as strong as the 0^+ state on this assumption, due to the number of available neutrons. The experimental ratio of spectroscopic factors is 1.75 ± 0.45, which agrees within the limits of error.

If the ground state of C^{13} were deformed, the presence of a $d_{5/2}^2$ configuration would not be so surprising. For values of Nilsson's deformation parameter $\delta>0.3$, the $K=\frac{1}{2}$ member of the $d_{5/2}$ band is brought down quite far from the level at $\delta=0$; practically degenerate with the $p_{1/2}$ level.[25]

The 10.84 MeV, 1^- and 11.83 MeV, (1^-) States

Both of these states are excited by $s_{1/2}$ pickup, and are wide. The agreement with the DWBA predictions was poor for the 10.84 MeV state, but $l=0$ gave the best fit. One possible reason for the failure of DWBA for these two states lies in their short lifetime which is of the same order of magnitude as the interaction time.

It seems improbable that these states would be formed by pickup of a $1s_{1/2}$ neutron, since the corresponding particle-hole excitation energy found by Vinh-Mau and Brown[22] is 31 MeV. It appears more likely that these states are excited through a configuration of the type $p_{3/2}^{-2}\,2s_{1/2}^2p_{1/2}$.

[25] S. G. Nilsson and B. R. Mottelson, Kgl. Danske Vidensk Selskab, Mat. Fys. Skrifter 1, No. 8 (1959).

ACKNOWLEDGMENTS

We would like to express our appreciation for the assistance rendered by Dr. R. H. Bassel and the Oak Ridge National Laboratory in obtaining the optical-model fits and the DWBA predictions. We are also grateful to Dr. C. M. Fou for help in data-taking and analysis. James Joyce and William Focht, as well as the operating staff of the Tandem Accelerator Laboratory, deserve mention for their help in obtaining the data.

PHYSICAL REVIEW VOLUME 152, NUMBER 3 16 DECEMBER 1966

Excitation of Generalized Giant Collective Multipole States by Electron Scattering

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(Received 31 May 1966; revised manuscript received 28 July 1966)

A new collective model for finite nuclei is presented which treats all of the four types of excitations of nuclear matter in arbitrary orbital-angular-momentum states. The model avoids the normal assumption of a constant ground-state density by generating the collective motion through a coordinate-scale-factor transformation of the ground-state density distribution. The cross sections for excitation of these generalized collective states by inelastic electron scattering are calculated for monopole and quadrupole oscillations. A specific application to the 180° inelastic scattering of electrons from ^{16}O is given. It is shown that the model exhausts the corresponding multipole sum rules.

I. INTRODUCTION

THE nuclear photoeffect is strikingly dominated[1] by the giant electric dipole resonance which was first described by Goldhaber and Teller[2] as a dipole oscillation of the protons as a whole against the neutrons as a whole in the nucleus. If one considers nuclear matter as made up of four interacting fluids, spin-up and -down protons and spin-up and -down neutrons, three other types of "normal modes" are possible.[3] These three are the *compressional mode*, all four fluids in phase, the *spin mode*, spin-up nucleons against spin-down nucleons, and the *spin-isospin* mode, spin-up protons and spin-down neutrons against the other two fluids. Following this nomenclature, we shall designate the Goldhaber-Teller mode as the *isospin* mode.

A phenomenological quantized oscillator model of the

* Supported in part by a grant from the U. S. National Science Foundation.
† Supported in part by a grant from the U. S. Air Force Office of Scientific Research.

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isospin dipole state has been introduced by Walecka,[4] and was later generalized by Überall[5] (and independently by Lewis[6]) to include the spin-isospin dipole states. In a O^+, $T=0$ nucleus, the latter oscillation gives rise to 2^-, 1^-, or 0^- states (the excited state spin is 1) which are excited by the inelastic backscattering of electrons[7] and play an important role in μ^- capture[8,9] and in neutrino absorption[10] by complex nuclei. Recently, the 20.4-MeV state in ^{4}He has been identified as a compressional monopole state,[11] and the oscillator model was further generalized to describe this vibration. The Frankfurt group has considered isospin monopole and quadrupole[12] states and found some evidence for the latter.[13] It is clear that the number of different low-lying collective states can be quite large.

It is the purpose of this paper to completely generalize the oscillator model and to consider collective vibrations of the nuclear fluids in general multipole states. For each multipole the four modes give rise to states with the given L value, two with $S=0$ [compressional ($T=0$) and isospin ($T=1$)] and two with $S=1$ [spin ($T=0$) and spin-isospin ($T=1$)]. The spin-orbit splitting which occurs in nature removes the J degeneracy of the latter pair of states and, in general, one has eight collective states for each multipole.

The multipolarity of the electromagnetic transition connecting the collective state to the ground state should not be confused with the multipolarity of the nuclear fluid vibration. Since we are dealing only with $T=0$ nuclei with 0^+ ground states, a complete description of each state is obtained by specifying (a) the angular momentum and parity of the state, (b) which of the four collective modes of nuclear matter it represents, and (c) the multipole order of the fluid oscillations. For example, the familiar giant dipole state is termed the "1^- isospin dipole" while the three collective states of the spin-isospin mode with $L=2$ are the "$1^+(2^+,3^+)$ spin-isospin quadrupoles."

On the following pages we present, besides the density matrices describing oscillations with general L, explicit cross sections for the excitation of monopole and quadrupole oscillations of the four modes by inelastic electron scattering. (The dipole states have been treated already.[4,5]) A comparison with experiment is made for ^{16}O, for which purpose the energies and widths of the states were taken from particle-hole calculations.[14,15]

II. COLLECTIVE STATE-DESCRIPTION

The goal of any nuclear theory is the calculation of matrix elements of single-particle operators, O_j, which generate transitions between initial and final states,

$$M=\langle j_f,m_f,\alpha_f|\sum_{j=1}^{A}O_j|j_i,m_i,\gamma_i\rangle$$
$$=\mathrm{tr}\{\sum_{j=1}^{A}O_j|j_i,m_i,\alpha_i\rangle\langle j_f,m_f,\alpha_f|\}. \quad (1)$$

Here the trace is taken over the spin, isospin, and spatial coordinates of all A nucleons, and α represents quantum numbers other than the total angular momentum j which designate a state. If the final state is a collective state derived from the initial state one makes a phenomenological theory of the process represented in Eq. (1) by replacing $|j_i,m_i,\alpha_i\rangle\langle j_f,m_f,\alpha_f|$ by a density matrix $\Phi(j_f,m_f,\alpha_f)$ in spin, isospin, and configuration space:

$$M=\mathrm{tr}\{O\Phi(j_f,m_f,\alpha_f)\}. \quad (2)$$

The trace is taken with respect to a single spin, isospin, and space variable. In general, Φ is derived from the ground state density by a time-dependent displacement of some sort and is linear in a displacement parameter. If an oscillator Hamiltonian is defined in terms of the displacement and its conjugate momentum, the oscillation can be quantized and Φ becomes a combination of creation and annihilation operators. Then the matrix element M is defined by

$$M=\langle n\omega|\,\mathrm{tr}\{O\Phi(j_f,m_f,\alpha_f)\}\,|0\rangle, \quad (3)$$

where $|0\rangle$ is the vacuum state and $n\omega$ designates the number and frequency of oscillation quanta in the final state.

The density matrices Φ for the four modes can be obtained from the ground-state density distribution $\rho_0(\mathbf{r})$ in a straightforward way. In this derivation we assume (1) the ground-state matter distribution $\rho_0(\mathbf{r})$ is spherically symmetric and represents a nucleus with $Z=N=A/2$, (2) L and S are good quantum numbers, and (3) for given L and S, there is J degeneracy. This latter assumption will be given up in the applications.

Our ansatz is that the multipole oscillations are introduced by making a scale-factor transformation[16] of the ground-state density distribution, ρ_0 ,

$$\rho_0(r)\rightarrow\rho_0(r(1-\varphi))=\rho_0(r)+\rho'(\mathbf{r}),$$
$$\rho'(\mathbf{r})=-r\varphi(d/dr)\rho_0(r). \quad (4a)$$

[4] J. Goldemberg, Y. Torizuka, W. C. Barber, and J. D. Walecka, Nucl. Phys. **43**, 242, (1963).
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In the special case that ρ' represents a monopole vibration, conservation of matter requires that Eq. (4a) be modified to

$$\rho'(r)=-(1/r^2)\varphi(d/dr)r^3\rho_0(r). \tag{4b}$$

The scale factor φ is expanded in a multipole series

$$\varphi=\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\begin{cases}\kappa=0, & l=0\\ \kappa=2, & l>0\end{cases} \tag{5}$$

where the expansion coefficients α_{lm} are to become operators. To describe the four different modes the excess density must be a matrix in spin and isospin space as well as a function in coordinate space. Consider the isospin mode in which the protons are 180° out of phase with the neutrons. The density matrix Φ_i must take the form

$$\Phi_i=\tfrac{1}{4}A\tau_z\rho' \tag{6}$$

because it satisfies

$$\begin{aligned}\mathrm{tr}\{\tfrac{1}{2}(1+\tau_z)\Phi_i\}&=-\mathrm{tr}\{\tfrac{1}{2}(1-\tau_z)\Phi_i\}\\&=\tfrac{1}{2}A\rho'\end{aligned} \tag{7a}$$

and

$$\mathrm{tr}\{\tfrac{1}{2}(1+\sigma_z)\Phi_i\}=\mathrm{tr}\{\tfrac{1}{2}(1-\sigma_2)\Phi_i\}=0. \tag{7b}$$

In the above equation the trace is taken only over spin and isospin space. Similar considerations for the other three modes lead to the fundamental expressions for the compressional, spin, isospin, and spin-isospin density matrices, namely,

$$\begin{aligned}\Phi_c&=-\tfrac{1}{4}Ar\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\\ \Phi_s&=-\tfrac{1}{4}A\sigma_{m'}r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\\ \Phi_i&=-\tfrac{1}{4}A\tau_z r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\\ \Phi_{s-i}&=-\tfrac{1}{4}A\tau_z\sigma_{m'}r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}).\end{aligned} \tag{8}$$

The subscript m' to the spin matrix σ denotes the z projection of the total spin, ±1, 0.

Electrons are scattered by the transition charge, current and magnetic moment densities inherent in the Φ's of Eq. (8). The charge, $\rho(\mathbf{r})$, and magnetic moment $\boldsymbol{\mu}(\mathbf{r})$, densities are given by

$$\begin{aligned}\rho(\mathbf{r})&=\mathrm{tr}\{\tfrac{1}{2}(1+\tau_z)\Phi\},\\ \mu_j(\mathbf{r})&=\frac{1}{2m}\mathrm{tr}\{[\mu_p(\tfrac{1}{2}(1+\tau_z))+\mu_n(\tfrac{1}{2}(1-\tau_z))]\sigma_j\Phi\},\end{aligned} \tag{9}$$

where μ_p and μ_n are the magnetic moments of the proton and neutron in units of nuclear magnetons, $1/2m$. Using the expressions in Eqs. (5), (8), and (9) we find for the charge distributions

$$\begin{aligned}\rho_c(\mathbf{r})=\rho_i(\mathbf{r})&=-\frac{A}{2}r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\\ \rho_s(\mathbf{r})=\rho_{s-i}(\mathbf{r})&=0,\end{aligned} \tag{10a}$$

and for the magnetic moment distributions

$$\begin{aligned}\mu_{cj}(\mathbf{r})&=\mu_{ij}(\mathbf{r})=0,\\ \mu_{sj}(\mathbf{r})&=-\frac{A}{2m}\left(\frac{\mu_p+\mu_n}{2}\right)\delta_{jm'}r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}),\\ \mu_{s-i,j}(\mathbf{r})&=-\frac{A}{2m}\left(\frac{\mu_p-\mu_n}{2}\right)\delta_{jm'}r\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\alpha_{lm}Y_{lm}(\hat{r}).\end{aligned} \tag{10b}$$

The current density can be found if an expression for the velocity $\mathbf{v}$ is obtained which satisfies the equation of continuity

$$\dot{\rho}'=-\nabla\cdot(\mathbf{v}\rho)\cong-\nabla\cdot(\mathbf{v}\rho_0), \tag{11}$$

where

$$\begin{aligned}\dot{\rho}'&=-r\dot{\varphi}\frac{d}{dr}\rho_0(r)\\ &=-\frac{r}{\mu}\frac{d}{dr}\rho_0\sum_{l,m}(r/R)^{l-\kappa}\beta_{l,m}Y_{lm}(\hat{r}).\end{aligned}$$

We have introduced $\beta_{lm}=\mu\dot{\alpha}_{lm}$ which upon quantization of our oscillator will become the momentum conjugate to α_{lm}. We obtain the velocity be expanding $\mathbf{v}$ in a complete set of spherical vector harmonics[17] $\mathbf{Y}_{lL}{}^m$:

$$\mathbf{v}=\sum_{l,m,L}\frac{1}{\mu}\beta_{lm}A_{lL}\frac{r^{l-\kappa+1}}{R^{l-\kappa}}\mathbf{Y}_{lL}{}^m(\hat{r}), \tag{12}$$

and determine the expansion coefficients A_{lL} such that Eq. (11) is satisfied. It is shown in the Appendix that there is only one nonzero coefficient for a given l, that for $L=l-1$. Its value is

$$A_{l-}=\left(\frac{2l+1}{l}\right)^{1/2}. \tag{13}$$

For monopole vibrations

$$\mathbf{v}=(\beta_{00}/\mu)\mathbf{r}. \tag{14}$$

Vector density matrices $\mathbf{F}$ describing the matter flux densities for the four modes are constructed as in

[17] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

Eq. (8),

$$\begin{aligned}\mathbf{F}_c&=-\tfrac{1}{4}A\mathbf{v}\rho_0,\\ \mathbf{F}_s&=-\tfrac{1}{4}A\sigma_{m'}\mathbf{v}\rho_0,\\ \mathbf{F}_i&=-\tfrac{1}{4}A\tau_z\mathbf{v}\rho_0,\\ F_{s-i}&=-\tfrac{1}{4}A\tau_z\sigma_{m'}\mathbf{v}\rho_0.\end{aligned}\tag{15}$$

Current densities $\mathbf{j}(\mathbf{r})$ are obtained from

$$\mathbf{j}=\mathrm{tr}\{\tfrac{1}{2}(1+\tau_z)\mathbf{F}\}.\tag{16}$$

The results are

$$\begin{aligned}\mathbf{j}_c(\mathbf{r})&=\mathbf{j}_i(\mathbf{r})=-\tfrac{1}{2}A\mathbf{v}\rho_0,\\ \mathbf{j}_s(\mathbf{r})&=\mathbf{j}_{s-i}(\mathbf{r})=0.\end{aligned}\tag{17}$$

The magnitudes of matrix elements of the charge, current, and magnetization densities are fixed upon quantization of the displacements α_{lm} in Eq. (5). Because of the orthogonality of the $Y_{lm}(\hat{r})$, the oscillator Hamiltonian can be written as

$$H=\sum_{l,m}\frac{1}{2\mu}\beta_{lm}\dagger\beta_{lm}+\sum_{l,m}\tfrac{1}{2}\mu\omega^2\alpha_{lm}\dagger\alpha_{lm},\tag{18}$$

where α_{lm} and β_{lm} are defined in terms of creation and annihilation operators $a_{lm}\dagger$ and a_{lm} by

$$\begin{aligned}\alpha_{lm}&=\left(\frac{1}{2\mu\omega_l}\right)^{1/2}[a_{lm}\dagger+(-1)^m a_{l-m}],\\ \beta_{lm}&=i\left(\frac{\mu\omega_l}{2}\right)^{1/2}[a_{lm}-(-1)^m a_{l-m}\dagger].\end{aligned}\tag{19}$$

As usual, the a_{lm}'s obey the commutation rules

$$[a_{lm},a_{l'm'}\dagger]=\delta_{ll'}\delta_{mm'}.\tag{20}$$

The frequency ω_l corresponds to the energy assigned to the state and the effective mass μ is determined by the theory. The procedure is to calculate the kinetic energy of the oscillation using the kinetic-energy density implied by the velocity $\mathbf{v}$ in Eq. (12) and to equate this to the kinetic energy part of the Hamiltonian in Eq. (18). Thus

$$\begin{aligned}K&\cong\tfrac{1}{2}mA\int d\mathbf{r}\,v^2\rho_0(r)\\ &=\tfrac{1}{2}mA\sum_{\substack{l,m,L\\ l',m',L'}}\frac{1}{\mu^2}\beta_{l,m}\beta_{l'm'}A_{lL}A_{l'L'}\int\frac{r^{2l-2\kappa+4}}{R^{2l-2}}\rho_0(r)\,dr\\ &\qquad\times\int d\Omega\mathbf{Y}_{lL}{}^m\cdot\mathbf{Y}_{l'L'}{}^{m'}\\ &=\sum_{l,m}\frac{mA}{2\mu^2}\frac{\langle r^{2l-2\kappa+2}\rangle}{4\pi R^{2l-2\kappa}}(-1)^m\beta_{lm}\beta_{l-m}\sum_L A_{lL}{}^2,\end{aligned}\tag{21}$$

with R the rms radius of the charge distribution, $\langle r^n\rangle$ the nth moment of the charge distribution, and where the orthogonality of the $\mathbf{Y}_{lL}{}^m$ has been used. We see that the reduced mass becomes,[16] after using the relation $\beta_{lm}\dagger=\beta_{l-m}$,

$$\mu=\frac{2l+1}{l}\frac{Am}{4\pi}\frac{\langle r^{2l-2\kappa+2}\rangle}{R^{2l-2\kappa}}.\tag{22}$$

Hence, for l, m;

$$\langle 1\omega_l|\alpha_{lm}|0\rangle=\frac{R^{l-\kappa}}{\langle r^{2l-2\kappa+2}\rangle}\left(\frac{2\pi l}{(2l+1)mA\omega_l}\right)^{1/2}.\tag{23}$$

III. ELECTRON-SCATTERING CROSS SECTIONS

For sufficiently light nuclei the Born approximation can be used to calculate the cross section for inelastic electron scattering in which collective states of the nucleus are excited. The cross section is[18]

$$\begin{aligned}\frac{d\sigma}{d\Omega}=\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}\Big[&V_L(\theta)\sum_{J=0}^{\infty}\frac{1}{2J_i+1}|\langle J_f\|M_J(q)\|J_i\rangle|^2\\ &+V_T(\theta)\sum_{J=1}^{\infty}\frac{1}{2J_i+1}(|\langle J_q\|T_J{}^e(q)\|J_i\rangle|^2\\ &+|\langle J_q\|T_J{}^m(q)\|J_i\rangle|^2)\Big].\end{aligned}\tag{24}$$

In the above equation $\mathbf{k}_1$ and $\mathbf{k}_2$ are the incident and scattered momenta (the electron mass is neglected, as is nuclear recoil), $\alpha=1/137$, $\mathbf{q}=\mathbf{k}_2-\mathbf{k}_1$, $\Delta^2=q^2-(k_2-k_1)^2$, θ is the electron scattering angle, and the two angular functions $V_L(\theta)$ and $V_T(\theta)$ are given by

$$\begin{aligned}V_L(\theta)&=2k_1k_2(\Delta^4/q^4)\cos^2(\theta/2),\\ V_T(\theta)&=(2k_1k_2/q^2)[\sin^2(\theta/2)]\\ &\quad\times[(k_1+k_2)^2-2k_1k_2\cos^2(\theta/2)].\end{aligned}\tag{25a}$$

The reduced matrix elements are those of the operators

$$M_{JM}=\int d\mathbf{r}\,\rho(\mathbf{r})j_J(qr)Y_{JM}(\hat{r}),\tag{24a}$$

$$T_{JM}{}^{ej}=q^{-1}\int d\mathbf{r}\,\mathbf{j}(\mathbf{r})\cdot\nabla\times j_J(qr)\mathbf{Y}_{JJ}{}^M(\hat{r}),\tag{24b}$$

$$T_{JM}{}^{mj}=\int d\mathbf{r}\,j_J(qr)\mathbf{Y}_{JJ}{}^M(\hat{r})\cdot\mathbf{j}(\mathbf{r}),\tag{24c}$$

$$T_{JM}{}^{e\mu}=q\int d\mathbf{r}\,j_J(qr)\mathbf{Y}_{JJ}{}^M(\hat{r})\cdot\boldsymbol{\mu}(\mathbf{r}),\tag{24d}$$

$$T_{JM}{}^{m\mu}=\int d\mathbf{r}\,\boldsymbol{\mu}(\mathbf{r})\cdot\nabla\times j_J(qr)\mathbf{Y}_{JJ}{}^M(\hat{r}).\tag{24e}$$

M_{JM} is the Coulomb multipole operator while $T_{JM}{}^e$, and $T_{JM}{}^m$ are the transverse electric and magnetic multipole operators; $e\rho(\mathbf{r})$, $e\mathbf{j}(\mathbf{r})$, and $e\boldsymbol{\mu}(\mathbf{r})$ are the

[18] F. H. Lewis, Jr., and J. D. Walecka, Phys. Rev. 133, B849 (1964).

nucleon charge, current, and magnetic moment densities. The reduced matrix elements are defined by the Wigner-Eckart theorem[17] as

$$(2J_f+1)^{1/2}\langle J_fM_f|\mathfrak{M}_{JM}|J_iM_i\rangle = (J_iM_iJM|J_fM_f)\langle J_f\|M_J\|J_i\rangle. \quad (25b)$$

In the giant dipole region of nuclear levels, one can expect that states representing all four modes of nuclear excitation will occur and that monopole and quadrupole vibration will be present as well as dipole. For this reason we have calculated the matrix elements and cross sections for excitation of all those states which have a monopole or quadrupole spatial dependence. (The dipole states have been considered in earlier work). Using Eqs. (10) and (17) for the nuclear densities, the following nonzero reduced matrix elements are obtained:

Monopole

$$0^+; \quad \langle 0\|M_0\|0\rangle_c=\langle 0\|M_0\|0\rangle_i = \frac{1}{(4\pi)^{1/2}R}\left(\frac{A}{8m\omega}\right)^{1/2} qF'(q); \quad (26a)$$

$$1^+; \quad \langle 0\|T_1{}^{m\mu}\|0\rangle_{s-i}=\frac{\mu_p-\mu_n}{2im}\frac{1}{(4\pi)^{1/2}R} \times\left(\frac{A}{4m\omega}\right)^{1/2} q^2F'(q). \quad (26b)$$

Quadrupole

$$2^+; \quad \langle 2\|M_2\|0\rangle_c=\langle 2\|M_2\|0\rangle_i = -\frac{1}{(4\pi)^{1/2}R}\left(\frac{A}{4m\omega}\right)^{1/2} qF'(q), \quad (26c)$$

$$\langle 2\|T_2{}^{ej}\|0\rangle_c=\langle 2\|T_2{}^{ej}\|0\rangle_i = \frac{\omega}{(4\pi)^{1/2}R}\left(\frac{3A}{8m\omega}\right)^{1/2} F'(q); \quad (26d)$$

$$1^+; \quad \langle 1\|T_1{}^{m\mu}\|0\rangle_{s-i}=\frac{\mu_p-\mu_n}{2im}\frac{1}{(4\pi)^{1/2}R} \times\left(\frac{A}{20m\omega}\right)^{1/2} q^2F'(q); \quad (26e)$$

$$2^+; \quad \langle 2\|\mathrm{T}_2{}^{e\mu}\|0\rangle_{s-i}= \frac{\mu_p-\mu_n}{2m}\frac{1}{(4\pi)^{1/2}R} \times\left(\frac{A}{4m\omega}\right)^{1/2} q^2F'(q), \quad (26f)$$

$$3^+; \quad \langle 3\|T_3{}^{m\mu}\|0\rangle_{s-i}=-\frac{(\mu_p-\mu_n)}{2mi}\frac{1}{(4\pi)^{1/2}R} \times\left(\frac{\mathrm{A}}{5m\omega}\right)^{1/2} q^2F'(q). \quad (26g)$$

Matrix elements for the spin-wave excitations are not listed but are the same as those for the spin-isospin states except for a factor of $\mu_p+\mu_n$ rather than $\mu_p-\mu_n$. The smallness of the square of this factor makes them unobservable in electron scattering. The frequencies ω in the above equations refer to the observed energies and are, of course, different for each of the eight states listed above. In all cases $F'(q)$ is the derivative with respect to q of the elastic form factor normalized so that $F(0)=1$.

The cross sections for the states are as follows:

Monopole

$$0^+; \quad \left(\frac{d\sigma}{d\Omega}\right)_c=\left(\frac{d\sigma}{d\Omega}\right)_i=\frac{k_2}{k_1}\frac{\alpha^2}{\Delta^4}\frac{A[F'(q)]^2}{4R^2m\omega}q^2V_L(\theta); \quad (27a)$$

$$1^+; \quad \left(\frac{d\sigma}{d\Omega}\right)_{s-i}=\frac{k_2}{k_1}\frac{\alpha^2}{\Delta^4}\frac{A[F'(q)]^2}{2R^2m\omega} \times\left(\frac{\mu_p-\mu_n}{2}\right)^2\frac{q^4}{m^2}V_T(\theta). \quad (27b)$$

Quadrupole

$$2^+; \quad \left(\frac{d\sigma}{d\Omega}\right)_c=\left(\frac{d\sigma}{d\Omega}\right)_i=\frac{k_2}{k_1}\frac{\alpha^2}{\Delta^4}\frac{A[F'(q)]^2}{2R^2m\omega} \times[q^2V_L(\theta)+\tfrac{3}{2}\omega^2V_T(\theta)]; \quad (28a)$$

$$\left\{\begin{matrix}1^+\\2^+\\3^+\end{matrix}\right\}; \quad \left(\frac{d\sigma}{d\Omega}\right)_{s-i}=\frac{k_2}{k_1}\frac{\alpha^2}{\Delta^4}\frac{A[F'(q)]^2}{2R^2m\omega} \times\left(\frac{\mu_p-\mu_n}{2}\right)^2\frac{q^4}{m^2}V_T(\theta)\left\{\begin{matrix}\frac{1}{5}\\1\\\frac{4}{5}\end{matrix}\right\}. \quad (28b)$$

The above cross sections are rough upper limits on the physical cross sections because the matrix elements for the various multipole transitions exhaust approximate sum rules. If Q is a sum of single-particle operators and H is the Hamiltonian for a nuclear system, it is easy to show[19] that

$$\langle 0|[Q,[H,Q]]|0\rangle=2\sum_n(E_n-E_0)|\langle n|Q|0\rangle|^2. \quad (29)$$

In our case, the operator Q has the form

$$Q=\sum_{i=1}^{A}f(\mathbf{r}_i)\left\{\begin{matrix}1\\\tau_3{}^{(i)}\\\tau_3{}^{(i)}\sigma_m{}^{(i)}\end{matrix}\right\}. \quad (30)$$

One recognizes that the spin and isospin operators which appear are part of the set of 15 generators of the Lie group SU_4. If one makes the assumption that the nu-

[19] R. A. Ferrell, Phys. Rev. **107**, 1631 (1957).

clear potential has SU_4 symmetry (Wigner supermultiplet theory) and, further, that it is velocity-independent, the operator Q commutes with the potential-energy operator and one can show that

$$\sum_n \omega_n |\langle n|Q|0\rangle|^2 = \frac{1}{2m}\langle 0|\sum_{i=1}^{A}[\nabla_i f(\mathbf{r}_i)]^2|0\rangle = \frac{A}{2m}\langle 0|[\nabla f(\mathbf{r})]^2|0\rangle. \tag{31}$$

The sum rule is exactly valid if Q contains no spin and isospin operators.[20] The monopole, dipole, and quadrupole sum rules for those states that are excited by the Coulomb field of the electron are:

Monopole

$$\sum_n \omega_n |\langle n|\sum_i r_i^2|0\rangle|^2 = \sum_n \omega_n |\langle n|\sum_i r_i^2\tau_3^{(i)}|0\rangle|^2 = \sum_n \omega_n |\langle n|\sum_i r_i^2\tau_3^{(i)}\sigma_m|0\rangle|\,| = \frac{2A}{m}R^2, \tag{32a}$$

Dipole

$$\sum_n \omega_n |\langle n|\sum_i z_i|0\rangle|^2 = \sum_n \omega_n |\langle n|\sum_i z_i\tau_3^{(i)}|0\rangle|^2 = \sum_n \omega_n |\langle n|\sum_i z_i\tau_3^{(i)}\sigma_m{}^i|0\rangle|^2 = \frac{A}{2m}, \tag{32b}$$

Quadrupole

$$\sum_n \omega_n |\langle n|\sum_i f_Q(\mathbf{r}_i)|0\rangle|^2 = \sum_n \omega_n |\langle n|\sum_i f_Q(\mathbf{r}_i)\tau_3^{(i)}|0\rangle|^2 = \sum \omega_n |\langle n|\sum_i f_Q(\mathbf{r}_i)\tau_3^{(i)}\sigma_n{}^{(i)}|0\rangle|^2 = \frac{4A}{m}R^2, \tag{32c}$$

with $f_Q(\mathbf{r}) = 2z^2 - x^2 - y^2$. In the above equations, R^2 again represents the mean-square radius.

The proof that our Coulomb matrix elements obey these sum rules is straightforward. As an example, we treat the matrix element for the 2^+ compressional quadrupole state. We first note that Eq. (32c) can be rewritten, to lowest order in q^2, as

$$\sum_n \omega_n |\langle 2\|M_2\|0\rangle|^2 = \frac{4A}{m}R^2\frac{q^4}{9\times 64\pi}, \tag{33}$$

where the definitions of Eqs. (24a) and (25b) have been used. Returning to Eq. (26c), it is apparent that $\omega|\langle 2\|M_2\|0\rangle|^2$ is equal to the right-hand side of the above quadrupole sum rule upon taking

$$F'(q) = -\tfrac{1}{3}qR^2 + \cdots.$$

[20] M. Gell-Mann and V. Telegdi, Phys. Rev. **91**, 169 (1953).

Sum rules for the transverse matrix elements are somewhat more complicated, so we confine ourselves to the matrix element appropriate to the 1^+ spin-isospin monopole state. Because of the monopole distribution of the magnetization density $\mathfrak{y}(\mathbf{r})$, the appropriate sum rule is

$$\sum_n \omega_n |\langle n|\sum_i j_0(qr_i)\tau_3^{(i)}\sigma_3^{(i)}|0\rangle|^2 = \frac{A}{2m}\langle 0|(\nabla j_0(qr))^2|0\rangle. \tag{34}$$

The right-hand side, to lowest order in q^2 is $\frac{1}{9}q^4R^2(A/2m)$. The matrix element on the left-hand side is related to our model reduced matrix element [Eq. (26b)] by the equation

$$\langle 1^+|\sum_i j_0(qr_i)\tau_3^{(i)}\sigma_3^{(i)}|0\rangle = \frac{a^{-1}}{\sqrt{3}}\langle 1\|T_1{}^{m\mu}\|0\rangle_{s-i},$$

$$a = \frac{1}{2m}\frac{(\mu_p-\mu_n)}{2}\frac{iq}{(6\pi)^{1/2}}. \tag{35}$$

Upon substitution of the value of the model reduced matrix element, one sees that the sum rule of Eq. (34) is exhausted by our 1^+ spin-isospin monopole state.

IV. APPLICATION TO ^{16}O

The nucleus ^{16}O has been intensively studied both theoretically and experimentally, and one can hope to identify the various collective states in its spectrum. For this reason, we have selected it as an example. Since our model provides no energies, these must be obtained either from experiment or from shell-model calculations. However, there are certain guidelines which the collective model does provide.

In the first place, for a given collective nuclear mode a relationship between the energies of the multipole oscillations can be established. In the dipole and quadrupole oscillation no radial nodes are introduced, while a radial node does occur in the monopole oscillation. If one returns to a model with constant nuclear density and a sharp edge[21] and sets up a wave equation for the excess density $\rho_l'(\mathbf{r})$, the boundary condition for the three densities are

$$\rho_0'(r)|_R = 0,$$
$$\hat{r}\cdot\nabla\rho_1'(\mathbf{r})|_R = \hat{r}\cdot\nabla\rho_2'(\mathbf{r}) = 0. \tag{36}$$

Computing the lowest roots k_lR of the appropriate Bessel functions and setting $\omega_l \propto k_l$, one finds that

$$\omega_1:\omega_0:\omega_2 = 1:1.51:1.59. \tag{37}$$

Thus, the monopole and quadrupole oscillations are roughly degenerate and somewhat higher in energy than the dipole. The same ordering is, of course, also

[21] Helmut Steinwedel and J. Hans D. Jensen, Naturfrsch. **5A**, 413 (1950)

expected on the basis of simple shell model arguments where the unperturbed states would have the ratio 1:2:2.

A second general guideline is provided by the nuclear matter calculation of Glassgold *et al.*[3] These authors found that the compressional mode excitation is much lower in energy than the other three (in fact the energy is complex since it is in the region of the unperturbed particle and hole energies), that the spin mode is next lowest in energy, and that the isospin and spin-isospin modes lie highest and are nearly degenerate. This is precisely the ordering found in ^{4}He where the 0^+ compressional monopole state[22] is at 20.4 MeV, the 2^- spin dipole state at 22.2 MeV, and the $T=1$ dipole states at somewhat higher energies.[22–24]

Inelastic-electron scattering cross sections have been measured[25,26] for ^{16}O at an angle of 180°, and at this angle, only the transverse matrix elements contribute. For that reason we do not discuss the two purely Coulomb states, the 0^+ compressional monopole (the "breathing mode" state) and the 0^+ isospin monopole. The positions and widths of all the other states, except the 3^+ spin-isospin quadrupole can be estimated from the calculation of Spicer and Eisenberg.[14] The 2^+ compressional quadrupole and 2^+ isospin quadrupole states are characterized by having large E_2 transition strengths. Spicer and Eisenberg calculate a 2^+ $T=0$ state in ^{16}O at 14.48 MeV which has 58.8% of the strength and which contains as its principal configuration $(1p_{3/2})^{-1}(1f_{7/2})$. We associate this state with the 13.0-MeV peak in the data of Vanpraet.[26] Spicer and Eisenberg find the 2^+ $T=1$ state with large E_2 strength (44.9%) at 25.6 MeV which has as its principal configuration also $(1p_{3/2})^{-1}(1f_{7/2})$. The 2^+ spin-isospin quadrupole state is harder to obtain from the shell-model calculation since it has nonzero quadrupole strength only if LS coupling holds. However, Spicer and Eisenberg find a 2^+, $T=1$ state at 30.62 MeV whose major configuration is $(1p_{3/2})^{-1}(1f_{5/2})$, and which is the most likely candidate. [We note that states with a predominant configuration $(1p_{3/2})^{-1}(2p_{3/2})$, etc., are ruled out because of the radial node in the transition density.]

The positions of the 1^+ states are also somewhat more ambiguous than the 2^+ states with large quadrupole strength. These states cannot be excited strongly by low momentum-transfer reactions such as gamma absorption, so with either the shell model of the collective model, the matrix elements are very small. For this reason the matrix elements are not given by Spicer and Eisenberg and we must attempt to identify the collective states from their principal configuration. The 1^+ spin-isospin quadrupole state can be associated with the 1^+, $T=1$ level at 31.53 MeV which has as its main configuration $(1p_{3/2})^{-1}(1f_{5/2})$, since the only other possible state, the one which is chiefly $(1s_{1/2})^{-1}(1d_{3/2})$, is at 49.96 MeV. Since the monopole energies should be nearly degenerate with the quadrupole energies, the most reasonable choice for the 1^+ spin-isospin monopole state is the one at 28.82 MeV with configuration $(1p_{3/2})^{-1}(2p_{1/2})$. Another possibility is the state at 19.98 MeV which we have retained throughout our calculations as an alternate monopole 1^+ state.

Because of spin-orbit splitting the spin-isospin quadrupole state should appear as separate 1^+, 2^+, and 3^+ states. We have already considered the 1^+ and 2^+. Spicer and Eisenberg did not include 3^+ states in their calculations, so we must rely chiefly on experiment and on analogy with the 0^-, 1^-, 2^- dipole triplet.[7] We have put the 3^+ spin-isospin quadrupole state at 23.0 MeV so that it lies approximately the same distance below the 2^+ isospin state as the 2^+ spin-isospin state lies above, as in the dipole case. Its main configuration may be conjectured to be $(1p_{3/2})^{-1}(1f_{7/2})$.

A spectrum of ^{16}O showing our assumed energy assignments appears in Fig. 1. The negative parity dipole states from Ref. 7 have also been added. The two possible energy values for the 1^+ spin-isospin monopole state are indicated by dotted lines.

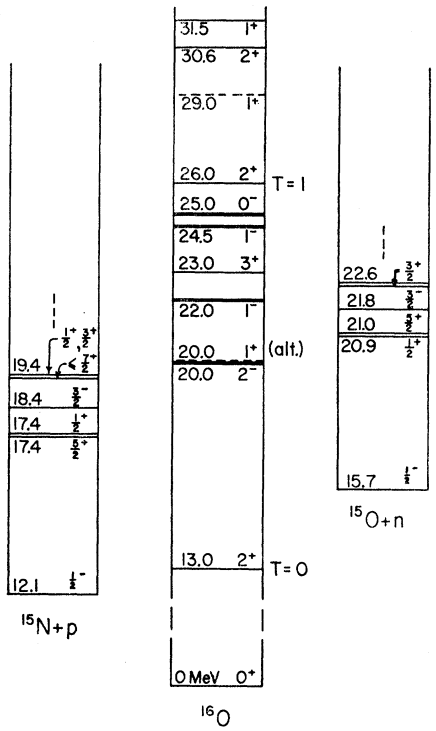

FIG. 1. Level scheme of collective states and their decay channels. The dotted lines show the two possible choices for the 1^+ spin-isospin monopole state. The negative parity states have been taken from Ref. 7.

[22] Paul Szydlik and Carl Werntz, Phys. Rev. **138**, B866 (1965); Erratum, **140**, A134 (1965).
[23] A. de-Shalit and J. D. Walecka, Phys. Rev. **147**, 763 (1966).
[24] T. A. Tombrello, Phys. Rev. **138**, B40 (1965).
[25] J. Goldemberg and W. L. Barber, Phys. Rev. **134**, B963 (1964).
[26] G. J. Vanpraet, Nucl. Phys. **74**, 219 (1965).

TABLE I. Particle widths and decay channel energies (both in MeV) of collective states in ^{16}O which can appear in 180° electron scattering. The upper line refers to transitions to the ground states, the lower line to transitions to the third excited states of the daughter nuclei. The last column gives the widths selected for the electroexcitation cross sections. The negative parity states are from Table II of Ref. 7.

State	Energy (MeV)	$E_n^{(i)}$	$\Gamma_n^{(i)}$	Γ_n	$E_p^{(i)}$	$\Gamma_p^{(i)}$	Γ_p	Γ_{tot}	Γ
1^+ *s-i Q*	31.5	15.8	0.14	3.5	19.4	0.14	4.3	7.8	2
		9.7	3.4		13.1	4.2			
2^+ *s-i Q*	30.6	14.9	0.14	2.2	18.5	0.16	2.9	5.1	2
		8.8	2.1		12.2	2.8			
1^+ *s-i M*	29.0	13.3	0.03	7.5	16.9	0.03	8.2	15.7	5
		7.2	7.5		10.6	8.2			
2^+ *i Q*	26.0	10.3	0.35	2.3	13.9	0.44	2.9	5.2	2
		4.2	2.0		7.6	2.5			
1^- *s-i D*	25.0	9.3	0.4	1.8	12.9	0.5	2.8	4.6	1.5
		3.2	1.4		6.6	2.3			
3^+ *s-i Q*	23.0								2
1^- *i D*	22.5	6.8	0.6	0.7	10.4	0.8	1.6	2.3	1.5
		0.7	0.1		4.1	0.8			
1^+ *s-i M* (alt)	21.0	5.3	6.1	6.1	8.9	7.4	7.4	13.5	5
					2.6				
2^- *s-i D*	20.0	4.3	0.2	0.2	7.9	0.3	0.7	0.9	1.0
					1.6	0.4			
2^+ *c Q*	13.0				0.9				1.0

In order to compare the calculated cross section with experiment, the neutron and proton widths of the levels must be known. These can be calculated from *R*-matrix theory[15] if the particle and hole configuration mixing coefficients are known, using the relation[7]

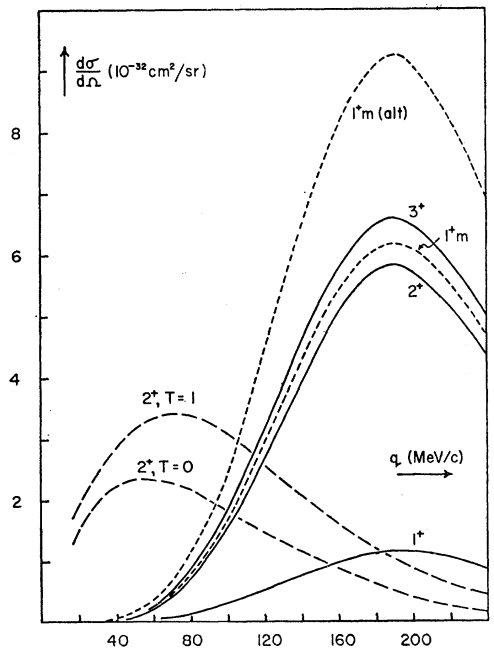

FIG. 2. Theoretical cross sections for inelastic electron excitation of the collective states in ^{16}O as a function of momentum transfer, $\theta=180°$.

(for $T=1$, $T_z=0$)

$$\Gamma_{jl}=\frac{1}{2}\left(\frac{11.1}{a}\right)^2 P_l\gamma_{jl}'^2 \tag{38}$$

The width Γ_{jl} is in MeV, the channel radius (in Fermis) is given by $a=1.4A^{1/3}$, P_l is the penetration factor, and the reduced widths are given by

$$\gamma_{jl}'=\left|\sum_{n_A}(-1)^{n_A-1}X_{aA}^{(n)}\right|, \tag{39}$$

where $X_{aA}^{(n)}$ are the particle-hole mixing coefficients defined in Ref. 15. In ^{16}O transitions can occur most easily from one-particle–one-hole states, with the hole in the p shell, to the odd parity $(1p_{1/2})^{-1}$ ground state and $(1p_{3/2})^{-1}$ third excited state in ^{15}O or ^{15}N. (The low-lying spectra of these nuclei also are shown in

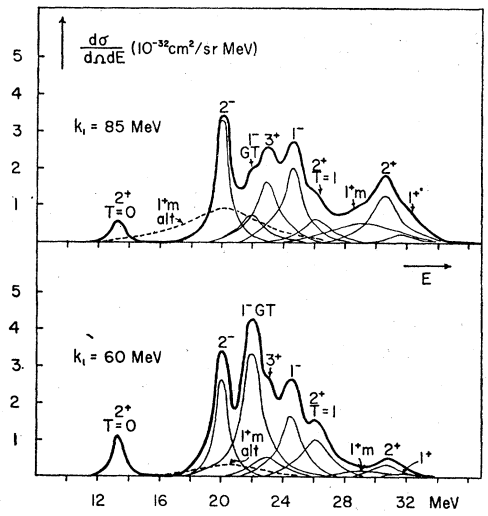

FIG. 3. Theoretical electroexcitations cross sections of the collective states in ^{16}O for primary electron energies of 60 and 85 MeV and $\theta=180°$.

Fig. 1.) Hence in calculating the widths one must calculate neutrons and proton widths for decay to both levels. The results of the calculation are shown in Table I (along with the results of Ref. 7 for the dipole states). It is apparent that the calculated widths are too large, a result which is due to the inherent uncertainties in applying *R*-matrix theory to the shell model and to the neglect of all but the main particle and hole configurations for the final state. The last column gives the value of Γ used in our calculation, where these have been chosen to conform with the observed experimental widths but roughly retain the relative magnitude indicated by the shell model states.

The expressions for the cross sections in Eqs. (27) and (28) have been used to calculate these as a function of momentum transfer for the states which are important in electron scattering through 180°, see Fig. 2. These cross sections, after being multiplied by Lorentz

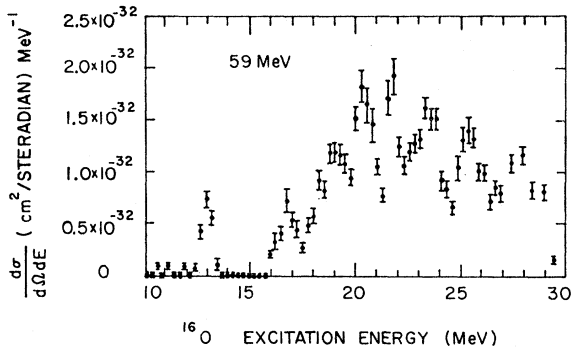

FIG. 4. Experimental electroexcitation cross section for ^{16}O in the giant-dipole region with an incident electron energy of 59 MeV and $\theta=180°$. The data points are taken from Vanpraet Ref. 26.

factors using the widths of Table I, have been used to calculate the differential spectrum for 180° inelastically scattered electrons, for different values of the incident electron energies. Figure 3 shows the cross section for incident energies of 60 and 85 MeV. The contribution of the alternate 1^+ spin-isospin monopole, indicated by the dashed line, was not included in the envelope of the states. For comparison of theory and experiment, the data of Vanpraet[26] for 59-MeV incident electron energy is reproduced in Fig. 4. One sees that our simple theory somewhat overestimates the total strength but provides a rough picture as to the distribution of strength.

In conclusion, we may say that at high momentum transfers a sizeable number of collective states, including higher multipoles, may contribute to the electron scattering cross section. While we cannot claim to have correctly identified every peak in the experimental inelastic-scattering cross sections, this cross section does not appear to have much more complexity than that given by the simple collective model we have put forward. Our model predicts in a straightforward way the relative strengths and q dependences of the lowest collective states generated by the four excited modes of nuclear matter and may be used as a guide to "what to look for" as inelastic electron experiments are pushed to higher momentum transfers.

APPENDIX

We want to find coefficients A_{lL} so that the equation of continuity, which takes the form

$$-\frac{d}{dr}\rho_0\sum_{l,m}\frac{1}{\mu}\frac{r^{l-1}}{R^{l-2}}\beta_{lm}Y_{lm}(\hat{r})=-\sum_{l,m,L}\frac{1}{\mu}\beta_{lm}A_{lL}\nabla\cdot\left[\frac{r^{l-1}}{R^{l-2}}\rho_0(r)\mathbf{Y}_{lL}{}^m(\hat{r})\right] \quad \text{(A1)}$$

is satisfied. Using standard equations given in Edmonds,[17] we find that

$$\nabla\cdot\left[\frac{r^{l-1}}{R^{l-2}}\rho_0\mathbf{Y}_{lL}{}^m\right]=-\left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{d}{dr}+\frac{l+1}{r}\right)\times\frac{r^{l-1}}{R^{l-2}}\rho_0Y_{lm}(\hat{r}),\quad L=l+1,$$

and (A2)

$$\nabla\cdot\left[\frac{r^{l-1}}{R^{l-2}}\rho_0\mathbf{Y}_{lL}{}^m\right]=\left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{d}{\mathrm{dr}}-\frac{l-1}{r}\right)\times\frac{r^{l-1}}{R^{l-2}}\rho_0Y_{lm}(\hat{r}),\quad L=l-1.$$

Thus, the right-hand side (R.H.S.) of Eq. (A1) can be written

$$\text{R.H.S.}=-\sum_{l,m}\frac{1}{\mu}\frac{\beta_{lm}}{R^{l-2}}\left\{-A_{l+}\left(\frac{l+1}{2l+1}\right)^{1/2}\times\left(\frac{d\rho_0}{dr}r^{l-1}+(2l+1)\rho_0r^{l-2}\right)Y_{lm}(\hat{r})+A_{l-}\left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{d\rho_0}{dr}r^{l-1}\right)Y_{lm}(\hat{r})\right. \quad \text{(A3)}$$

Comparing this to the left-hand side, we obtain the two equations

$$1=-A_{l+}\left(\frac{l+1}{2l+1}\right)^{1/2}+A_{l-}\left(\frac{l}{2l+1}\right)^{1/2}, \quad \text{(A4)}$$

$$0=-A_{l+}\left(\frac{l+1}{2l+1}\right)^{1/2}(2l+1),$$

with the solutions

$$A_{l+}=0,$$

$$A_{l-}=\left(\frac{2l+1}{l}\right)^{1/2}.$$