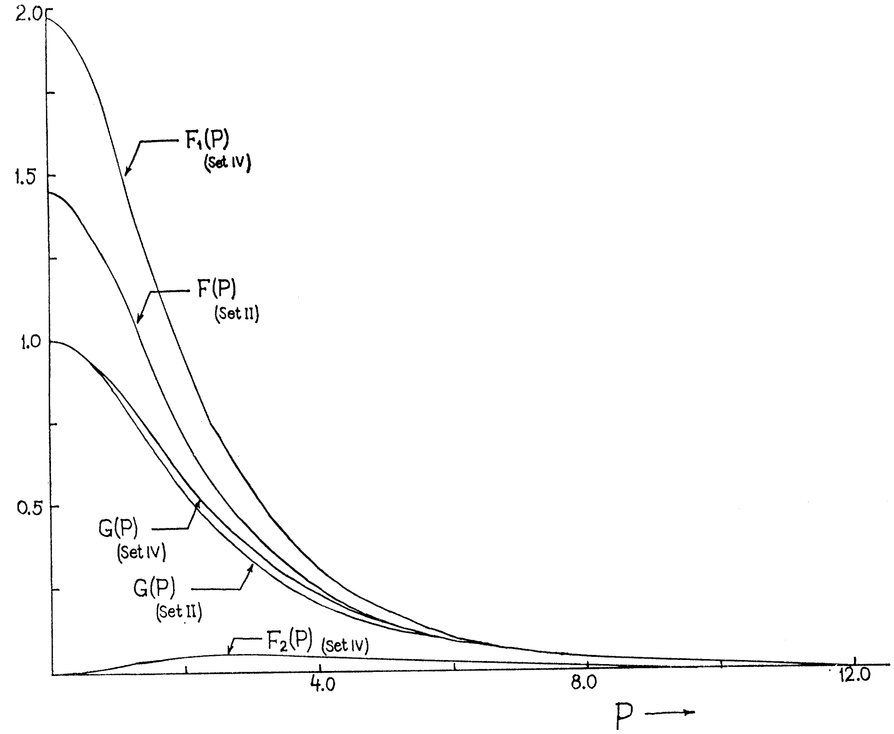

FIG. 2. The spectator functions (F,G) with set II and (F_1,F_2,G) with set IV, as functions of α. The curves are all normalized to $G(0)=1$.

(for the tensor case):

$$C(p)=(p^2+\beta_t^2)^{-1}, \tag{6.3}$$

$$T(p)=-tp^2(p^2+\gamma_t^2)^{-2}. \tag{6.4}$$

Several sets of the triplet and singlet parameters as given by Yamaguchi,[16] and subsequently in improved form by Naqvi,[19] which were used for the calculations are as shown in Table I. The actual curves obtained for the spectator function with Yamaguchi's parameters (sets I and III) and the corresponding curves with Naqvi's parameters (sets II and IV) are given in Fig. 1 and Fig. 2, respectively.

It is seen from these curves that even near the maximum of F_2, it is about 10% of the corresponding value of F_1 and only 2% of the maximum value of F_1 (which occurs at $P=0$). The singlet spectator function G, as expected, has a shape and magnitude similar to F or F_1.

For the calculation of the integrals, each of which involves two such spectator functions, it is most convenient to use the Feynman method of integration, since all the other factors (potential and denominator functions) have the structure of "propagators." For this purpose, the spectator functions must also be explicitly parametrized to such forms. Indeed, it is found that for the s-wave case, each of $F(P)$ and $G(P)$ can be accurately fitted by the general form

$$\frac{A}{P^2+\gamma^2}-B\left[\frac{\gamma_1^2}{P^2+\gamma_1^2}-\frac{\gamma_2^2}{P^2+\gamma_2^2}\right], \tag{6.5}$$

where A, B, γ, γ_1, and γ_2 are suitable constants. Again for the tensor case, $F_1(P)$ and $G(P)$ are equally well represented by the above form, with suitably adjusted constants. However, $F_2(P)$ needs the following alternative representation:

$$F_2(P)=CP^2(P^2+\delta_1^2)^{-1}(P^2+\delta_2^2)^{-1}(P^2+\delta_3^2)^{-1}. \tag{6.6}$$

While the fit (6.6) for F_2 is not as good as (6.5) for F_1, F, G, it should be recognized that F_2 itself is appreciably smaller than F_1 or G, so that the over-all effect of the approximation is considerably weighted down. Typical fits for $F_1(P)$ and $F_2(P)$ are shown in Table II. Table III gives the values of the different parameters obtained for all the potentials listed in Table I.

With these functional forms, the various integrals can be evaluated in a semi-analytic manner for which the approximation techniques employed are briefly described in the Appendix. The body form factors which are now evaluated with the help of these integrals

TABLE I. The potential parameters of various central and tensor forces used for the calculations. The Yamaguchi (Ref. 16) and Naqvi (Ref. 19) parameters are distinguished by the suffixes Y and N, respectively. S represents the $^1S_0$ potential and $C^{\rm eff}$ the effective $^3S_1$ force. α is the deuteron binding-energy parameter. See text for other notation.

Set potential	β_S/α	β_t/α	γ_t/α	t	λ_{13}/α^3	λ_{31}/α^3
I $C_Y^{\rm eff}+S_Y$	6.255	6.255	...	...	23.4306	33.29
II C_N+S_N	5.8	5.8	...	...	18.9	22.9
III $(C+T)_Y+S_Y$	6.255	5.759	6.771	1.784	23.4306	20.0378
IV $(C+T)_N+S_N$	5.8	5.8	5.8	0.9519	18.9	22.9

[19] J. H. Naqvi, Nucl. Phys. 36, 578 (1962).

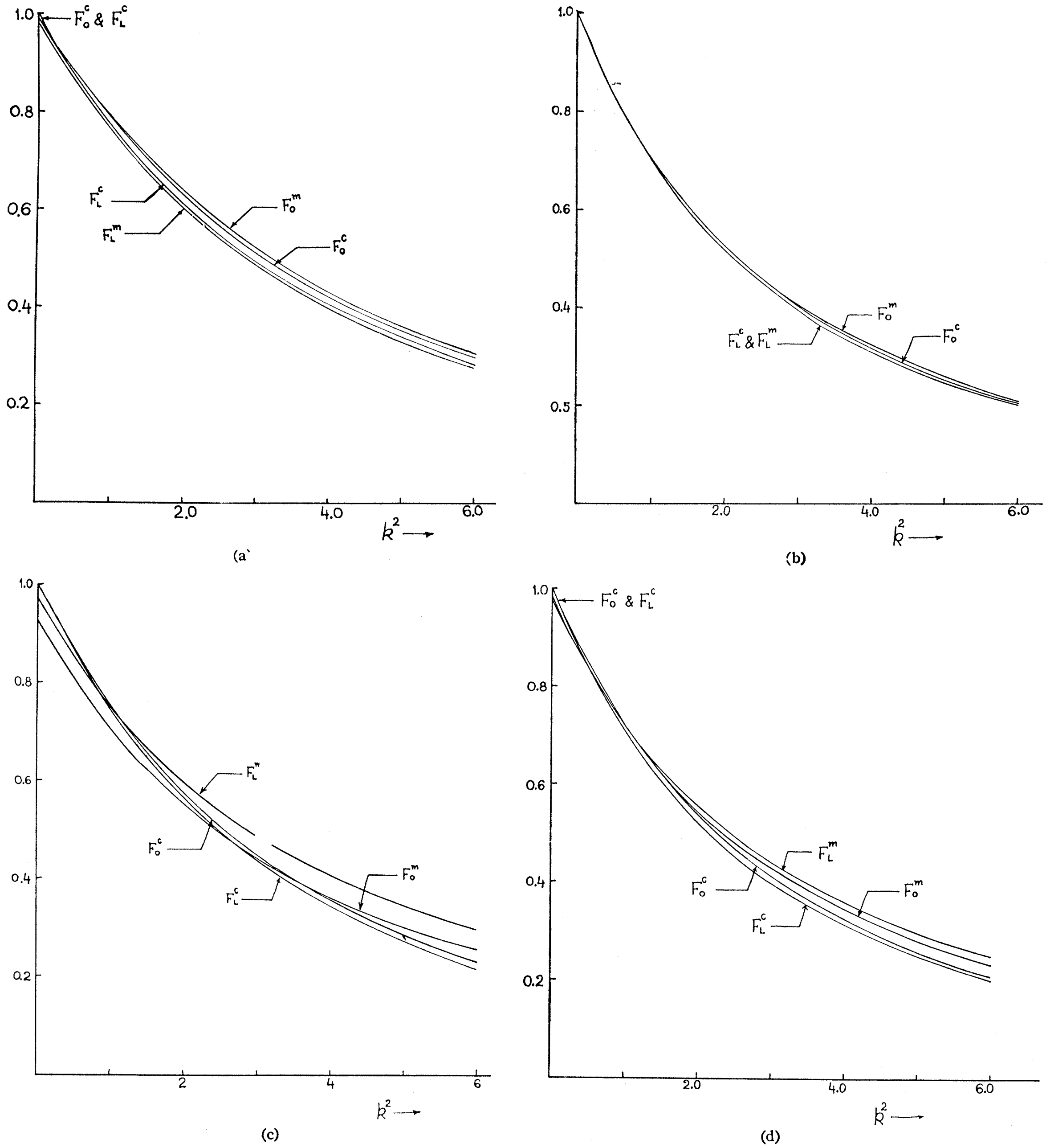

FIG. 3. Curves for the body form factors $F_O{}^c$, $F_O{}^m$, $F_L{}^c$, $F_L{}^m$ for the charge and magnetic distributions as functions of the square of the momentum transfer (k^2) in units of F^{-2}. The curves (a), (b), (c), and (d) refer to the potential parameters corresponding to sets I, II, III, and IV, respectively.

according to Eqs. (5.11)–(5.14), are given in Figs. 3(a) to 3(d), for the different potentials used. The radii corresponding to these form factors (which are evaluated by numerical interpolation, using third-degree polynomials in the variable k^2, in the region of low momentum transfers) are listed in Table IV.

Finally, the form factors of H^3 and He^3 are evaluated with the help of Eqs. (4.17), (4.20)–(4.22) using known values of the nucleon form factors as given by de Vries *et al.*[20] While we omit the actual curves for these quantities, it may be of interest to reproduce in Table V the radii of these nuclei obtained with the help of Table IV and the values for the nucleon radii as given in Ref. 20.[21]

[20] C. de Vries, R. Hofstadter, A. Johansson, and R. Herman, Phys. Rev. **134**, B848 (1964).

[21] Another set of data by de Vries *et al.* used a smaller magnitude for $a_n{}^2$(ch), but the fit to the charge radii of H^3 and He^3 with this set is even poorer [C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev. Letters **8**, 381 (1962)].

TABLE II. A typical fit to the central and tensor spectator functions $F_1(P)$ and $F_2(P)$, corresponding to set III of Table I, by the parametric forms (6.5) and (6.6), respectively. The momentum is in the units of the deuteron binding-energy parameter α.

Momentum P	$F_1(P)$ actual	$F_1(P)$ fitted	$F_2(P)$ actual	$F_2(P)$ fitted
6.24783×10^{-2}	1.73336	1.73352	7.5213×10^{-5}	6.9074×10^{-5}
0.327732	1.69478	1.69469	2.0200×10^{-3}	1.86318×10^{-3}
0.799007	1.51930	1.51851	$1.0672\times L0^{-2}$	1.0035×10^{-2}
1.46637	1.16315	1.16270	2.6879×10^{-2}	2.6220×10^{-2}
2.31560	0.753551	0.754416	4.1467×10^{-2}	4.1801×10^{-2}
3.32861	0.434461	0.434866	4.5933×10^{-2}	4.6657×10^{-2}
4.48381	0.235207	0.234494	4.0892×10^{-2}	4.0913×10^{-2}
5.75658	0.123704	0.122999	3.1496×10^{-2}	3.0874×10^{-2}
7.11979	6.4491×10^{-2}	6.4516×10^{-2}	2.2074×10^{-2}	2.1454×10^{-2}
8.54438	3.3792×10^{-2}	3.4425×10^{-2}	1.4613×10^{-2}	1.4401×10^{-2}
10.0000	1.7985×10^{-2}	1.8837×10^{-2}	9.405×10^{-3}	9.639×10^{-3}
12.8802	5.523×10^{-3}	6.093×10^{-3}	3.883×10^{-3}	4.594×10^{-3}
15.5162	1.976×10^{-3}	2.110×10^{-3}	1.745×10^{-3}	2.509×10^{-3}
18.5336	6.244×10^{-4}	3.943×10^{-4}	7.144×10^{-4}	1.359×10^{-3}
19.9375	3.634×10^{-4}	2.408×10^{-5}	4.753×10^{-4}	1.047×10^{-3}

7. DISCUSSION AND CONCLUSION

Before we discuss the comparison with experiments we wish to say a few words about the normalizations. While F_L^c and F_O^c are by definition normalized to unity, as can be seen from Eqs. (4.17) and (4.20), F_L^m and F_O^m need not be so. Indeed, as can be clearly seen from Eq. (11) of Schiff's paper,[4] the inclusion of the S'^2 terms would have given

$$F_L^m=F_1-\tfrac{1}{3}F_2-(5/9)F_3\,, \tag{7.1}$$

$$F_O^m=F_1+\tfrac{2}{3}F_2-(2/9)F_3\,, \tag{7.2}$$

where

$$F_3=\int d\mathbf{r}_i[2\exp(i\mathbf{k}\cdot r_1)v_2^2 +\exp(i\mathbf{k}\cdot r_2)(3^{1/2}v_1+v_2)^2]\,. \tag{7.3}$$

Note that F_3 is a positive-definite quantity which does not vanish at zero momentum transfer. Therefore, Eqs. (7.1) and (7.2) show that F_L^m and F_O^m not only cannot be normalized to unity, but that their values at $k^2=0$ would be somewhat different from each other, because of the terms $(5/9)F_3(0)$ and $\frac{2}{9}F_3(0)$, respectively. Indeed, for the two S-wave cases represented by sets I and II, the normalized quantities $F_{L,O}^m(0)$ are found to be the following:

Set I:

$$F_L^m(0)=1-0.01291\,,\quad F_O^m(0)=1-0.00516\,, \tag{7.4}$$

Set II:

$$F_L^m(0)=1-0.00353\,,\quad F_O^m(0)=1-0.00141\,, \tag{7.5}$$

which brings out the amounts by which these quantities fall short of unity. The deviations from unity are indeed in the ratio of 5:2, as required by Eqs. (7.1) and (7.2). For the case of tensor forces, represented by sets III and IV, there are further corrections to the normalization (due to D waves), not merely expressible by the simple equations like (7.1) and (7.2). Indeed, for the Yamaguchi tensor case (set III), characterized by a high D-state probability ($\sim$5.3%), the net deviation of $F_O^m(0)$ from unity is as much as 0.07033 and that of $F_L^m(0)$ is 0.03023, which are appreciably larger corrections than shown in (7.4). For the Naqvi potential set (IV), which yields a smaller D-state probability ($\sim$2.7%), the corresponding net corrections are 0.01862 and 0.02215, respectively.

TABLE III. The various constants $(A,B,\gamma,\gamma_1,\gamma_2)$ and $(C,\delta_1,\delta_2,\delta_3)$ of the parametric fits (6.5) and (6.6) to the spectator functions (F,F_1,G) and F_2, respectively, for the different sets of potentials used.

Set	Spectator function	A	B	γ	γ_1	γ_2	C	δ_1	δ_2	δ_3
I	$F(P)$	12.0509	5.89500	2.41225	5.48621	5.28832	⋯	⋯	⋯	⋯
	$G(P)$	8.67804	3.97000	2.94518	6.11353	5.91494	⋯	⋯	⋯	⋯
II	$F(P)$	6.22489	3.23500	2.05477	5.27229	5.07600	⋯	⋯	⋯	⋯
	$G(P)$	5.19530	2.81000	2.27841	5.58283	5.39815	⋯	⋯	⋯	⋯
III	$F_1(P)$	8.19268	3.54200	2.17301	4.99079	4.72709	⋯	⋯	⋯	⋯
	$G(P)$	7.21425	3.29000	2.68519	5.68163	5.46496	⋯	⋯	⋯	⋯
	$F_2(P)$	⋯	⋯	⋯	⋯	⋯	205.000	6.94000	36.5000	47.5000
IV	$F_1(P)$	8.17284	8.83400	2.03419	4.69662	4.58371	⋯	⋯	⋯	⋯
	$G(P)$	6.28318	3.44000	2.50582	5.44632	5.25880	⋯	⋯	⋯	⋯
	$F_2(P)$	⋯	⋯	⋯	⋯	⋯	154.000	4.95000	33.9000	44.5000

TABLE IV. The radii (in fermis) of different body form factors, a_L(ch), a_0(ch), a_L(mag), a_0(mag), for the various sets of Table I.

Set	a_L(ch)	a_0(ch)	a_L(mag)	a_0(mag)
I	1.284	1.244	1.297	1.215
II	1.537	1.512	1.545	1.495
III	1.413	1.370	1.313	1.340
IV	1.502	1.454	1.421	1.419

For a comparison with experiment, the important features of the H^3 and He^3 form factors are, (1) the actual magnitudes of the various radii, and (2) an appreciable difference between the charge radii of He^3 and H^3, as may be seen from the following experimental values[11]:

$$a_{ch}(H^3)=1.70\pm0.05\text{ F}=a_{mag}(H^3),$$
$$a_{ch}(He^3)=1.87\pm0.05\text{ F},\quad a_{mag}(He^3)=1.74\pm0.1\text{ F}. \tag{7.6}$$

As for the magnitudes of the radii, pure S-wave forces yield rather small values, as may be seen from set I of Table V for the effective S-wave Yamaguchi force. The results with set II, which represents merely the S-wave part of the total triplet (central plus tensor) force, are included in Table V just for an estimate of the tensor force contribution to the sizes of He^3 and H^3. The larger values of the radii compared with set IV indicate that the tensor force, while not so important as a central force in the binding of a three-body system, has nevertheless an appreciable role to play in determining the size of the triton.

A substantial improvement in the radii of the body form factors F_O and F_L is achieved with the tensor force, as may be seen from the results of sets III and IV in Table IV. Here again, as was found for the binding energy of H^3,[13] the Naqvi parameters (set IV) yield definitely better results than Yamaguchi's. The effect of this improvement in F_L and F_O reflects itself in a corresponding improvement in the actual radii (charge and magnetic) of H^3 and He^3, as calculated in Table V.[22] The results with set IV are particularly encouraging, in that they fall short of the experimental figures by not more than 10%, even in the "worst case" of the He^3 charge radius. To explain a discrepancy of this order of magnitude, the most natural candidate should be the effects of the hard core. Unfortunately, no concrete data are as yet available with both tensor and hard-core effects taken into account in a realistic manner. However, a model calculation by Tabakin[23] had shown that the hard core could decrease the binding energy of H^3 by numbers ranging between 0.5 and 0.9 MeV, depending on the model chosen. This reduction, being about 5 to 10% in the binding energy of H^3, should result in a corresponding *increase* in the sizes of H^3 and He^3, as a crude argument based on the asymptotic properties of the three-body wave functions would suggest.[24] This correction should perhaps be taken in conjunction with relativistic corrections,[25] as in the case of the binding energy of H^3.[26] Of course, this argument is no substitute for an exact evaluation which, while extremely involved would still be of great interest from the point of view of understanding detailed three-body effects with realistic two-body forces.

TABLE V. The charge and magnetic radii (in fermis), a_{He^3}(ch), a_{H^3}(ch), a_{He^3}(mag), a_{H^3}(mag), for the various sets of Table I, obtained from the data of de Vries *et al.* (Ref. 20) for the nucleon charge and magnetic distributions.

Set	a_{He^3}(ch)	a_{H^3}(ch)	a_{He^3}(mag)	a_{H^3}(mag)
I	1.520	1.421	1.538	1.527
II	1.739	1.662	1.754	1.752
III	1.631	1.533	1.555	1.589
IV	1.708	1.608	1.646	1.667

We recall in this connection, the recent results of Amado[27] for the radii of F_O and F_L using pure S-wave forces. While he of course recognifies the importance of hard-core effects, his values of a_L and a_O are much too large to be expected from any realistic S-wave force. We have traced this important discrepancy with our results to his large S' probability ($\sim$7%) which does not conform to any reasonable physical requirements for this parameter.[5–7] A smaller S' state should clearly have given a smaller radius, since a correspondingly larger probability for the totally symmetric state would have been more effective in bringing three nucleons together. We therefore feel that our poor S-wave results for the radii are at least realistic (with S' probability $\sim$1%) and that there is no escape from the tensor force to get the right magnitudes.

As for the difference in charge radii of He^3 and H^3, which represents another important experimental quantity, we note that it depends strongly on what is assumed about the neutron charge distribution, according to the formula

$$a_{He^3}^2(\text{ch})-a_{H^3}^2(\text{ch})=a_L^2(\text{ch})-a_O^2(\text{ch})-\tfrac{3}{2}a_n^2(\text{ch}), \tag{7.7}$$

which can be easily derived from Eqs. (4.17) and (4.20). Now while Table IV shows that $a_L^2(\text{ch})>a_O^2(\text{ch})$,[28] as

[22] The results of set II in Table V can not be discussed for physical comparison since it is an "incomplete" potential, used only for assessing the importance of the tensor force.

[23] F. Tabakin, Phys. Rev. **137**, B75 (1965).

[24] It is known that for the deuteron problem, the asymptotic wave function (which gives excellent results for its size), depends only on its (small) binding energy. For the present three-body case, the exact form of the asymptotic wave function is no doubt much more complicated, yet the square root of the binding energy is still appreciably lower than the inverse range parameters of the various forces. This would imply that the size should be governed more strongly by the binding-energy parameter than by the inverse range parameters of the forces. Therefore, to the extent that Tabakin's estimate of the hard-core effect gives 5–10% reduction in the binding energy, the effect seems to be enough to increase the radii by the order of magnitude required.

[25] G. B. West, Phys. Rev. **139**, B1246 (1965).

[26] V. K. Gupta, B. S. Bhakar, and A. N. Mitra, Phys. Rev. Letters **15**, 974 (1965).

[27] R. D. Amado, Phys. Rev. **141**, 902 (1966).

[28] Incidentally, the result $a_L^2>a_O^2$ shows *a fortiori* that the sign of the S' amplitude with respect to the S amplitude is obviously the "correct" one according to Schiff's analysis (Ref. 4).

required by the experimental value (0.607 F^2) of the left-hand side of (7.7), the excess $a_L^2(\text{ch})-a_O^2(\text{ch})$ is not enough to explain the latter. We must, in other words, invoke a *negative* value of $a_n^2(\text{ch})$ (i.e., a positive slope for the neutron charge distribution). This conclusion agrees with the results of Levinger and Srivastava[29] for the three-nucleon form factors using a variational wave function. We note further that a positive slope for the neutron charge form factor is also indicated by the nucleon form-factor analysis of de Vries *et al.*,[20] in terms of Clementel-Villi–type formulas,[30] in relation to the data for inelastic electron-deuteron scattering.[3] The data in Table V are based on $a_n^2(\text{ch})=-0.123$ F^2, but apparently this explains only a part ($\sim$0.10 F) of the experimental difference (0.17 F) between $a_{\text{He}^3}(\text{ch})$ and $a_{\text{H}^3}(\text{ch})$. To explain the full difference, we formally require $a_n^2(\text{ch})=-0.30$ F, which, however, would be rather too large to account for the inelastic electron-deuteron scattering data.

It would perhaps be more reasonable to ascribe the remaining discrepancy of 0.07 F between the two charge radii to other neglected effects. Of these, the hard core which has already been mentioned in connection with the actual sizes of these nuclei, could well play a differential role with respect to a_L and a_O. The other possibilities are Coulomb corrections for He^3, various exchange moment contributions, and a small admixture of $T=\frac{3}{2}$ states.[9] It is, however, premature to talk about these effects in any quantitative terms. As for three-body forces, we believe that while these could exist in principle, they should have a much lower priority for consideration (in view of the success already achieved with two-body forces) than the other effects mentioned in this paragraph.

To summarize, we have found that the inclusion of tensor forces gives a significant improvement over the S-wave results for the three-body radii, and leaves a fairly small margin between theory and experiment. It is argued that hard-core effects could be a promising candidate for explaining the gap. Further, the experimental difference between the charge radii of He^3 and H^3 requires a positive slope for the neutron charge distribution, again in agreement with the analysis of deuteron data.

ACKNOWLEDGMENTS

We have enjoyed interesting discussions with Dr. V. S. Bhasin on this subject. We are also indebted to Professor R. C. Majumdar for his interest and encouragement.

APPENDIX

We describe first the evaluation of the integrals in the S-wave case. The integrals I_α in Eqs. (5.6)–(5.9) involve the potential shapes (6.1) and (6.2) and the form (6.5) for the spectator functions. Each integral involves three pairs of like factors which we shall refer to as "potential", "spectator," and "denominator," respectively. Our first task is to combine the numbers of each pair by a "Feynman variable," according to

$$\frac{1}{ab}=\int_0^1 du[au+b(1-u)]^{-2}, \tag{A1}$$

and express it in the approximate form

$$\frac{1}{ab}\approx\frac{4}{(a+b)^2}\left[1+\frac{(a-b)^2}{(a+b)^2}+\cdots\right], \tag{A2}$$

where the expansion (A2) provides the necessary background to the approximations used for the problem. Since *like pairs* are being combined, their differences are expected to be small compared with their sums, the first nonvanishing correction providing an estimate of the error involved in neglecting the higher order corrections. The differences $(a-b)$ are of two types, arising from (i) small differences between the parameters β_s^2 and β_t^2 in the potentials, and (ii) certain angular correlations between the momenta $\mathbf{p}$, $\mathbf{q}$, $\mathbf{k}$, which would usually appear with *opposite signs* in a particular pair of like functions. In any case, the analytical structures of the sums $(a+b)$ can be made much simpler (by using such considerations) than those of a or b individually. In this manner we are left with expressions whose principal terms have the structures

$$(a_1+b_1)^{-2}(a_2+b_2)^{-2}(a_3+b_3)^{-2}, \tag{A3}$$

and the correction terms involve merely higher (negative) powers of one or more of the factors (a_i+b_i). Since higher powers of the same quantity do not involve additional "Feynmann variables," it is enough to discuss the evaluation of the principal terms only.

We illustrate this procedure in some detail with special reference to two specific integrals, say, I_1 and I_3 in the S-wave case. A typical I_1 integral has the form

$$\begin{aligned}&[(p^2+\beta_1^2)(p^2+\beta_2^2)]^{-1}[\{(\mathbf{q}+\tfrac{1}{3}\mathbf{k})^2+\gamma_1^2\}\\&\quad\times\{(\mathbf{q}-\tfrac{1}{3}\mathbf{k})^2+\gamma_2^2\}]^{-1}[\{p^2+\tfrac{3}{4}(\mathbf{q}+\tfrac{1}{3}\mathbf{k})^2+\alpha_T^2\}\\&\quad\quad\times\{p^2+\tfrac{3}{4}(\mathbf{q}-\tfrac{1}{3}\mathbf{k})^2+\alpha_T^2\}]^{-1},\end{aligned} \tag{A4}$$

where the groupings of the three pairs have been explicitly shown. It is clear from these expressions that the quantities (a_i+b_i) and (a_i-b_i) are of the forms indicated below:

$$\begin{aligned}&2p^2+(\beta_1^2+\beta_2^2), && \beta_1^2-\beta_2^2,\\&2(q^2+k^2)+(\gamma_1^2+\gamma_2^2), && (\gamma_1^2-\gamma_2^2)+\tfrac{4}{3}(\mathbf{q}\cdot\mathbf{k}),\\&2p^2+\tfrac{3}{2}(q^2+\tfrac{1}{9}k^2)+2\alpha_T^2, && (\mathbf{q}\cdot\mathbf{k}).\end{aligned}$$

Since in all the cases discussed in the text, β_1^2 and β_2^2 differ little from each other, and γ_1^2 and γ_2^2 do likewise, an expansion like (A2) should be physically quite

[29] J. S. Levinger and B. K. Srivastava, Phys. Rev. 137, B426, (1965).
[30] E. Clementel and C. Villi, Nuovo Cimento 4, 1207 (1956).

justified. The angular terms like $(\mathbf{q}\cdot\mathbf{k})$ lend themselves to even better justification for expansion, since the variables $\mathbf{p}$ and $\mathbf{q}$ are eventually going to be integrated out, so that only the *isotropic* parts of their *even* powers would survive. For the I_1 integral, we are therefore left with an expression of the form

$$\int\int d\mathbf{p}d\mathbf{q}[p^2+(\beta^2)_{\mathrm{av}}]^{-2}[q^2+\tfrac{1}{9}k^2+(\gamma^2)_{\mathrm{av}}]^{-2} \times(p^2+\tfrac{3}{4}q^2+\tfrac{1}{12}k^2+\alpha_T{}^2)^{-2} \quad (\mathrm{A5})$$

plus "correction terms" involving similar integrals but with higher powers for the various factors. This integral can be analytically evaluated in one of the variables $\mathbf{p}$ or $\mathbf{q}$, but the other needs numerical evaluation for different values of k^2 of physical interest.

For an estimate of the accuracy of this procedure, the second-order corrections to the principal terms were examined in detail for several I_1 integrals, and found to provide about 10–15% effects for the highest values considered for k^2 (viz. $\sim$6 F^{-2}). Since these corrections were explicitly taken into account, the higher order effects (e.g., fourth order) are not expected to exceed 5% at the highest k^2, which represents the degree of accuracy of our calculation.

For the integral I_3, the structure of the principal term, after appropriate expansion in the differences (a_i-b_i), is

$$\int\int d\mathbf{p}d\mathbf{q}[\tfrac{1}{4}p^2+(9/16)q^2+(1/16)k^2+\tfrac{1}{4}(\mathbf{p}\cdot\mathbf{k})+(\beta^2)_{\mathrm{av}}]^{-2} \times[p^2+\tfrac{3}{4}q^2+(1/12)k^2+\alpha_T{}^2]^{-2} \times[p^2+\tfrac{1}{4}q^2+(1/36)k^2-\tfrac{1}{3}(\mathbf{p}\cdot\mathbf{k})+(\gamma^2)_{\mathrm{av}}]^{-2}. \quad (\mathrm{A6})$$

Since in this case these factors still involve the angles through $(\mathbf{p}\cdot\mathbf{k})$, a further translation in $\mathbf{p}$ is necessary after combining the three factors by two Feynman variables, say (u,v). The resultant integral in $\mathbf{p}$ and $\mathbf{q}$ is then of the form

$$120\int_0^1 u(1-u)du\int_0^1 v^3(1-v)dv \times\int\int d\mathbf{p}d\mathbf{q}[Ap^2+Bq^2+C]^{-6}, \quad (\mathrm{A7})$$

where A, B, C are now functions of u, v, and k^2. The evaluation of the $\mathbf{p}$ and $\mathbf{q}$ integrations then yields a two-dimensional integral of the form

$$\int_0^1 u(1-u)du\int_0^1 v^3(1-u)du\, A^{-3/2}B^{-3/2}C^{-3}, \quad (\mathrm{A8})$$

which is most conveniently evaluated numerically for several input values of k^2. The correction terms are also of the form (A7), except for (i) the replacement $6\rightarrow 6+2n$ (n integral) in the exponent of the integrand, (ii) suitable additional factors in u, $(1-u)$, v, $(1-v)$, in the numerators arising from Feynman parametrizations, and (iii) certain angular functions in the numerator (which present no difficulty). The integrals I_2 and I_4 are evaluated in manners identical to the I_1 and I_3 cases, respectively.

For the tensor case, the procedure is quite similar, except that the structure of some of the principal terms, e.g., those which involve the potential $T(p)$, are like the correction terms in the S-wave case. Here again, the "second-order corrections" to the integrals have been taken into account in complete details, to the same order of accuracy as in the S-wave case.