Phenomenological Λ-Nucleon Potentials from S-Shell Hypernuclei. I. Dependence on Hard-Core Size*

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The binding-energy data of the s-shell hypernuclei and the total cross sections of the Λ-proton scattering are examined with two-body, spin-dependent, charge-independent, central Λ-nucleon potentials which have an intrinsic range of 1.5 F and a hard core with a radius of 0, 0.3, 0.45, or 0.6 F. The main purpose is to see whether there exists a hard core in the Λ-nucleon interaction. The results show that a hard core very likely does exist and has a radius greater than about 0.3 F. Also, it is found that even when the hard-core radius is as large as 0.6 F, the Λ-nucleon interaction is not strong enough to bind together a two-body Λ-hypernuclear system. However, it does seem to be strong enough to allow the formation of a particle-stable excited state in the hypernucleus ${}_\Lambda H^4$ with a small binding energy.

I. INTRODUCTION

WITHIN the past ten years, a large number of analyses has been performed on the s-shell hypernuclei for the purpose of obtaining some information about the basic features of the Λ-nucleon interaction.[1–13] In most of these analyses, efforts have been concentrated on the hypertriton which is the lightest hypernucleus known to date. Because of mathematical complexity, the four-body hypernucleus ${}_\Lambda H^4$ and the five-body hypernucleus ${}_\Lambda He^5$ have, in most cases, been treated rather crudely. For these two hypernuclei, pair correlations have been taken into account only in the calculations of the present authors,[11] Dietrich *et al.*,[10] and Beck and Gutsch.[12] In the work of the present authors, the mathematical difficulty was alleviated by the use of a Monte Carlo technique.[14] In the calculations of Dietrich *et al.* and Beck and Gutsch, the independent-pair method used by Mang and Wild[15] for light nuclei has been employed. As has been discussed previously,[11] this latter method may not be too accurate when the binding energy of the Λ particle is small, which, unfortunately, happens to be the case for the s-shell hypernuclei. Thus even though these authors have included the effect of pair correlations in their calculations, the accuracy in their results may still be somewhat questionable.

Our previous calculation on the s-shell hypernuclei[11] was not an extensive one, since only a Λ-nucleon interaction with a hard-core radius of 0.4 F and an intrinsic range of 1.5 F has been considered. In this investigation, we extend the calculation by using Λ-nucleon potentials with a hard core of radius ranging from 0 to 0.6 F. In this way, we hope to obtain information about whether or not there is a repulsive core in the Λ-nucleon potential. The intrinsic range will still be chosen as 1.5 F. In a later publication, we shall report on results which will be obtained using longer intrinsic ranges of 2.0 and 2.5 F.

The nucleon-nucleon potential used here will be that which was employed in our recent study on nuclear two-, three-, and four-body systems.[16] It has a hard core of radius 0.45 F, followed by an attractive part of exponential shape. This particular potential is preferred, since it yields not only a satisfactory fit to the two-nucleon low-energy effective-range parameters but also a good agreement with the experimentally determined binding energies and body form factors of H^3 and He^4.

In Sec. II, the results of our analysis on the hypernuclear systems with $A=3-5$ will be presented. From these results, we determine the strength of the Λ-nucleon interaction in the triplet and singlet states. Section III is devoted mainly to a study concerning the necessity of having a repulsive core in the Λ-nucleon potential. From this study, we do get an indication that a hard core of radius greater than about 0.3 F seems to be quite necessary in order to explain the binding-energy data on ${}_\Lambda H^3$, ${}_\Lambda H^4$, and ${}_\Lambda He^5$. In Sec. IV, we compute the Λ-nucleon scattering cross sections yielded by the various potentials. Here, too, we find that the experi-

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mental data on Λ-proton scattering supports the above-mentioned statement about the size of the hard core. Finally, in Sec. V, we discuss and summarize the results of this investigation.

II. ANALYSIS OF S-SHELL HYPERNUCLEI

The nucleon-nucleon potential is assumed to be central and charge-independent. It has the form

$$V_{ik}=[(1+P_{ik}^{\sigma})/2]V_t(r_{ik}) +[(1-P_{ik}^{\sigma})/2]V_s(r_{ik})+V_c(r_{ik})\epsilon_{ik}, \quad (1)$$

where P_{ik}^{σ} denotes the spin-exchange operator and the last term represents the Coulomb interaction, with ϵ_{ik} equal to 1 if i and k are protons, and 0 otherwise. The quantities $V_t(r)$ and $V_s(r)$ are the triplet and singlet potentials in the even states and are chosen to be of the following exponential type:

$$\begin{aligned} V_t(r)&=\infty, & (r<r_{NN})\\ &=-V_{0t}\exp[-\kappa_t(r-r_{NN})], & (r>r_{NN}) \quad (2)\\ V_s(r)&=\infty, & (r<r_{NN})\\ &=-V_{0s}\exp[-\kappa_s(r-r_{NN})], & (r>r_{NN}) \end{aligned}$$

with $r_{NN}=0.45$ F, $V_{0t}=549.26$ MeV, $V_{0s}=277.07$ MeV, $\kappa_t=2.735$ F^{-1}, and $\kappa_s=2.211$ F^{-1}.[16] It is not necessary to specify the potential in the odd states, since, in this investigation, we shall assume that the trial function is symmetric with respect to the space exchange of all the nucleons.

With this nucleon-nucleon potential, we obtain not only a good fit to the effective-range parameters but also satisfactory values for the binding energies and rms radii of the nuclei H^3 and He^4.[16] This is shown in Table I, where E represents the ground-state energy and R_{rms} is the rms radius.[17]

For the Λ-nucleon potential, we use a spin-dependent central potential of the form

$$U_{i\Lambda}=[(1+P_{i\Lambda}^{\sigma})/2]U_t(r_{i\Lambda})+[(1-P_{i\Lambda}^{\sigma})/2]U_s(r_{i\Lambda}), \quad (3)$$

with

$$\begin{aligned} U_t(r)&=\infty, & (r<r_{\Lambda N})\\ &=-U_{0t}\exp[-\lambda(r-r_{\Lambda N})], & (r>r_{\Lambda N}) \quad (4)\\ U_s(r)&=\infty, & (r<r_{\Lambda N})\\ &=-U_{0s}\exp[-\lambda(r-r_{\Lambda N})], & (r>r_{\Lambda N}) \end{aligned}$$

where $r_{\Lambda N}$ represents the radius of the hard core. For an intrinsic range of 1.5 F, the various values of $r_{\Lambda N}$ and the corresponding values of λ considered in this investigation are listed in Table II.

With a trial function which is symmetric with respect to the space exchange of all the nucleons, the depths of the spin-averaged Λ-nucleon potentials in the s-shell hypernuclei can be expressed in terms of the triplet depth U_{0t} and the singlet depth U_{0s}. Depending upon whether $U_{0s}>U_{0t}$ or $U_{0s}<U_{0t}$, we have the following relations:

$$\begin{aligned} U_{0s}>U_{0t}:\quad & U_{03}=\tfrac{1}{4}U_{0t}+\tfrac{3}{4}U_{0s},\\ & U_{04}=\tfrac{1}{2}U_{0t}+\tfrac{1}{2}U_{0s}, \quad (5)\\ & U_{05}=\tfrac{3}{4}U_{0t}+\tfrac{1}{4}U_{0s}; \end{aligned}$$

$$\begin{aligned} U_{0t}>U_{0s}:\quad & U_{03}=U_{0t},\\ & U_{04}=\tfrac{5}{6}U_{0t}+\tfrac{1}{6}U_{0s}, \quad (6)\\ & U_{05}=\tfrac{3}{4}U_{0t}+\tfrac{1}{4}U_{0s}, \end{aligned}$$

where the symbol U_{0A} denotes the depth of the spin-averaged Λ-nucleon potential in the hypernucleus $_{\Lambda}Z^A$. From the binding-energy data of $_{\Lambda}H^3$ and $_{\Lambda}He^5$, we will obtain the values of U_{03} and U_{05}. Using these values, two sets of values for U_{0t} and U_{0s} can be determined, depending on whether Eq. (5) or Eq. (6) is used. Both of these sets are then checked to see if the binding energy of $_{\Lambda}H^4$ is given correctly. In this way, we hope to get information about the size of the hard core and whether the triplet or the singlet interaction is the stronger one in the Λ-nucleon potential.

The trial wave function is written as

$$\Psi=\psi\chi, \quad (7)$$

with ψ and χ being the spatial and the appropriate spin functions, respectively. The function ψ will be chosen as

$$\psi=[\prod_{i<j=1}^{A-1} g(r_{ij})][\prod_{i=1}^{A-1} f(r_{i\Lambda})], \quad (8)$$

with i and j representing the nucleons. For the function $f(r)$, we adopt a form which has been used in a number of our previous calculations concerning nuclear and hypernuclear few-body problems[14]; it is

$$\begin{aligned} f(r)&=u_f(r)/r, & (r<d_f)\\ &=A_f r^{n_f}[\exp(-\alpha_f r)+B_f\exp(-\beta_f r)], & (r>d_f) \quad (9) \end{aligned}$$

TABLE I. Ground-state energies and rms radii.

Nucleus	E (MeV)	R_{rms} (F)
H^2	−2.225	1.92
H^3	−7.42±0.06	1.68
He^4	−28.31±0.19	1.44

TABLE II. Values of $r_{\Lambda N}$ in the Λ-nucleon potentials.

Potential type	$r_{\Lambda N}$ (F)	λ (F^{-1})
A	0	2.361
B	0.30	3.935
C	0.45	5.902
D	0.60	11.804

[17] In the case of H^3 and He^4, the values of E listed are actually the upper bounds, but, these are rather close to the ground-state eigenvalues computed with the potential $V_c(r)=\frac{1}{2}[V_t(r)+V_s(r)]$ [Y. C. Tang, E. W. Schmid, and R. C. Herndon, Nucl. Phys. 65, 203 (1965)].

where $u_f(r)$ is a solution of the equation

$$-\frac{\hbar^2}{2\mu_f}\frac{d^2}{dr^2}u_f(r)+[V_f(r)-e_f]u_f(r)=0, \qquad (10)$$

with μ_f being the reduced mass of the nucleon and the Λ particle. The potential $V_f(r)$ is the spin-averaged Λ-nucleon potential effective in the hypernucleus $_\Lambda Z^A$; it is equal to $U_A(r)$ where

$$U_A(r)=\infty, \qquad (r<r_{\Lambda N})$$
$$=-U_{0A}\exp[-\lambda(r-r_{\Lambda N})]. \quad (r>r_{\Lambda N}), \qquad (11)$$

The constants A_f and B_f in Eq. (9) are adjusted to insure continuity at the separation distance d_f for the function $f(r)$ and its first derivative. The function $g(r)$ is defined in an analogous manner, except that μ_f is replaced by μ_g, the reduced mass of two nucleons, and the potential function in Eq. (10) is replaced by the potential $V_g(r)$ which is equal to $V_t(r)$ for $_\Lambda H^3$ and equal to $\frac{1}{2}[V_t(r)+V_s(r)]$ for $_\Lambda H^4$ and $_\Lambda He^5$. In total, there are ten variational parameters in our trial function; these are α_f, β_f, e_f, d_f, n_f, α_g, β_g, e_g, d_g, and n_g.

In practice, we have not varied the parameters n_f and n_g. Instead, we have simply set $n_f=-1/(A-1)$ and $n_g=-1/(A-2)$.[18] From the experience which has been gained in similar calculations, we know that with this simplification, the upper bound will be only very slightly worse than that which could be obtained by varying all ten parameters.

To find the depths of the spin-averaged Λ-nucleon potentials, the following procedure will be adopted. We take two suitably chosen values of U_{0A} and compute the corresponding values of the ground-state energy E_A. From the value of E_A, the energy of the core nucleus given in Table I is then subtracted off; the negative of the resultant is thus the binding energy B_Λ of the Λ particle. With these two sets of values for U_{0A} and B_Λ, we find the constants a_A and b_A in the interpolation formula[19]

$$U_{0A}=a_A+b_A B_\Lambda^{1/2}, \qquad (12)$$

from which the depths U_{0A} corresponding to the observed values of the binding energies can be determined.

The results of this calculation for $_\Lambda H^3$, $_\Lambda H^4$, and $_\Lambda He^5$ are given in Tables III, IV, and V, respectively. In these tables, the statistical accuracy in the value of E_A is obtained with 50 000 estimates for $_\Lambda H^3$ and 200 000 estimates for $_\Lambda H^4$ and $_\Lambda He^5$. The quantity $\langle r_{NN}^2\rangle^{1/2}$ represents the rms distance of separation between two nucleons; its listed uncertainty is mostly not statistical, but comes from the uncertainty in the optimum values of the variational parameters.

The most recent values of the binding energies of $_\Lambda H^3$ and $_\Lambda He^5$ are[20]

$$B_\Lambda(_\Lambda H^3)=0.32\pm0.17 \text{ MeV},$$
$$B_\Lambda(_\Lambda He^5)=3.04\pm0.03 \text{ MeV}. \qquad (13)$$

Using these numbers and the values of a_A and b_A given in Table VI, we obtain from Eq. (12) the values of U_0

TABLE III. Results of the variational calculation for $_\Lambda H^3$.

Potential type	U_{03} (MeV)	α_f (F^{-1})	β_f (F^{-1})	e_f (MeV)	d_f (F)	α_g (F^{-1})	β_g (F^{-1})	e_g (MeV)	d_g (F)	E_3 (MeV)	B_Λ (MeV)	$\langle r_{NN}^2\rangle^{1/2}$ (F)
A	184.0	0.070	5.0	−2.0	1.2	0.23	2.2	−10.0	1.2	−2.45±0.08	0.23±0.08	3.32±0.07
	192.0	0.085	4.5	−2.0	1.2	0.24	2.2	−8.0	1.2	−2.72±0.07	0.49±0.07	3.14±0.07
B	650.0	0.065	7.5	−20.0	1.15	0.23	2.2	−9.0	1.2	−2.44±0.09	0.21±0.09	3.36±0.07
	665.0	0.080	6.5	−20.0	1.15	0.235	2.2	−8.0	1.2	−2.69±0.08	0.46±0.08	3.22±0.07
C	1630.0	0.075	5.5	−25.0	1.1	0.22	2.3	−7.0	1.2	−2.55±0.08	0.33±0.08	3.36±0.07
	1645.0	0.090	5.0	−25.0	1.1	0.23	2.3	−7.0	1.2	−2.71±0.08	0.48±0.08	3.18±0.07
D	7080.0	0.070	5.5	−30.0	1.1	0.22	2.3	−7.0	1.2	−2.40±0.09	0.18±0.09	3.40±0.07
	7120.0	0.090	5.0	−30.0	1.1	0.22	2.3	−7.0	1.2	−2.61±0.09	0.39±0.09	3.25±0.07

TABLE IV. Results of the variational calculation for $_\Lambda H^4$.

Potential type	U_{04} (MeV)	α_f (F^{-1})	β_f (F^{-1})	e_f (MeV)	d_f (F)	α_g (F^{-1})	β_g (F^{-1})	e_g (MeV)	d_g (F)	E_4 (MeV)	B_Λ (MeV)	$\langle r_{NN}^2\rangle^{1/2}$ (F)
A	150.0	0.10	3.0	−2.0	1.0	0.28	2.3	−22.0	1.2	−8.91±0.13	1.49±0.14	2.63±0.05
	160.0	0.12	3.5	−3.0	1.0	0.28	2.4	−23.0	1.2	−9.65±0.12	2.23±0.13	2.56±0.05
B	580.0	0.10	3.5	−7.0	1.0	0.29	2.5	−24.0	1.2	−8.58±0.13	1.16±0.14	2.58±0.05
	600.0	0.12	3.5	−8.0	1.0	0.29	2.6	−25.0	1.2	−9.28±0.13	1.86±0.14	2.53±0.05
C	1520.0	0.11	4.0	−17.0	1.0	0.29	2.4	−27.0	1.2	−8.71±0.13	1.29±0.14	2.58±0.05
	1550.0	0.12	4.0	−18.0	1.0	0.29	2.4	−28.0	1.2	−9.40±0.14	1.98±0.15	2.56±0.05
D	6900.0	0.11	4.0	−25.0	1.0	0.29	2.5	−31.0	1.2	−8.78±0.16	1.36±0.17	2.58±0.05
	6950.0	0.13	4.0	−26.0	1.0	0.29	2.5	−31.0	1.2	−9.35±0.17	1.93±0.18	2.54±0.05

[18] The choice of n_g for $_\Lambda H^3$ is different from that in our previous calculation (Ref. 11).
[19] See Appendix A.
[20] C. Mayeur, J. Sacton, P. Vilain, G. Wilquet, D. Stanley, P. Allen, D. H. Davis, E. R. Fletcher, D. A. Garbutt, M. A. Shaukat, J. E. Allen, V. A. Bull, A. P. Conway, and P. V. March, Nuovo Cimento 43, 180 (1966).

TABLE V. Results of the variational calculation for $_\Lambda He^5$.

Potential type	U_{05} (MeV)	α_f (F^{-1})	β_f (F^{-1})	e_f (MeV)	d_f (F)	α_g (F^{-1})	β_g (F^{-1})	e_g (MeV)	d_g (F)	E_5 (MeV)	B_Λ (MeV)	$\langle r_{NN}^2\rangle^{1/2}$ (F)
A	119.0	0.13	4.0	−1.0	1.0	0.30	3.3	−23.0	1.2	−31.16±0.45	2.85±0.49	2.19±0.04
	126.0	0.13	4.0	−3.0	1.0	0.30	3.5	−24.0	1.2	−32.11±0.47	3.80±0.51	2.18±0.04
B	540.0	0.11	4.0	−2.0	1.0	0.29	3.1	−14.0	1.2	−31.11±0.43	2.80±0.47	2.24±0.04
	550.0	0.12	4.0	−4.0	1.0	0.29	3.5	−16.0	1.2	−31.76±0.43	3.45±0.47	2.22±0.04
C	1466.0	0.12	4.0	5.0	1.0	0.29	2.9	−16.0	1.2	−31.35±0.44	3.04±0.48	2.25±0.04
	1500.0	0.13	4.0	2.5	1.0	0.29	3.0	−18.0	1.2	−32.84±0.46	4.53±0.50	2.24±0.04
D	6800.0	0.11	4.0	17.0	1.0	0.29	3.0	−17.0	1.2	−31.48±0.56	3.17±0.59	2.26±0.04
	6840.0	0.12	4.0	15.0	1.0	0.29	3.1	−19.0	1.2	−32.30±0.58	3.99±0.61	2.25±0.04

and U_{05} which correspond to the experimental binding energies. With U_{03} and U_{05} determined, we can then calculate the depths U_{0s} and U_{0t} of the Λ-nucleon potentials. Depending upon whether $U_{0s}>U_{0t}$ or $U_{0s}<U_{0t}$, their values, together with those of U_{03} and U_{05}, are given in Table VII, where the values of the well-depth parameters s_s and s_t are also listed.

From Table VII, it is seen that with the type of Λ-nucleon potentials considered here, the values of s_s and s_t are all less than one, which means that for a Λ-nucleon potential with a hard-core radius less than or equal to 0.6 F, a bound Λ-*N* system does not exist.

III. Λ-NULCEON POTENTIALS AND BINDING ENERGY OF $_\Lambda H^4$

With U_{03} and U_{05} determined from the binding energies of $_\Lambda H^3$ and $_\Lambda He^5$, we can calculate U_{04} with either Eq. (5) or Eq. (6). Using Eq. (12) and the values of a_4 and b_4 given in Table VI, the values of B_Λ for $_\Lambda H^4$ can then be computed. These are tabulated in Table VIII.

The experimental binding energies of the four-body hypernuclei, as given by Mayeur *et al.*,[20,21] are

$$B_\Lambda(_\Lambda H^4)=1.95\pm0.14\text{ MeV},$$
$$B_\Lambda(_\Lambda He^4)=2.07\pm0.09\text{ MeV}. \qquad (14)$$

The difference between these two quantities is

$$\Delta B_\Lambda=B_\Lambda(_\Lambda He^4)-B_\Lambda(_\Lambda H^4)=0.12\pm0.17\text{ MeV}. \qquad (15)$$

TABLE VI. Values of a_A and b_A.

Hypernucleus	Potential type	a_A (MeV)	b_A (MeV)$^{1/2}$
$_\Lambda H^3$	*A*	167.0	35.7
	B	619.0	67.6
	C	1561.6	120.0
	D	6996.3	199.0
$_\Lambda H^4$	*A*	105.2	36.7
	B	504.9	69.7
	C	1394.4	110.6
	D	6638.6	224.2
$_\Lambda He^5$	*A*	73.8	26.8
	B	449.1	54.3
	C	1312.0	88.4
	D	6471.9	184.3

[21] In Ref. 20, a discussion is given about possible sources of error for these B_Λ values.

The fact that ΔB_Λ is positive indicates that there is a charge-symmetry-breaking (CSB) component in the Λ-nucleon interaction. As was mentioned by Dalitz and von Hippel[22] and Downs and Phillips,[23] ΔB_Λ would be a negative quantity due to Coulomb effects if the Λ-nucleon interaction were completely charge symmetric. A crude estimate based on our calculated difference in the matter radii of H^3 and He^3 leads to the value[24]

$$(\Delta B_\Lambda)_{\text{Coulomb}}=-0.3\pm0.1\text{ MeV}. \qquad (16)$$

Together with Eq. (15), this means that the CSB component of the Λ-nucleon interaction would be required to account for a value of ΔB_Λ equal to 0.42±0.20 MeV.

Since in our calculation the Λ-nucleon interaction is assumed to be charge-symmetric, we should not compare the calculated values of $B_\Lambda(_\Lambda H^4)$ given in Table VIII with the experimental value of Eq. (14). Rather, a comparison should be made with the modified value of

$$B_\Lambda(_\Lambda H^4)=2.16\pm0.10\text{ MeV}, \qquad (17)$$

obtained by adding one-half of the CSB contribution (0.42±0.20 MeV) to the measured value of Eq. (14).[25]

A comparison between the values of $B_\Lambda(_\Lambda H^4)$ in Table VIII and Eq. (17) makes it evident that the case with $U_{0s}<U_{0t}$ can be ruled out. On the other hand, for the case with $U_{0s}>U_{0t}$, the calculated and experimental values are quite consistent with each other, except when the potential is of type *A* with $r_{\Lambda N}$ equal to zero. Thus, from this comparison, we obtain the following conclusions: (i) the singlet interaction is stronger than the triplet interaction, and (ii) there is likely a repulsive core in the Λ-nucleon potential, with a core radius greater than about 0.3 F. From the first conclusion, it can be immediately inferred that the spins of $_\Lambda H^3$, $_\Lambda H^4$, and $_\Lambda He^5$ are $\frac{1}{2}$, 0, and $\frac{1}{2}$, respectively,

[22] R. H. Dalitz and F. von Hippel, Phys. Letters **10**, 153 (1964).
[23] B. W. Downs and R. J. N. Phillips, Nuovo Cimento **41**, 374 (1966).
[24] The value of $(\Delta B_\Lambda)_{\text{Coulomb}}$ given here is somewhat different from that given by Downs (Ref. 34). The reason for this is that the difference in the matter radii of H^3 and He^3 (≈0.03 F) obtained by us with a variational calculation is only about half as much as that obtained by Downs with the help of the "naive model" of R. H. Dalitz and T. W. Thacker, Phys. Rev. Letters **15**, 204 (1965).
[25] This has been previously pointed out by Downs (Ref. 34).

TABLE VII. Depths of triplet and singlet Λ-nucleon potentials.

Potential type	U_{03} (MeV)	U_{05} (MeV)	$U_{0s}>U_{0t}$ U_{0s} (MeV)	U_{0t} (MeV)	s_s	s_t	$U_{0s}<U_{0t}$ U_{0s} (MeV)	U_{0t} (MeV)	s_s	s_t
A	187.1± 5.9	120.5± 3.8	220.5± 9.1	87.2± 6.4	0.715±0.030	0.283±0.021	−79.5± 23.3	187.1± 5.9	−0.258±0.076	0.607±0.019
B	657.2±11.4	543.8± 7.4	713.9±17.5	487.1±12.5	0.835±0.020	0.570±0.015	203.6± 45.2	657.2±11.4	0.238±0.053	0.769±0.013
C	1629.4±20.0	1466.0±12.2	1711.0±30.6	1384.3±20.9	0.890±0.016	0.720±0.011	976.0± 77.3	1629.4±20.0	0.508±0.040	0.848±0.010
D	7108.7±33.1	6793.2±30.9	7266.5±52.0	6635.5±47.2	0.945±0.007	0.862±0.006	5846.7±158.6	7108.7±33.1	0.760±0.021	0.924±0.004

which is in agreement with the experimental finding on the spins of these hypernuclei.[26,27]

Since the conclusion about the relative strength in the triplet and singlet states is a rather definite one, we shall in the following consider only the case where U_{0s} is greater than U_{0t}.

It is interesting to point out that because of the short-range nature of the Λ-nucleon potential, the nuclear cores in the s-shell hypernuclei are significantly compressed. The rms values of the separation distance between two nucleons, as given in Tables III, IV, and V, are about equal to 3.33, 2.54, and 2.25 F in $_{\Lambda}H^3$, $_{\Lambda}H^4$, and $_{\Lambda}He^5$, respectively. Comparing with the corresponding values of 3.84, 2.91, and 2.35 F in the free nuclei H^2, H^3, and He^4, we note that the amount of core compression is 13, 13, and 4% in these three hypernuclei, respectively. Thus, the present study shows that, in general, core compression needs to be considered in hypernuclear studies, if accurate results are desired; it is only in these cases where the nuclear core has a compressibility as low as that of the alpha particle that such compression effect may be neglected.

The values of the effective-range parameters (a_s,r_{0s},a_t,r_{0t}) of the Λ-nucleon potentials are listed in Table IX.[28] From this table, we note that the singlet parameters are relatively insensitive to the radius of the hard core, which is, however, not the case for the triplet parameters.

With U_{0s} and U_{0t} given in Table VII, we can determine the binding energy of a possible particle-stable excited state of $J=1$ for the hypernucleus $_{\Lambda}H^4$. For this state, the spin-averaged well-depth is

$$U_{04}^*=\tfrac{5}{6}U_{0t}+\tfrac{1}{6}U_{0s}. \qquad (18)$$

With U_{04}^* determined, the value of B_{Λ}^* can be computed by using Eq. (12); for potential A, B, C, and D, B_{Λ}^* turns out to be equal to 0.01, 0.08, 0.16, and 0.21 MeV, respectively.[29]

Recently, Bodmer[13,30] has suggested that the effective Λ-nucleon interaction in hypernuclei may differ from the free Λ-nucleon interaction through a suppression effect which arises from a modification of the coupling between the ΛN and the ΣN channels. This suppression is possibly quite important in $_{\Lambda}He^5$, since in this case, the virtual process $He^4+\Lambda \rightarrow He^4+\Sigma$ is forbidden because of isospin conservation and the coupling can occur only through $T=1$ states of the alpha particle, which have rather large excitation energies of more than 20 MeV.[31] For $_{\Lambda}H^3$ and $_{\Lambda}H^4$, on the other hand, one would expect that this effect should be relatively unimportant, with the consequence that the effective Λ-nucleon interaction in these hypernuclei is essentially the same as the free Λ-nucleon interaction.

From this calculation, we can estimate the importance of the suppression effect in the following manner. The scattering lengths a_s and a_t given in Table IX are obtained from the binding-energy data of $_{\Lambda}H^3$ and $_{\Lambda}He^5$ in Eq. (13). Similarly, we can calculate these quantities by using instead the binding-energy data of $_{\Lambda}H^3$ and $_{\Lambda}H^4$ in Eqs. (13) and (17). A comparison between these two

TABLE VIII. Calculated values of B_{Λ} for $_{\Lambda}H^4$.

Potential type	$U_{0s}>U_{0t}$ U_{04} (MeV)	$B_{\Lambda}(_{\Lambda}H^4)$ (MeV)	$U_{0s}<U_{0t}$ U_{04} (MeV)	$B_{\Lambda}(_{\Lambda}H^4)$ (MeV)
A	153.8± 3.5	1.76±0.23	142.7± 3.2	1.04±0.18
B	600.5± 6.8	1.88±0.25	581.6± 6.2	1.21±0.20
C	1547.7±11.7	1.92±0.27	1520.5±10.5	1.30±0.22
D	6951.0±22.7	1.94±0.29	6898.4±23.4	1.34±0.25

TABLE IX. Low-energy effective-range parameters of the Λ-nucleon potentials.

Potential type	Singlet parameters a_s (F)	r_{0s} (F)	Triplet parameters a_t (F)	r_{0t} (F)
A	−2.84±0.40	2.01±0.08	−0.47±0.05	4.59±0.35
B	−3.08±0.47	1.93±0.08	−0.60±0.06	4.26±0.31
C	−3.13±0.50	1.91±0.08	−0.71±0.06	3.74±0.26
D	−3.18±0.52	1.89±0.08	−0.78±0.07	3.43±0.23

[26] M. M. Block, R. Gessaroli, J. Kopelman, S. Ratti, M. Schneeberger, L. Grimellini, T. Kikuchi, L. Lendinara, L. Monari, W. Becker, and E. Harth, in *Proceedings of the International Conference on Hyperfragments, St. Cergue, Switzerland, 1963* (CERN, Geneva, 1964), p. 63.

[27] R. H. Dalitz and L. Liu, Phys. Rev. **116**, 1312 (1959).

[28] We wish to mention that the values of the effective-range parameters given in Table VI of Ref. 13 have not been computed quite correctly. For those cases where the Λ-nucleon potential used does not have a hard core, the correct magnitudes of the scattering lengths, for instance, are 10–15% less than the listed values. Thus, when a comparison is made with our values given here, this should be kept in mind.

[29] These values of B_{Λ}^* are computed without considering the contribution from the CSB component of the Λ-nucleon interaction.

[30] See also R. H. Dalitz, an invited paper presented at the Topical Conference on the Use of Elementary Particles in Nuclear Structure Studies, Brussels, 1965 (unpublished).

[31] J. Cerny, C. Détraz, and R. H. Pehl, Phys. Rev. Letters **15**, 300 (1965).

TABLE X. Values of a_s and a_t from different methods of computation.

	a_s (F)			a_t (F)		
Potential type	From $_\Lambda H^3$ and $_\Lambda H^4$	From $_\Lambda H^3$ and $_\Lambda He^5$	% difference	From $_\Lambda H^3$ and $_\Lambda H^4$	From $_\Lambda H^3$ and $_\Lambda He^5$	% difference
A	−2.63	−2.84	8.0	−0.60	−0.47	21.7
B	−2.89	−3.08	6.6	−0.70	−0.60	14.3
C	−2.98	−3.13	5.0	−0.79	−0.71	10.1
D	−3.07	−3.18	3.6	−0.85	−0.78	8.2

sets of values for the scattering lengths would then give a measure of the suppression effect. In Table X, we list the values of a_s and a_t obtained from these two different methods of computation. From this table, we see that the suppression effect seems to be only of minor importance, especially for those cases where the Λ-nucleon potential has a hard core of radius equal to 0.3 F or more. The two values of a_t differ by an average of about 15%, which is somewhat smaller than the amount estimated by Bodmer[13] using a rough theoretical procedure and a reasonable value for the average excitation energy of the alpha-particle $T=1$ states.

It should be emphasized that the above discussion can only yield a qualitative conclusion that the suppression effect is relatively unimportant.[32] To obtain a more quantitative estimate, it is necessary to know first the radius of the hard core. But, this latter information is exactly what we would like to obtain from this investigation. In this sense, therefore, the possible existence of a suppression effect creates a rather unfortunate situation. Without this effect, we will be able to conclude from the binding-energy data of the *s*-shell hypernuclei that a hard core with a radius greater than about 0.3 F exists in the Λ-nucleon potential. With this effect, such a conclusion can no longer be made and other means of determining the hard-core radius must be sought.

IV. Λ-NUCLEON SCATTERING

In this section, we shall compute the Λ-nucleon total cross sections yielded by the various potentials in the low-energy region. The results will be compared with the recent experimental data on Λ-proton scattering obtained by Alexander *et al.* in the c.m. energy range of 3 to 20 MeV.[33]

First, we should mention that the contribution from the CSB component in the Λ-nucleon interaction to the low-energy effective-range parameters has recently been considered by Downs.[34] He found that with the CSB interaction taken into account, the scattering lengths in the Λ-proton case are changed by about 10% and the effective ranges are almost unchanged. For a change of this magnitude, we have found by using the effective-range approximation that the corresponding change in the total cross section is only a few percent in the energy region of interest (3–20 MeV), which is much less than the percentage error in the experimental data. Thus, in this calculation, we have simply ignored the CSB contribution and compared the calculated total cross sections directly with the experimental Λ-proton cross sections.

The total cross sections (σ) as a function of c.m. energy (E) for the various potentials are shown in Fig. 1.[35] To obtain these cross sections, the central values of U_{0t} and U_{0s} given in Table VII have been used. From this figure, we see that when the Λ-nucleon potential is of type *A* without a hard core, the cross sections are considerably smaller than the experimental values. On the other hand, for potential *B*, *C*, and *D*, although the values of σ are still below σ_{exp}, the discrepancy is no longer too bad. In all three cases, the calculated points are within two standard deviations of

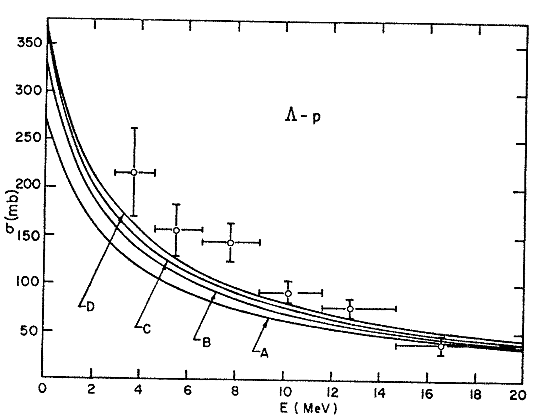

FIG. 1. Total Λ-proton elastic scattering cross section as a function of c.m. energy. For the calculated curves, the central values of U_{0t} and U_{0s} given in Table VII are used.

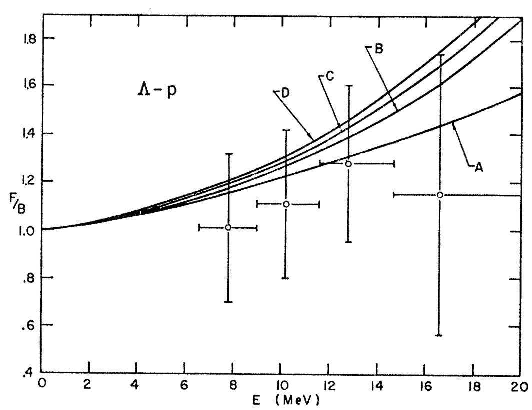

FIG. 2. Forward-to-backward ratio of Λ-proton scattering as a function of c.m. energy.

[32] To reach this conclusion, we have assumed that other effects, such as those due to tensor and three-body *ΛNN* forces, are not important.

[33] G. Alexander, O. Benary, U. Karshon, A. Shapiro, G. Yekutieli, R. Engelmann, H. Filthuth, A. Fridman, and B. Schiby, Phys. Letters **19**, 715 (1966).

[34] B. W. Downs, Nuovo Cimento **43**, 459 (1966).

[35] As in our previous calculation (Ref. 11), we have computed triplet and singlet phase shifts up to δ_3.

the experimental points. Thus, from this comparison, even though a definitive conclusion cannot be prudently made, we do obtain a strong indication that a hard core very likely exists in the Λ-nucleon potential, with a radius greater than about 0.3 F.

Also, it is seen from Fig. 1 that the experimental cross sections seem to fall off more rapidly with increasing energy than do the calculated curves. This suggests that the intrinsic range of 1.5 F used in this investigation is possibly too small, and larger values of 2.0 and 2.5 F might very well be more appropriate.

The behavior of the forward-to-backward ratio (F/B) is shown in Fig. 2 for the various potentials. From this figure, we see that the calculated values increase with energy, which is due to the assumption of an ordinary nature for the Λ-nucleon interaction. Experimentally, the data are too crude to confirm this definitely, but there does seem to be a rising tendency.[33]

To make sure that the conclusion made above about the size of the hard core in the Λ-nucleon potential is reasonable, we have examined the uncertainty in σ due to the uncertainties associated with U_{03} and U_{05}. What we do is to use the upper limits of U_{03} and U_{05} to compute U_{0s} and U_{0t} and then calculate σ. The results are shown in Fig. 3, where again we find that potential B, C, and D yield acceptable fits to the experimental data, while potential A does not.

Also, it is necessary to examine how the suppression effect discussed in Sec. III might affect the values of the total cross section. For this purpose, we shall calculate σ for a number of cases where U_{0t} is increased over that given in Table VII, while at the same time keeping U_{03} fixed. This has been done for all the potentials considered, but, for simplicity, we shall give here only the result for the case with potential C. The various combinations of U_{0s} and U_{0t} which have been examined are listed in Table XI and the result is shown in Fig. 4. Here, we note that the change in σ is relatively unimportant even for an increase in U_{0t} by 4%, which is a

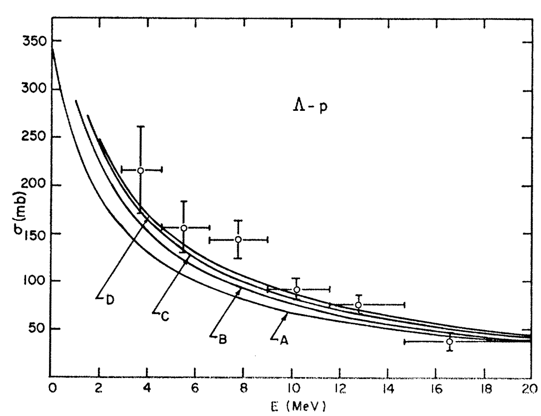

FIG. 3. Total Λ-proton elastic scattering cross section as a function of c.m. energy. For the calculated curves, the values of U_{0t} and U_{0s} used are computed from the upper limits of U_{03} and U_{05}.

TABLE XI. Potential combinations to study the influence of the suppression effect.

Potential type	U_{0s} (MeV)	U_{0t} (MeV)	% increase in U_{0t}
C	1711.0	1384.3	0
$C1$	1692.6	1439.7	4
$C2$	1674.1	1495.1	8
$C3$	1655.7	1550.5	12

rather large amount, since the value of $B_\Lambda({}_\Lambda H^4)$ corresponding to this particular combination of U_{0s} and U_{0t} (i.e., potential $C1$), being 2.41 MeV, is already larger than the value given in Eq. (17).[36] Thus, we conclude from this study that with the type of Λ-nucleon potentials considered here the suppression effect does not seem to be important in a calculation on Λ-nucleon scattering cross sections.

V. CONCLUSION

In this investigation, the binding-energy data of the s-shell hypernuclei and the total cross sections of the Λ-nucleon scattering have been examined with two-body, spin-dependent, charge-independent, central Λ-nucleon potentials which have an intrinsic range of 1.5 F and a hard core with a radius ranging from 0 to 0.6 F. The main purpose is to find out whether there exists a hard core in the Λ-nucleon interaction. The results show that even though it is not possible to make a definitive conclusion from this study, there is still a strong indication that a hard core does exist and has a radius greater than about 0.3 F.

It is interesting to note that the above conclusion agrees with the result which we have recently obtained from an analysis of the hypernuclei ${}_\Lambda Be^9$ and ${}_\Lambda C^{13}$.[37] In this latter analysis, it was found that to obtain agreement with the experimentally determined values of

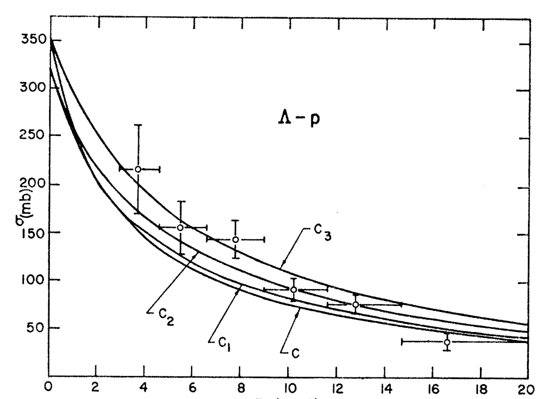

FIG. 4. Total Λ-proton elastic scattering cross section as a function of c.m. energy. The combinations of U_{0t} and U_{0s} used are given in Table XI.

[36] For potential $C2$ and $C3$, the values of $B_\Lambda({}_\Lambda H^4)$ are 2.96 and 3.56 MeV, respectively.

[37] R. C. Herndon and Y. C. Tang, Phys. Rev. **149**, 735 (1966).

B_Λ, the Λ-nucleon potential employed should have an attractive part with an intrinsic range less than about 1 F. In this respect, we should point out that the Λ-nucleon potentials used in this investivation have an intrinsic range for the attractive part less than this value when they have a hard-core radius greater than 0.3 F.

This study also shows that the hypernuclear results do have a fairly sensitive dependence on the radius of the hard core in the Λ-nucleon potential. The total cross section of the Λ-nucleon scattering, for example, increases by about 30% when the radius is increased from 0 to 0.6 F.

Also, we have found that when the hard-core radius is less than or equal to 0.6 F, a hyperdeuteron does not exist. Since this value (0.6 F) represents very likely an upper limit for the core radius, we can quite definitely conclude that the Λ-nucleon interaction is not strong enough to bind a two-body Λ-hypernuclear system.

From our calculation, it appears that there is a particle-stable excited state for the hypernucleus $_\Lambda H^4$, but its binding energy is rather small. Even in the most favorable case when the Λ-nucleon potential has a core radius of 0.6 F, the value of B_Λ^* is only about 0.2 MeV.[38]

The trial wave function used here has been assumed to be totally space symmetric with respect to the nucleon coordinates. In particular, the effect of S'-state mixing in $_\Lambda H^3$, as discussed by Bodmer,[13] has not been taken into consideration. This can inject some uncertainty into our results, but we do not feel that the uncertainty is a large one. In our investigation, all the Λ-nucleon potentials which have been found to be of interest, i.e., those with a hard-core radius greater than 0.3 F, have an attractive part with an intrinsic range similar to or shorter than that of the K-meson exchange. From Bodmer's calculation, it has indeed been found that when the Λ-nucleon potential has such a short range, the effect of S'-state mixing is quite insignificant.

A comparison between the calculated and experimental values for the Λ-nucleon total cross sections gives an indication that the intrinsic range of 1.5 F adopted here is probably too small and larger values of 2.0 or 2.5 F might be more appropriate.[39] This is presently being investigated in detail, and the results will be given in a forthcoming publication.

APPENDIX A: THE INTERPOLATION FORMULA

In this appendix, it will be shown that for a short-range potential $-Vf(r)$ which has a bound s state with a small binding energy B, the two quantities V and B are approximately related by the equation

$$V=V_0+cB^{1/2}, \tag{A1}$$

where c is a constant depending on the shape of the potential and V_0 is the depth required to form a bound state of zero binding.

To show this, we write down the Schrödinger equations

$$\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+Vf(r)u(r)=Bu(r), \tag{A2}$$

and

$$\frac{\hbar^2}{2m}\frac{d^2u_0}{dr^2}+V_0f(r)u_0(r)=0. \tag{A3}$$

Multiplying Eq. (A2) by u_0 and Eq. (A3) by u and subtracting, we obtain

$$\frac{\hbar^2}{2m}\frac{d}{dr}\left(u_0\frac{du}{dr}-u\frac{du_0}{dr}\right)+(V-V_0)f(r)uu_0=Buu_0. \tag{A4}$$

After integrating over r from zero to infinity and applying the appropriate boundary conditions, the above equation becomes

$$(V-V_0)\int_0^\infty f(r)uu_0dr=B\int_0^\infty uu_0dr. \tag{A5}$$

Under the condition that B is very much smaller than V, the function $u(r)$ is almost the same as $u_0(r)$ inside the potential well; thus, the left-hand side of Eq. (A5) is approximately equal to

$$(V-V_0)\int_0^\infty f(r)u_0^2dr,$$

where the important point to note is that the factor multiplying $(V-V_0)$ does not depend on V but only on the shape factor $f(r)$. The right-hand side of Eq. (A5) can be handled in the following manner. For $r<d$, where d is the range of the potential, both u and u_0 will behave roughly like $\sin(\pi r/2d)$. For $r>d$, the function $u_0(r)$ is equal to 1 and $u(r)$ is approximately given by

$$u(r)=\exp[\alpha(d-r)], \tag{A6}$$

with

$$\alpha=(2mB/\hbar^2)^{1/2}. \tag{A7}$$

Using these forms of $u(r)$ and $u_0(r)$ in the right-hand side of Eq. (A5), the result is

$$B\int_0^\infty uu_0dr=B\left(\frac{d}{2}+\frac{1}{\alpha}\right), \tag{A8}$$

which is nearly equal to B/α if

$$\tfrac{1}{2}\alpha d\ll 1. \tag{A9}$$

[38] In our calculation, a number of small effects, such as that due to charge asymmetry in the Λ-nucleon interaction (see Ref. 34), have not been taken into account. For this reason, we feel that the values of B_Λ^* determined here should be considered as having only semiquantitative significance.

[39] Similar remark has also been made by B. W. Downs and R. J. N. Phillips, Nuovo Cimento 36, 120 (1965), and by Dalitz (Ref. 30).

Thus, Eq. (A5) is reduced to

$$(V-V_0)\int_0^\infty f(r)u_0^2dr=B/\alpha=\left(\frac{h^2}{2m}\right)^{1/2}B^{1/2}, \quad \text{(A10)}$$

which has the form of Eq. (A1).

For the s-shell hypernuclei, the use of Eq. (12) as an interpolation formula is based on the following observations: (1) the Λ particle can be approximately regarded as moving in a potential well created by its interaction with the individual nucleons, with the depth of the well determined by the strength of the Λ-nucleon interaction, and (2) the condition expressed by Eq. (A9) is fairly well satisfied. There is a slight complication arising from the fact that the shape of the well depends somewhat on the depth of the Λ-nucleon potential, but we do not think that this can seriously affect the results obtained by using Eq. (12). In any case, we have taken the extra precaution of always choosing one value of U_{0A} which yields a value for B_Λ close to that determined experimentally.

There is also another piece of evidence which shows that a two-parameter interpolation formula is quite sufficient for the s-shell hypernuclei. In Ref. 11, we have used a more careful procedure involving three values of U_{0A} and a three-parameter interpolation formula. But, this was later found to be unnecessary, since a two-parameter formula would have yielded very nearly the same results as that from a three-parameter formula, if the values of U_{0A} are chosen properly.

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Nature of Hartree-Fock Calculations in Light Nuclei*

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The role of the two-body force, its exchange mixture, and the spin-orbit force in their effect on the Hartree-Fock wavefunctions and spectra is investigated. It is shown that the main features of the Hartree-Fock single-particle field are determined almost completely by the long-range part of the two-body force. The solutions for a long-range model are derived for various systems of different neutron excesses, and the exchange dependence of the energy "gap" between occupied and unoccupied levels is particularly considered. The main effect of the spin-orbit force and the finite range of the two-body force is to mix the orbitals. In the cases where the energy "gap" is large, the mixing is only of the occupied orbitals among themselves. Out of this study it emerges that the most natural representation for the Hartree-Fock single-particle orbitals is that associated with the axially symmetric deformed harmonic oscillator where one takes linear combinations of degenerate orbitals which are time-reversal eigenstates. This prescription results often in nonaxially-symmetric nuclei and is consistent with the results found in exact calculations with realistic forces.

I. INTRODUCTION

In recent years the method of self-consistent deformed orbitals has been successfully applied to various nuclear structure problems. In particular, there now exists a number of papers[1–5] dealing with the rotational and vibrational aspects of the low-lying spectra of nuclei in the $1p$ and $2s$, $1d$ shells. Intershell problems[6–9] such as the O^{16} spectrum involving $1p$ holes and $2s$, $1d$ particles as well as the dipole giant resonances involving the $1p$, $2s$, $1d$, and $2p$, $1f$ shells have also been treated by the method of deformed orbitals.

The success of the above calculations certainly indicate that the underlying Hartree-Fock (HF) approximation has considerable validity in light nuclei. It is the purpose of this paper to discuss the main physical features of these calculations and to investigate the role of the two-body force and its exchange mixture and the spin-orbit force in their effect on the Hartree-Fock wave functions and spectra. Usually these points are obscured

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