Antinormally Ordered Correlations and Quantum Counters*

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The properties of a photon counter functioning by stimulated emission—rather than by absorption—of photons in an external field are examined. A possible scheme for such a quantum counter is described, and it is shown that correlation measurements performed with a number of quantum counters correspond to antinormally ordered products of field operators. These correlations, unlike the normally ordered ones, are always positive definite and depend explicitly on the number of radiation modes per unit volume to which the counter is coupled. It is shown that the antinormally ordered correlations carry useful information about the field only when the average photon occupation number per mode is large. A general expression for the probability that the quantum counter registers *n* counts in a certain time interval is derived and is shown to be related in an interesting way to the corresponding expression for the photoelectric detector. The variance of the probability distribution is evaluated for some simple states of the field.

1. INTRODUCTION

IN the quantum theory of optical coherence the configuration space creation and annihilation operators $\hat{\mathbf{A}}^\dagger(x)$ and $\hat{\mathbf{A}}(x)$ are used a great deal. Of all functions of these operators, the normally ordered ones have hitherto played a preferred role. The justification for this preferred role rests on the fact that measurements of the field are normally carried out with photo-electric detectors, and that the rate of delayed multiple coincidence counting of N photoelectric detectors exposed to the field at the space-time points $x_1, \cdots x_N$ is proportional to the expectation value of the normally ordered product[1,2]

$$\Gamma^{(N,N)}(x_1,\cdots,x_N) = \langle :\hat{\mathbf{A}}^\dagger(x_1)\cdot\hat{\mathbf{A}}(x_1)\cdots\hat{\mathbf{A}}^\dagger(x_N)\cdot\hat{\mathbf{A}}(x_N):\rangle. \quad (1)$$

Here $:\hat{O}:$ denotes normal ordering of the operator $\hat{O}$. The functions $\Gamma^{(N,N)}(x_1,\cdots,x_N)$ therefore describe correlation properties of the field with respect to photoelectric measurements. Some properties of both the alternating operator products $\langle\hat{\mathbf{A}}^\dagger(x_1)\cdot\hat{\mathbf{A}}(x_1)\cdots\hat{\mathbf{A}}^\dagger(x_N)\times\hat{\mathbf{A}}(x_N)\rangle$ and of the antinormally ordered operators $\langle``\hat{\mathbf{A}}^\dagger(x_1)\cdot\hat{\mathbf{A}}(x_1)\cdots\hat{\mathbf{A}}^\dagger(x_N)\cdot\hat{\mathbf{A}}(x_N)"\rangle$ ("$\hat{O}$" stands for antinormal ordering of the operator $\hat{O}$) have also been examined,[3–9] but the physical significance of the latter has not so far become very clear. This is the problem we shall examine in the following.

Now, in principle at least, there exists an alternative method of measuring electromagnetic fields, in which the photo-electric detectors are replaced by atomic counting devices whose mode of action resembles the operation of the laser amplifier. These counters function not by absorption but by stimulated emission of photons. For convenience we shall refer to them as quantum counters, but their operation is somewhat different from the atomic counting devices described by Bloembergen,[10] Basov *et al.*,[11] and others, which are often called quantum counters. A possible form of quantum counter will be described in outline. It will be seen that correlations of the electromagnetic field which are measured with the help of quantum counters correspond to expectation values of antinormally ordered operators of the form

$$\tilde{\Gamma}^{(N,N)}(x_1,\cdots,x_N) = \langle``\hat{\mathbf{A}}(x_1)\cdot\hat{\mathbf{A}}^\dagger(x_1)\cdots\hat{\mathbf{A}}(x_N)\cdot\hat{\mathbf{A}}^\dagger(x_N)"\rangle. \quad (2)$$

Superficially there appears to be a certain symmetry between the two correlations $\Gamma^{(N,N)}$ and $\tilde{\Gamma}^{(N,N)}$, and between the two kinds of measurement. There is however an important asymmetry connected with the fact that emission, unlike absorption can occur spontaneously, or in the presence of a vacuum field. As a result, quite apart from any practical difficulties connected with the construction, quantum counters are not useful for measuring a large class of fields, and we shall see that this fact is reflected in the properties of $\tilde{\Gamma}^{(N,N)}$. Moreover, whereas the photo-electric detector can, at least in principle, respond to an unlimited band of frequencies of the radiation field, the quantum counter can respond only to frequencies determined by its own atomic-energy-level structure. It follows that the number of radiation modes μ of the field in a normalization volume L^3 which are coupled to the quantum counter is in principle finite, and this number appears explicitly in expressions for correlations in terms of antinormally ordered operators.

We shall see that, unlike normally ordered correla-

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[8] C. L. Mehta, *Lectures in Theoretical Physics* (University of Colorado Press, 1965), p. 345.

[9] R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. deWitt *et al.* (Gordon and Breach Science Publishers, New York, 1965), p. 65.

[10] N. Bloembergen, Phys. Rev. Letters **2**, 84 (1959).

[11] N. G. Basov, O. N. Krokhin, and Yu. M. Popov, Usp. Fiz. Nauk **72**, 161 (1960) [English transl.: Soviet Phys.—Usp. **3**, 702 (1961)].

tions, the antinormally ordered field correlations are always positive definite. This fact is also reflected in the nature of the particular phase-space functional which allows the quantum correlations to be expressed in the same form as the classical ones.[5–9] Nevertheless this functional is not the classical probability functional. In the classical limit, when the average photon occupation number per mode becomes very large, $\Gamma^{(N,N)}$ and $\tilde{\Gamma}^{(N,N)}$ become equal, and the results of measurements performed with quantum counters and photodetectors become indistinguishable. As we shall see, there is an interesting correspondence between the probability distributions of the counts registered by photodetectors and quantum counters, although the distributions are very different in general.

2. MEASUREMENT OF ANTINORMALLY ORDERED CORRELATIONS WITH QUANTUM COUNTERS

Consider an atomic system having an energy-level structure as indicated in Fig. 1. Here "a" represents a terminal energy level and "b" a metastable level which is radiatively coupled to a broad energy band "c," corresponding to a very short-lived state. We suppose that the atomic system will make spontaneous radiative transitions from c to the terminal level a. It will be noted that some features of the energy-level scheme of Fig. 1 bear a superficial resemblance to the energy-level structure of ruby as used in the laser.

Let us suppose that we have such an atomic system prepared in the state b and located at the point $\mathbf{x}$ at time t in a radiation field. Although the method of preparing the state will not concern us here, we may assume that there exists a broad energy band "d" above b, and that the system will make nonradiative transitions from d to b. The state b can then be prepared beforehand by the usual method of optical pumping from the ground state to the level d. We suppose moreover that the interval $(E_b-E_c)/\hbar c$, defined by the energy levels E_b and E_c, is of the same order as the wave number of a typical mode of the external field, and that $E_c-E_a>E_b-E_c$.

Under the influence of the external field the system may be induced to make a stimulated transition to the energy level c with the emission of a photon, and, since level c is very short lived, it will decay spontaneously from c to a with the further emission of a photon. Since E_c-E_a is always greater than E_b-E_c, the latter photon is clearly distinguishable from the former, and a neighboring transparent photodetector with a sufficiently high photoelectric threshold will register the second photon alone. The combination of the photodetector with a large number of such atomic systems evidently acts as a quantum counter for the external field, functioning by the stimulated emission of radiation. We note that the photodetector here plays an auxiliary role only,

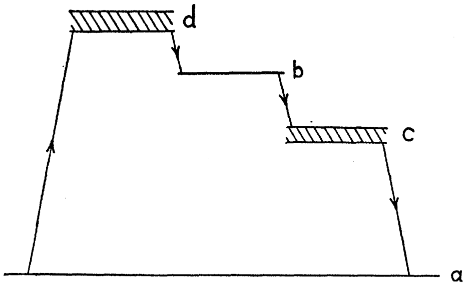

FIG. 1. Energy level scheme for a quantum counter. The counter responds to frequencies within the interval $(E_b-E_{c\,\min})/h$ to $(E_b-E_{c\,\max})/h$.

and that the external field is actually "measured" by means of the first induced transition.

Just as $\hat{\mathbf{A}}(\mathbf{x},t)$ is the operator corresponding to the photoelectric measurement of the field at the space time point $(\mathbf{x},t)$, so the "observable" which most nearly corresponds to a measurement of the field with the quantum counter is the photon creation operator $\hat{\mathbf{A}}^\dagger(\mathbf{x},t)$ defined by

$$\mathbf{A}^\dagger(\mathbf{x},t)=(1/L^{3/2})\sum_{[\mathbf{k},s]}\hat{a}_{\mathbf{k},s}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k},s}^*\exp[-i(\mathbf{k}\cdot\mathbf{x}-ckt)]. \quad (3)$$

Here L^3 is the normalization volume, $\hat{a}_{\mathbf{k},s}^\dagger$ is the creation operator for a photon of wave vector, spin mode $\mathbf{k}$, s, $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ is the unit polarization vector, and the symbol $[\mathbf{k},s]$ denotes the set of all modes of the field to which the quantum counter responds. If E'' and E' are the upper and lower bounds of E_b-E_c, then $\hbar ck$ is constrained to lie between E'' and E'. There may also be constraints on the polarization s determined by the induced dipole moments of the atomic system making up the quantum counter, although these constraints disappear if the counter contains a large number of randomly oriented atomic systems. For simplicity we assume from here on that all the occupied modes of the external radiation fields in which we are interested are modes to which the counter responds.

Now consider a quantum counter which is allowed to interact with a radiation field in a state represented by the density operator $\hat{\rho}$ at the space-time point x. If $\hat{\rho}$ has a diagonal representation in terms of a complete set of states $|s_1\rangle$ in the form

$$\hat{\rho}=\sum_{s_1} p(s_1)|s_1\rangle\langle s_1|, \quad (4)$$

and $|s_2\rangle$ is any possible final state of the field, then the probability that a count will be registered at x is proportional to

$$\sum_{s_2}\sum_{s_1} p(s_1)|\langle s_2|\hat{\mathbf{A}}^\dagger(x)|s_1\rangle|^2,$$

where the sum is taken over the complete set of all possible final states $|s_2\rangle$. By expanding the square we find, by an argument similar to that given by Glauber,[1] that

$$\sum_{s_2}\sum_{s_1} p(s_1)|\langle s_2|\hat{\mathbf{A}}^\dagger(x)|s_1\rangle|^2=\sum_{s_1}\sum_{s_2} p(s_1)\langle s_1|\hat{\mathbf{A}}(x)|s_2\rangle\cdot\langle s_2|\hat{\mathbf{A}}^\dagger(x)|s_1\rangle$$
$$=\sum_{s_1} p(s_1)\langle s_1|\hat{\mathbf{A}}(x)\cdot\hat{\mathbf{A}}^\dagger(x)|s_1\rangle=\mathrm{Tr}[\hat{\rho}\hat{\mathbf{A}}(x)\cdot\hat{\mathbf{A}}^\dagger(x)]. \quad (5)$$

Thus the antinormally ordered operator $\hat{\mathbf{A}}(x)\cdot\hat{\mathbf{A}}^\dagger(x)$ here plays the role of an intensity operator with respect to measurements carried out with the quantum counter, just as $\hat{\mathbf{A}}^\dagger(x)\cdot\hat{\mathbf{A}}(x)$ behaves as an intensity operator with respect to photoelectric measurements, when the sum in (3) is taken over all modes of the field to which the quantum counter responds.

In a similar way it may be seen that the joint probability that counts will be registered by N quantum counters at the space-time points $x_1, x_2, \cdots, x_N$ is proportional to

$$\sum_{s_2}\sum_{s_1} p(s_1)|\langle s_2|\mathbf{A}^\dagger(x_1)\cdots\hat{\mathbf{A}}^\dagger(x_N)|s_1\rangle|^2=\sum_{s_1}\sum_{s_2} p(s_1)\langle s_1|\hat{\mathbf{A}}(x_1)\cdots\hat{\mathbf{A}}(x_N)|s_2\rangle\cdot\langle s_2|\hat{\mathbf{A}}^\dagger(x_1)\cdots\hat{\mathbf{A}}^\dagger(x_N)|s_1\rangle$$
$$=\sum_{s_1} p(s_1)\langle s_1|``\hat{\mathbf{A}}(x_1)\cdot\hat{\mathbf{A}}^\dagger(x_1)\cdots\hat{\mathbf{A}}(x_N)\cdot\hat{\mathbf{A}}^\dagger(x_N)"|s_1\rangle=\mathrm{Tr}[\rho``\hat{\mathbf{A}}(x_1)\cdot\hat{\mathbf{A}}^\dagger(x_1)\cdots\hat{\mathbf{A}}(x_N)\cdot\hat{\mathbf{A}}^\dagger(x_N)"]. \quad (6)$$

Evidently these antinormally ordered correlations will bear a similar relation to measurements with quantum counters, as do the more familiar normally ordered correlations to photoelectric measurements.

The expressions in (5) and (6) are proportional to the differential counting probabilities. The constants of proportionality depend on the matrix elements for the atomic transition and the density of atomic states. When the field is in the form of quasimonochromatic plane waves incident normally on the detector, as is usually the case in practice, it may be shown by an argument similar to that used in connection with the photoelectric detector,[9,12,13] that the expectation value of the number of counts n registered by the quantum counter of sensitive surface area S in a time interval from t to $t+T$ is given by

$$\langle n\rangle=\alpha c\int_S\int_t^{t+T}\langle\mathbf{A}(\mathbf{x},t')\cdot\mathbf{A}^\dagger(\mathbf{x},t')\rangle d^2x dt'. \quad (7)$$

α is a dimensionless parameter involving the properties of the quantum counter, which we may regard as a measure of the quantum efficiency of the process. The calculation leading to Eq. (7) is a standard calculation based on perturbation theory, with the usual electromagnetic interaction Hamiltonian, and will not be given here.

We can express this result in another form by introducing the commutator of $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{\mathbf{A}}^\dagger(\mathbf{x},t)$. From Eq. (3) and its Hermitian conjugate, together with the well known commutation rules obeyed by the $\hat{a}_{\mathbf{k},s}$ operators, we find

$$[\hat{\mathbf{A}}(\mathbf{x},t),\cdot\hat{\mathbf{A}}^\dagger(\mathbf{x},t)]=\frac{1}{L^3}\sum_{[\mathbf{k},s]}(\boldsymbol{\varepsilon}_{\mathbf{k},s}\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}^*)$$
$$=\mu/L^3, \quad (8)$$

where μ is the number of modes of the set $[\mathbf{k},s]$. With the help of this relation we can express Eq. (7) in the form

$$\langle n\rangle=\alpha c\int_S\int_t^{t+T}\left[\langle\hat{\mathbf{A}}^\dagger(\mathbf{x},t')\cdot\hat{\mathbf{A}}(\mathbf{x},t')\rangle+\frac{\mu}{L^3}\right]d^2x dt',$$

and, since the external field is in the form of plane waves, this may also be written

$$\langle n\rangle=\alpha\int_{\mathcal{V}}\left[\langle\hat{\mathbf{A}}^\dagger(\mathbf{x},t')\cdot\hat{\mathbf{A}}(\mathbf{x},t')\rangle+\frac{\mu}{L^3}\right]d^3x, \quad (9)$$

where the volume $\mathcal{V}$ of integration is in the form of a cylinder whose base is the sensitive surface S of the quantum counter and whose height is cT. It is convenient to introduce the operator

$$\hat{n}_{\mathcal{V},t}=\int_{\mathcal{V}}\hat{\mathbf{A}}^\dagger(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t)d^3x, \quad (10)$$

which has recently been shown[14] to play the role of a photon-number operator in configuration space, provided the linear dimensions of $\mathcal{V}$ are all large compared with the wavelengths of all modes of the set $[\mathbf{k},s]$. This condition is almost always satisfied in practice and, with the help of Eq. (10), we can express (9) in the simple form

$$\langle n\rangle=\alpha\langle``\hat{n}_{\mathcal{V},t}"\rangle. \quad (11)$$

For comparison we note that the expectation value of the number of counts registered in the same time by a photoelectric detector having the same geometry is of the form $\alpha\langle\hat{n}_{\mathcal{V},t}\rangle$. We see from Eq. (9) that the rate of counting of the quantum counter is expected to exceed that of the photodetector. The difference depends on the number of modes of the field $\mu\mathcal{V}/L^3$ corresponding to the volume $\mathcal{V}$, and will be recognized as due to spon-

[12] L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) **84**, 435 (1964).

[13] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

[14] L. Mandel, Phys. Rev. **144**, 1071 (1966).

taneous transitions[15] of the atomic system shown in Fig. 1 from energy level b to level c. The rate of counting due to spontaneous transitions may be extremely high, unless the counting surface is very small and the optical bandwidth $\Delta\nu$ centered on frequency ν_0 to which the counter responds is very small also. The ratio (μ/L^3) is of order $8\pi\nu_0^2\Delta\nu/c^3$. These extra counts are of course undesirable from the point of view of a measurement of the field, but they are inevitable in a counter functioning by emission rather than by absorption of photons. As can be seen from Eq. (9), the counts due to spontaneous emission persist even in a vacuum field.

Corresponding to Eq. (6), it can be shown that the correlation of the numbers of counts $n_1, n_2, \cdots, n_N$ registered by N separate quantum counters in time intervals t_1 to t_1+T_1, t_2 to t_2+T_2, etc., is given by

$$\langle n_1\cdots n_N\rangle = \alpha_1\cdots\alpha_N c^N \int_{S_1}\cdots\int_{S_N}\int_{t_1}^{t_1+T_1}\cdots\int_{t_N}^{t_N+T_N}\langle ``\hat{\mathbf{A}}(\mathbf{x}_1,t_1)\cdot \times\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)\cdots\hat{\mathbf{A}}(\mathbf{x}_N,t_N)\cdot\hat{\mathbf{A}}^\dagger(\mathbf{x}_N,t_N)"\rangle \times d^2x_1\cdots d^2x_N dt_1\cdots dt_N. \quad (12)$$

For disjoint space-time regions, such that all events registered by any one counter have a space-like separation from all events registered by any other counter, this may also be written

$$\langle n_1\cdots n_N\rangle = \alpha_1\cdots\alpha_N\langle ``\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}"\rangle, \quad (13)$$

where the volumes $\mathcal{V}_1, \mathcal{V}_2, \cdots\mathcal{V}_N$ are defined as for Eq. (10). Thus, correlation measurements carried out with disjoint quantum counters are expressible in terms of antinormally ordered products of the $\hat{n}_{\mathcal{V},t}$ operators, just as similar measurements with photoelectric detectors are expressible in terms of the corresponding normally ordered products. However, when the disjointness condition is not satisfied, the correlation Eq. (12) cannot be expressed in such a simple form as Eq. (13).

3. ANTINORMALLY ORDERED CORRELATIONS OF $\hat{n}_{\mathcal{V},t}$

We shall now examine the antinormally ordered correlations of the type appearing in Eq. (13), and express them in terms of moments of the $\hat{n}_{\mathcal{V},t}$ operators. By writing Eq. (13) in the form

$$\langle ``\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}"\rangle = \sum_{i_1}\cdots\sum_{i_{N-1}}\int_{\mathcal{V}_1}\cdots\int_{\mathcal{V}_{N-1}}\langle\hat{A}_{i_1}(\mathbf{x}_1,t_1) \cdots\hat{A}_{i_{N-1}}(\mathbf{x}_{N-1},t_{N-1})``\hat{n}_{\mathcal{V}_N,t_N}"\hat{A}_{i_{N-1}}^\dagger(\mathbf{x}_{N-1},t_{N-1}) \cdots\hat{A}_{i_1}^\dagger(\mathbf{x}_1,t_1)\rangle d^3x_1\cdots d^3x_{N-1} \quad (14)$$

and moving the "$\hat{n}_{\mathcal{V}_N,t_N}$" operator repeatedly to the right, we can express the Nth order antinormal products of the $\hat{n}_{\mathcal{V},t}$ operators in terms of ordinary products. However, in order to make the transformation we need to know the commutator of "$\hat{n}_{\mathcal{V},t}$" or $\hat{n}_{\mathcal{V},t}$ and $\hat{\mathbf{A}}(\mathbf{x},t')$.

The properties of $\hat{n}_{\mathcal{V},t}$ have recently been examined in some detail,[14] and it has been shown that, with the previously mentioned restriction on the linear dimensions of $\mathcal{V}$, the commutator

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{\mathcal{V},t'}] = \frac{1}{L^{3/2}}\sum_{[\mathbf{k},s]}\hat{a}_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)] \times U[\mathbf{x}-c\mathbf{k}(t-t')/k;\mathcal{V}], \quad (15)$$

where the function

$$U[\mathbf{x};\mathcal{V}]=1 \text{ if } \mathbf{x} \text{ lies within the volume } \mathcal{V}, \quad =0 \text{ otherwise.}$$

In particular, if $(\mathbf{x},t)$ and $(\mathcal{V},t')$ are disjoint in the sense that $U[\mathbf{x}-c\mathbf{k}(t-t')/k;\mathcal{V}]=0$ in Eq. (15) for all modes of the field, then

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{\mathcal{V},t'}]=0. \quad (16)$$

On the other hand, when $(\mathbf{x},t)$ is conjoint with $(\mathcal{V},t')$, in the sense that $U[\mathbf{x}-c\mathbf{k}(t-t')/k;\mathcal{V}]=1$ in Eq. (14) for all modes of the field, then

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{\mathcal{V},t'}]=\hat{\mathbf{A}}(\mathbf{x},t). \quad (17)$$

For photo-electric detectors illuminated normally by a plane wave radiation field, the disjointness condition has been shown to apply whenever the detectors are located side by side.[14] However the same is not true for quantum counters, since the spontaneous emission may generate photons in modes which were previously unoccupied. Only if the different counters are separated sufficiently so that the different measurements do not influence each other does Eq. (16) apply. Equation (17) has no immediate relevance to the quantum counter, since the correlation (12) is not expressible in the form (13) when the regions are not disjoint.

We can now make of Eqs. (8) and (16) together with their Hermitian conjugates to move the "$\hat{n}_{\mathcal{V}_N,t_N}$" operator in Eq. (14) repeatedly to the right. We then find

$$\langle ``\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}"\rangle = \langle ``\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_{N-1},t_{N-1}}"[\hat{n}_{\mathcal{V}_N,t_N}+(\mu\mathcal{V}_N/L^3)]\rangle, \quad (18)$$

when the disjointness condition applies. Since Eq. (18) holds for any state of the field, it can be regarded as a recurrence relation between the operators themselves, and repeated application of the relation then leads to

$$\langle ``\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}"\rangle = \langle(\hat{n}_{\mathcal{V}_1,t_1}+\mu\mathcal{V}_1/L^3)\cdots(\hat{n}_{\mathcal{V}_N,t_N}+\mu\mathcal{V}_N/L^3)\rangle \quad (19)$$

when the disjointness condition applies. This equation may be compared with the corresponding equation for

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[15] See, for example, W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Company, Inc., New York, 1964), p. 189.

normally ordered products of the $\hat{n}_{\mathcal{V},t}$ operators,[14]

$$\langle:\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}:\rangle=\langle\hat{n}_{\mathcal{V}_1,t_1}\cdots\hat{n}_{\mathcal{V}_N,t_N}\rangle. \quad (20)$$

Although, as we have pointed out, the correlations (12) are not expressible in the form (13) in general, it is nevertheless of some interest to examine the nature of Eq. (14) when all the regions $(\mathcal{V}_1,t_1),\cdots,(\mathcal{V}_N,t_N)$ coincide. We can again move the "$\hat{n}_{\mathcal{V},t}$" operator repeatedly to the right, this time with the help of the commutator (17) for conjoint regions. We then find that

$$\langle``\hat{n}_{\mathcal{V},t}{}^{N}"\rangle=\langle``\hat{n}_{\mathcal{V},t}{}^{N-1}"[\hat{n}_{\mathcal{V},t}+(\mu\mathcal{V}/L^3)+N-1]\rangle \quad (21)$$

and this may again be regarded as a recurrence relation between the operators. Repeated application of the recurrence relation then leads to

$$\langle``\hat{n}_{\mathcal{V},t}{}^{N}"\rangle=\left\langle\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}\right)\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}+1\right)\cdots\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}+N-1\right)\right\rangle. \quad (22)$$

This may be compared with the corresponding relation for normally ordered operators[14]

$$\langle:\hat{n}_{\mathcal{V},t}{}^{N}:\rangle=\langle\hat{n}_{\mathcal{V},t}(\hat{n}_{\mathcal{V},t}-1)\cdots(\hat{n}_{\mathcal{V},t}-N+1)\rangle. \quad (23)$$

It will be noted that successive factors in Eq. (23) are decreased by unity, whereas the ones in Eq. (22) are increased by unity. The difference may be regarded as a reflection of the fact that normally ordered correlations correspond to photon absorptions, whereas antinormally ordered correlations correspond to photon emissions.

A number of interesting properties of the antinormally ordered correlations can be seen by inspection of Eq. (19). Since the $\hat{n}_{\mathcal{V},t}$ operators are non-negative definite,[14] and since each of the factors on the right-hand side of Eq. (19) contains a positive number, it follows that these antinormally ordered correlations are always positive definite. This property is to be compared with the property of the normally ordered correlations given by (20), which vanish for states having fewer than N photons. The difference can again be understood in terms of the behavior of the quantum counter, for the spontaneous emission ensures that the results of counting correlation measurements will always be positive.

Examination of Eq. (19) also shows that, if the numbers $\mu\mathcal{V}/L^3$ are very large, the correlations may become very insensitive to the state of the field. Let us suppose that the different volumes $\mathcal{V}_1$, $\mathcal{V}_2$, etc., are comparable, and that $\langle\hat{n}_{\mathcal{V}_i,t_i}\rangle\simeq m$ for any i. Now $\mu\mathcal{V}/L^3$ will normally be a large number, and it is clear from inspection of Eq. (19) that the correlations are very insensitive to the state of the field when $m\ll\mu\mathcal{V}/L^3$, for the field is then too weak to produce many stimulated transitions of the atomic system in the quantum counter. The counts registered by the quantum counter will be very largely due to spontaneous emission, which is independent of the external field. Under these conditions it is clear that measurements of the field made with the quantum counter will be almost useless. Since m is the mean number of photons localized in the volume $\mathcal{V}$, and $\mu\mathcal{V}/L^3$ is the number of modes associated with the volume $\mathcal{V}$, the ratio

$$m/(\mu\mathcal{V}/L^3)\equiv\delta \quad (24)$$

is a measure of the average photon occupation number per mode or of the degeneracy parameter.[2] For nondegenerate fields, such as those from familiar thermal sources, δ is always much less than unity,[2,16] and the quantum counter will not be a useful measuring device. For laser beams, on the other hand, δ may become very large and this objection is no longer valid. However, for sufficiently large values of δ and moderate values of N, it will be seen from Eqs. (19) and (20) [also (22), (23)], that the difference between normally ordered and antinormally ordered correlations ceases to be important, since the role of spontaneous emission then becomes unimportant. The state of the field then approaches the classical limit and both photoelectric detectors and quantum counters will give similar results.

4. GENERATORS FOR ANTINORMALLY ORDERED OPERATORS

If Eq. (22) holds for all N, it may be used to obtain the generating function for the antinormally ordered products of $\hat{n}_{\mathcal{V},t}$. Thus

$$\begin{aligned}\langle``\exp(i\hat{n}_{\mathcal{V},t}x)"\rangle&=1+\sum_{N=1}^{\infty}\langle``\hat{n}_{\mathcal{V},t}{}^{N}"\rangle\frac{(ix)^N}{N!}\\&=1+\sum_{N=1}^{\infty}\frac{(ix)^N}{N!}\left\langle\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}\right)\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}+1\right)\cdots\left(\hat{n}_{\mathcal{V},t}+\frac{\mu\mathcal{V}}{L^3}+N-1\right)\right\rangle\\&=\langle[1-ix]^{-n_{\mathcal{V},t}+(\mu\mathcal{V}/L^3)}\rangle. \end{aligned}\quad (25)$$

Actually, since the validity of Eq. (22) rests on the assumption that the linear dimensions of $\mathcal{V}$ are large compared with the wavelength of any field mode contributing to $\hat{n}_{\mathcal{V},t}$, the validity and the convergence of the foregoing series when N tends to infinity ought to be investigated. It seems reasonable to suppose that Eq. (25) holds for sufficiently small values of x, but we will not go into the question of convergence here. By putting

$$1-ix=e^{-iv}, \quad (26)$$

[16] L. Mandel, J. Opt. Soc. Am. 51, 797 (1961).

we can write (25) in the alternative form

$$\langle\exp\{iy[\hat{n}_{\mathcal{V},t}+(\mu\mathcal{V}/L^3)]\}\rangle=\langle\text{``}\exp[\hat{n}_{\mathcal{V},t}(1-e^{-iy})]\text{''}\rangle,$$

or

$$\langle\exp(iy\text{``}\hat{n}_{\mathcal{V},t}\text{''})\rangle=\langle\text{``}\exp[\hat{n}_{\mathcal{V},t}(1-e^{-iy})]\text{''}\rangle. \quad (27)$$

This can be compared with the corresponding expression for normally ordered correlations[14]

$$\langle\exp(iy\hat{n}_{\mathcal{V},t})\rangle=\langle:\exp[\hat{n}_{\mathcal{V},t}(e^{iy}-1)]:\rangle, \quad (28)$$

and shows an interesting correspondence. But while the expansion of the right-hand side of Eq. (28) may terminate after N terms, the corresponding expansion of Eq. (27) is always infinite.

5. THE PHASE SPACE DISTRIBUTION FOR ANTINORMALLY ORDERED CORRELATIONS

The antinormally ordered correlations may be expressed in a form that is characteristic of classical correlations, with the help of a certain phase space distribution that has many of the properties of a classical probability.[5–9] We introduce the so-called "coherent" eigenstates $|\{v_{\mathbf{k},s}\}\rangle$ of the $\hat{\mathbf{A}}(\mathbf{x},t)$ operator,[1,2] defined by

$$\hat{\mathbf{A}}(\mathbf{x},t)|\{v_{\mathbf{k},s}\}\rangle=\mathbf{V}(\mathbf{x},t)|\{v_{\mathbf{k},s}\}\rangle, \quad (29a)$$

where $\{v_{\mathbf{k},s}\}$ stands for the entire set of $v_{\mathbf{k},s}$, and where the eigenvalue $\mathbf{V}(\mathbf{x},t)$ has the expansion

$$\mathbf{V}(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{[\mathbf{k},s]} v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)]. \quad (29b)$$

The states $|v_{\mathbf{k},s}\rangle$ can be used to furnish a resolution of the unit operator in the form

$$1=\prod_{\{\mathbf{k},s\}}\frac{1}{\pi}\int|v_{\mathbf{k},s}\rangle\langle v_{\mathbf{k},s}|\,d^2v_{\mathbf{k},s}. \quad (30)$$

If $\hat{\rho}$ is the density operator of the field, and if we introduce the operator

$$\hat{\mathfrak{N}}(S,t,T)\equiv\int_S\int_t^{t+T}\hat{\mathbf{A}}^\dagger(\mathbf{x},t')\cdot\hat{\mathbf{A}}(\mathbf{x},t')d^2x\,dt',$$

we can express the antinormally ordered correlation in Eq. (12) in the form

$$\langle\text{``}\hat{\mathfrak{N}}(S_1,t_1,T_1)\cdots\hat{\mathfrak{N}}(S_N,t_N,T_N)\text{''}\rangle=\mathrm{Tr}\sum_{i_1}\cdots\sum_{i_N}\int_{S_1}\int_{t_1}^{t_1+T_1}\cdots\int_{S_N}\int_{t_N}^{t_N+T_N}\hat{\rho}\hat{A}_{i_1}(\mathbf{x}_1,t_1')\cdots\hat{A}_{i_N}(\mathbf{x}_N,t_N')$$
$$\times\hat{A}_{i_N}^\dagger(\mathbf{x}_N,t_N')\cdots\hat{A}_{i_1}^\dagger(\mathbf{x}_1,t_1')d^2x_1dt_1'\cdots d^2x_Ndt_N'. \quad (31)$$

We now introduce the unit operator given by (30) between the $\hat{A}_{i_N}(\mathbf{x}_N,t_N')$ and $\hat{A}_{i_N}^\dagger(\mathbf{x}_N,t_N')$ operators and make repeated use of the eigenvalue relation (29) and its Hermitian conjugate. We then find

$$\langle\text{``}\hat{\mathfrak{N}}(S_1,t_1,T_1)\cdots\hat{\mathfrak{N}}(S_N,t_N),T_N\text{''}\rangle$$
$$=\mathrm{Tr}\sum_{i_1}\cdots\sum_{i_N}\int_{S_1}\int_{t_1}^{t_1+T_1}\cdots\int_{S_N}\int_{t_N}^{t_N+T_N}\hat{\rho}\hat{A}_{i_1}(\mathbf{x}_1,t_1')\cdots\hat{A}_{i_N}(\mathbf{x}_N,t_N')$$
$$\times\prod_{\{\mathbf{k},s\}}\left[\frac{1}{\pi}\int|v_{\mathbf{k},s}\rangle\langle v_{\mathbf{k},s}|d^2v_{\mathbf{k},s}\right]\hat{A}_{i_N}^\dagger(\mathbf{x}_N,t_N')\cdots\hat{A}_{i_1}^\dagger(\mathbf{x}_1,t_1')d^2x_1\cdots d^2x_Ndt_1'\cdots dt_N'$$
$$=\int_{S_1}\int_{t_1}^{t_1+T}\cdots\int_{S_N}\int_{t_N}^{t_N+T_N}\prod_{\{\mathbf{k},s\}}\left[\frac{1}{\pi}\int d^2v_{\mathbf{k},s}\langle v_{\mathbf{k},s}|\right]\hat{\rho}|\{v_{\mathbf{k},s}\}\rangle$$
$$\times\mathbf{V}(\mathbf{x}_1,t_1')\cdot\mathbf{V}^*(\mathbf{x}_1,t_1')\cdots\mathbf{V}(\mathbf{x}_N,t_N')\cdot\mathbf{V}^*(\mathbf{x}_N,t_N')d^2x_1\cdots d^2x_Ndt_1'\cdots dt_N'$$
$$=\int\prod_{\{\mathbf{k},s\}}\left[\frac{1}{\pi}\langle v_{\mathbf{k},s}|\right]\hat{\rho}|\{v_{\mathbf{k},s}\}\rangle U_1\cdots U_Nd^2\{v_{\mathbf{k},s}\}$$
$$\equiv\langle\langle U_1\cdots U_N\rangle\rangle, \quad (32)$$

where

$$U_i\equiv U(S_i,t_i,T_i)\equiv\int_{S_i}\int_{t_i}^{t_i+T}\mathbf{V}(\mathbf{x}_i,t_i')\cdot\mathbf{V}^*(\mathbf{x}_i,t_i')d^2x_id^2t_i'. \quad (33)$$

The expression on the right-hand side of Eq. (32) is essentially a correlation of the classical functions U_1, $U_2,\cdots$, etc., averaged with respect to

$$p(\{v_{\mathbf{k},s}\})=\prod_{\{\mathbf{k},s\}}[(1/\pi)\langle v_{\mathbf{k},s}|]\hat{\rho}|\{v_{\mathbf{k},s}\}\rangle \quad (34)$$

used as weighting functional. It has the form of an ordinary classical correlation, since, as has been shown,[5–9] $p(\{v_{\mathbf{k},s}\})$ is positive definite, bounded, and normalized to unity, and therefore behaves as a classical probability. This is also emphasized by the notation

$\langle\langle\ \rangle\rangle$ introduced in Eq. (32). Thus the expectation value of an antinormally ordered product of field operators is expressible in a classical form. Nevertheless, it is easy to see that the functional $p(\{v_{\mathbf{k},s}\})$ is not the probability distribution arising in the classical description of the field. Thus, consider a field in the vacuum state, for which $\hat{\rho}=|\{0\}\rangle\langle\{0\}|$. For this field

$$p(\{v_{\mathbf{k},s}\})=\prod_{\{\mathbf{k},s\}}\frac{1}{\pi}|\langle 0|v_{\mathbf{k},s}\rangle|^2$$

$$=\prod_{\{\mathbf{k},s\}}\frac{1}{\pi}\exp(-|v_{\mathbf{k},s}|^2), \tag{35}$$

which is a multivariate Gaussian distribution with $\langle\langle|v_{\mathbf{k},s}|^2\rangle\rangle=1$, and is certainly not the classical ensemble distribution for the vacuum.

Equation (32) is to be compared with the corresponding relation for normally ordered operators,[19]

$$\langle:\hat{\mathfrak{N}}(S_1,t_1,T_1)\cdots\hat{\mathfrak{N}}(S_N,t_N,T_N):\rangle$$

$$=\int\phi(\{v_{\mathbf{k},s}\})U_1\cdots U_N d^2\{v_{\mathbf{k},s}\}, \tag{36}$$

where $\phi(\{v_{\mathbf{k},s}\})$ is the generalized functional that appears in the "diagonal" representation of the density operator ρ in terms of coherent states,[17,1,2]

$$\hat{\rho}=\int\phi(\{v_{\mathbf{k},s}\})|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|d^2\{v_{\mathbf{k},s}\}. \tag{37}$$

As has been emphasized,[2,17,18] the functional $\phi\{v_{\mathbf{k},s}\}$, which is also normalized to unity, is a generalized weighting functional, which coincides with the classical ensemble distribution whenever it is positive definite. On the other hand, $\phi(\{v_{\mathbf{k},s}\})$ may be negative and highly singular.

The difference between the distributions $p\{v_{\mathbf{k},s}\}$ and $\phi(\{v_{\mathbf{k},s}\})$ may appear puzzling, but some aspects of the difference can be understood by reference to Eqs. (32) and (36).[19] As we have shown, the antinormally ordered correlations given by Eq. (32) are always positive definite, while the normally ordered ones given by Eq. (36) may vanish for certain states of the field. Since the $U(S_i,t_i,T_i)$ are all positive quantities, Eq. (32) can always be satisfied by a positive definite weighting functional, while Eq. (36) clearly cannot.

In the classical limit, when the average photon occupation number δ per mode tends to infinity, we have seen that the normally ordered and antinormally ordered correlations tend to equality, and from Eqs. (32) and (36), we would expect the functionals $p(\{v_{\mathbf{k},s}\})$ and $\phi(\{v_{\mathbf{k},s}\})$ to tend to equality also. We can get a rough idea how this convergence comes about by noting that, from (34) and (37)

$$p(\{v_{\mathbf{k},s}\})$$

$$=\int\phi(\{v_{\mathbf{k},s}'\})\prod_{[\mathbf{k},s]}\left[\frac{1}{\pi}|\langle v_{\mathbf{k},s}'|v_{\mathbf{k},s}\rangle|^2\right]d^2\{v_{\mathbf{k},s}'\}$$

$$=\int\phi(\{v_{\mathbf{k},s}'\})\times\prod_{[\mathbf{k},s]}\left[\frac{1}{\pi}\exp(-|v_{\mathbf{k},s}-v_{\mathbf{k},s}'|^2)\right]d^2\{v_{\mathbf{k},s}'\}. \tag{38}$$

Now, if we are dealing with a highly degenerate field near the classical limit, $\phi(\{v_{\mathbf{k},s}'\})$ will be very small or zero for all $\{v_{\mathbf{k},s}'\}$ except those for which $|v_{\mathbf{k},s}'|^2\gg 1$. In view of the sharply peaked form of the function $\exp(-|v_{\mathbf{k},s}-v_{\mathbf{k},s}'|^2)$, the principal contributions to the integral will come from values $v_{\mathbf{k},s}'$ in the neighborhood of $v_{\mathbf{k},s}$, and, if $\phi(\{v_{\mathbf{k},s}'\})$ is a sufficiently smooth function, we can write

$$p(\{v_{\mathbf{k},s}\})\simeq\phi(\{v_{\mathbf{k},s}\}). \tag{39}$$

But, even when this equality is not valid, the normally ordered and antinormally ordered correlations will tend to coincide in the classical limit.

6. THE PROBABILITY DISTRIBUTION OF QUANTUM COUNTS

We will now calculate the probability $p(n;t,t+T)$ that n counts will be registered in the time interval t to $t+T$ when the quantum counter is exposed to a plane beam of quasi-monochromatic light to which the quantum counter can respond. The statistical approach will be similar to one adopted previously in the calculation of $p(n;t,t+T)$ for a photoelectric detector,[20] and we shall see that there is an interesting correspondence between the two results.

We divide the time interval t to $t+T$ into $T/\delta T$ equal intervals δT, which are short compared with the reciprocal frequency spread $\Delta\nu$ over which the counter responds, but still long compared with a typical period of the light. We label the intervals $i=1, 2\cdots T/\delta T$, so that the ith interval extends from $t_i-\frac{1}{2}\delta T$ to $t_i+\frac{1}{2}\delta T$. From Eqs. (6) and (12) the joint probability that n counts will be registered in the i_1th, i_2th, $\cdots$, and i_nth interval is

$$P_n(t_{i_1},\cdots t_{i_n})=\langle``\alpha c\hat{J}(t_{i_1})\delta T\cdots\alpha c\hat{J}(t_{i_n})\delta T"\rangle, \tag{40}$$

where the operator $\hat{J}(t)$ stands for

$$\hat{J}(t)\equiv\int_S\hat{\mathbf{A}}^\dagger(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)d^2x, \tag{41}$$

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[17] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[18] L. Mandel and E. Wolf, Phys. Rev. **149**, 1033 (1966).
[19] For a discussion of the relationship see Refs. 7 and 8.
[20] See, for example, L. Mandel, in *Progress in Optics*, edited by E. Wolf (North Holland Publishing Company, Amsterdam, 1963), Vol. 2, p. 181.

and S is the sensitive surface of the quantum counter. It is implicit in this formula that the population of excited atoms is not significantly depleted by successive counts.

Now let $\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}})$ denote the joint probability that counts will be registered in the i_1th, i_2th, $\cdots$ and i_nth time intervals, but *not* in the i_{n+1}th time interval. Then from the unitarity condition for probabilities we must have

$$\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}}) + P_{n+1}(t_{i_1},\cdots,t_{i_n},t_{i_{n+1}}) = P_n(t_{i_1},\cdots,t_{i_n})\,, \qquad (42)$$

so that, from Eq. (40),

$$\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}}) = \langle\text{“}\alpha c\hat{J}(t_{i_1})\delta T\cdots\alpha c\hat{J}(t_{i_n}) \times\delta T[1-\alpha c\hat{J}(t_{i_{n+1}})\delta T]\text{”}\rangle. \qquad (43)$$

Similarly, if $\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}},t_{i_{n+2}})$ denotes the joint probability that counts will be registered in the i_1th, i_2th$\cdots$, and i_nth time intervals, but not in the i_{n+1}th and i_{n+2}th intervals, then

$$\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}},t_{i_{n+2}}) + \tilde{P}_{n+1}(t_{i_1},\cdots t_{i_n},t_{i_{n+1}};\, t_{i_{n+2}}) = \tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+2}})\,, \qquad (44)$$

and from (40) and (43)

$$\tilde{P}_n(t_{i_1},\cdots t_{i_n};\, t_{i_{n+1}},t_{i_{n+2}}) = \langle\text{“}\alpha c\hat{J}(t_{i_1})\delta T\cdots\alpha c\hat{J}(t_{i_n})\delta T \times[1-\alpha c\hat{J}(t_{i_{n+1}})\delta T][1-\alpha c\hat{J}(t_{i_{n+2}})\delta T]\text{”}\rangle. \qquad (45)$$

By proceeding in this manner we readily see that the probability of obtaining n counts in the time intervals labeled by $i_1, i_2,\cdots,i_n$, but no counts in any of the other $(T/\delta T-n)$ time intervals, is

$$\tilde{P}_n(t_{i_1},\cdots t_{i_n};\,\text{no other}) = \langle\text{“}\prod_{r=1}^{n}[\alpha c\hat{J}(t_{i_r})\delta T] \prod_{\substack{s=1\\ \text{except } s=i_1,\cdots i_n}}^{T/\delta T}[1-\alpha c\hat{J}(t_s)\delta T]\text{”}\rangle. \qquad (46)$$

The required probability $p(n;\, t,\, t+T)$ can now be obtained from (46) by summing over all possible time intervals $i_1, i_2,\cdots,i_n$ in which the counts can occur, dividing by the number of permutations of the equal intervals, and proceeding to the limit $\delta T\to 0$. Thus:

$$p(n;\, t,\, t+T) = \lim_{\delta T\to 0}\frac{1}{n!}\sum_{i_1=1}^{T/\delta T}\cdots\sum_{i_n=1}^{T/\delta T}\langle\text{“}\prod_{r=1}^{n}[\alpha c\hat{J}(t_{i_r})\delta T] \times\prod_{\substack{S=1\\ \text{except } S=i_1,\cdots i_n}}^{T/\delta T}[1-\alpha c\hat{J}(t_s)\delta T]\text{”}\rangle. \qquad (47)$$

Actually, the relation (40) and subsequent equations are valid only when the time intervals δT are much longer than a typical period of the light, so that strictly speaking, we are not entitled to proceed to the mathematical limit $\delta T\to 0$. However, since typical periods of a light beam are shorter than 10^{-14} sec, which is far beyond the limit of resolution of available detectors, it is clear that, under normal circumstances, we may regard the case $\delta T\sim 10^{-14}$ sec as a very good approximation to the limit, provided the intensity is not excessively high, with $\alpha c\langle\hat{J}(t)\rangle\delta T\ll 1$ for $\delta T\sim 10^{-14}$ sec, and provided the rate of counting due to spontaneous transitions is not excessive, with $\alpha cS\delta T(\mu/L^3)\approx 8\pi\alpha\delta TS\nu_0^2\Delta\nu/c^2\ll 1$ for $\delta T\sim 10^{-14}$ sec. The last condition would be satisfied if the response bandwidth $\Delta\nu$ of the quantum counter were limited to about 10^{10} counts/sec and the surface area S to about 10^{-6} cm^2, with $\alpha<1$. These numbers give some idea of the rate of spontaneous counting.

Now if n is not too large, and if the foregoing restrictions hold, we may replace the product

$$\prod_{\substack{S=1\\ \text{except } S=i_1,\cdots i_n}}^{T/\delta T}[1-\alpha c\hat{J}(t_s)\delta T] \quad\text{by}\quad \prod_{s=1}^{T/\delta T}[1-\alpha c\hat{J}(t_s)\delta T]$$

in Eq. (47). Moreover, by expanding the product it may be shown[20]

$$\prod_{s=1}^{T/\delta T}[1-\alpha c\hat{J}(t_s)\delta T]\to\exp[-\sum_{s=1}^{T/\delta T}\alpha c\hat{J}(t_s)\delta T] \to\exp\left[-\int_t^{t+T}\alpha c\hat{J}(t')dt'\right] \qquad (48)$$

for sufficiently short δT, provided any particular ordering of the operators is preserved in the expansion. Similarly,

$$\sum_{i_1=1}^{T/\delta T}\cdots\sum_{i_n=1}^{T/\delta T}\prod_{r=1}^{n}[\alpha c\hat{J}(t_{i_r})\delta T]\to\left[\int_t^{t+T}\alpha c\hat{J}(t')dt'\right]^n \qquad (49)$$

when δT is very small. By introducing (48) and (49) into (47), we finally arrive at

$$p(n;\, t,\, t+T) = \frac{1}{n!}\left\langle\text{“}\left[\int_t^{t+T}\alpha c\hat{J}(t')dt'\right]^n \times\exp\left[-\int_t^{t+T}\alpha c\hat{J}(t')dt'\right]\text{”}\right\rangle. \qquad (50)$$

Since the operator on the right of Eq. (50) is in antinormal order, we may make use of the result embodied in Eq. (32), which holds for any antinormally ordered product. Thus we replace the $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{\mathbf{A}}^\dagger(\mathbf{x},t)$ operators by their eigenvalues (corresponding to right and left eigenstates, respectively) and average with respect to the weighting functional $\phi(\{v_{\mathbf{k},s}\})$ given by Eq. (34). We then obtain

$$p(n;\, t,\, t+T) = (1/n!)\langle\langle W^n e^{-W}\rangle\rangle\,, \qquad (51)$$

where

$$W = \alpha c\int_S\int_t^{t+T}\mathbf{V}^*(\mathbf{x},t')\cdot\mathbf{V}(\mathbf{x},t')d^2x dt'\,, \qquad (52)$$

and $\langle\langle g(\{v_{\mathbf{k},s}\})\rangle\rangle$ stands for the average of $g(\{v_{\mathbf{k},s}\})$ evaluated with respect to $p(\{v_{\mathbf{k},s}\})$.

Equation (51) has a very close formal similarity to the well-known expression for the probability distribution $p'(n;t,t+T)$ of the number of counts registered by a photo-electric detector,[2,9,12,13,20]

$$p'(n;t,t+T)=(1/n!)\langle W^n e^{-W}\rangle\,, \tag{53}$$

which holds under rather less restrictive conditions than Eq. (51), where $\langle g(\{v_{\mathbf{k},s}\})\rangle$ stand for the average of $g(\{v_{\mathbf{k},s}\})$ calculated with respect to $\phi(\{v_{\mathbf{k},s}\})$ as weighting functional. Because of the formal similarity of the probability distributions (51) and (53), a number of relations between the moments of n and of W, which are already well known in connection with Eq. (53), can be written down at once. Thus[2,12,20]

$$\langle n\rangle_n=\langle\langle W\rangle\rangle\,, \tag{54}$$

$$\langle(\Delta n)^2\rangle_n=\langle n\rangle_n+\langle\langle(\Delta W)^2\rangle\rangle\,, \tag{55}$$

$$\langle \exp inx\rangle_n=\langle\langle \exp W(e^{ix}-1)\rangle\rangle\,, \tag{56}$$

where the symbol $\langle\ \rangle_n$ denotes the statistical average over the ensemble of n with respect to $p(n;t,t+T)$ given by Eq. (51). Since the weighting functional $p(\{v_{\mathbf{k},s}\})$ is positive definite, $\langle\langle(\Delta W)^2\rangle\rangle\geqslant 0$, and the fluctuations of n will exceed or equal those corresponding to a Poisson distribution for all states of the field. This is not necessarily the case for a photoelectric detector.

By Fourier inversion of Eq. (56), with the substitution $(e^{ix}-1)=iy$, we obtain[21]

$$\mathcal{P}(W)=\int_{-\infty}^{\infty}\langle(1+iy)^n\rangle_n\exp(-iyW)dy\,, \tag{57}$$

where $\mathcal{P}(W)$ is the probability density of W defined by

$$\mathcal{P}(W)=\int\delta\bigg[W-\alpha c\int_S\int_t^{t+T}\mathbf{V}^*(\mathbf{x},t')\cdot\mathbf{V}(\mathbf{x},t')d^2x dt'\bigg]\times p(\{v_{\mathbf{k},s}\})d^2\{v_{\mathbf{k},s}\}\,. \tag{58}$$

However, the formal similarity of Eqs. (51) and (53) and of the relations derived from them should not obscure the fact that the counting distributions $p(n;t,t+T)$ and $p'(n;t,t+T)$ will in general be very different. Since the explicit evaluation of $p(n;t,t+T)$ from Eq. (51) tends to be somewhat involved even for simple states of the field, we will illustrate some properties of the quantum counter by calculating the variance of the number of counts from Eq. (55).

7. SOME EXAMPLES OF $\langle(\Delta n)^2\rangle_n$ FOR A QUANTUM COUNTER

Let us consider a radiation field in a coherent state $|\{v_{\mathbf{k},s}'\}\rangle$. Let the field be in the form of plane waves incident normally on the quantum counter, such that all nonzero values of $v_{\mathbf{k},s}'$ correspond to modes to which the quantum counter is sensitive. In this case we find from Eq. (34) that

$$p(\{v_{\mathbf{k},s}\})=\prod_{\{\mathbf{k},s\}}\bigg[\frac{1}{\pi}\exp(-|v_{\mathbf{k},s}-v_{\mathbf{k},s}'|^2)\bigg]\,. \tag{59}$$

For simplicity we suppose that the counting interval T is short compared with the reciprocal frequency spread of all modes of the set $[\mathbf{k},s]$. It then follows from Eqs. (52) and (29) that

$$W=\alpha cT\int_S\mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)d^2x\,, \tag{60}$$

so that

$$\begin{aligned}\langle\langle W\rangle\rangle&=\alpha cT\int\int_S\prod_{\{\mathbf{k},s\}}\bigg[\frac{1}{\pi}\exp(-|v_{\mathbf{k},s}-v_{\mathbf{k},s}'|^2)\bigg]\mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)d^2xd^2\{v_{\mathbf{k},s}\}\\&=\alpha cT\int\int_S\prod_{\{\mathbf{k},s\}}\bigg[\frac{1}{\pi}\exp(-|v_{\mathbf{k},s}|^2)\bigg][\mathbf{V}^*(\mathbf{x},t)+\mathbf{V}'^*(\mathbf{x},t)]\cdot[\mathbf{V}(\mathbf{x},t)+\mathbf{V}'(\mathbf{x},t)]d^2xd^2\{v_{\mathbf{k},s}\}\,,\end{aligned} \tag{61}$$

where

$$\mathbf{V}'(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{[\mathbf{k},s]}v_{\mathbf{k},s}'\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)] \tag{62}$$

is the complex wave amplitude corresponding to the coherent state $|\{v_{\mathbf{k},s}'\}\rangle$. The weighting functional under the integral in Eq. (61) will be recognized as the form of $p(\{v_{\mathbf{k},s}\})$ given by Eq. (34) for the vacuum field. If we denote the "vacuum expectation" with respect to this functional by ${}^0\langle\langle\ \rangle\rangle^0$ we can write (61) in the abbreviated form

$$\begin{aligned}\langle\langle W\rangle\rangle&=\alpha cT\int_S{}^0\langle\langle[\mathbf{V}^*(\mathbf{x},t)+\mathbf{V}'^*(\mathbf{x},t)]\cdot[\mathbf{V}(\mathbf{x},t)+\mathbf{V}'(\mathbf{x},t)]\rangle\rangle^0d^2x\\&=\alpha cTS[{}^0\langle\langle I(\mathbf{x},t)\rangle\rangle^0+I'(\mathbf{x},t)]\,,\end{aligned} \tag{63}$$

[21] Compare E. Wolf and C. L. Mehta, Phys. Rev. Letters 13, 705 (1964).

where $\mathbf{x}$ is any point on the surface S, and

$$\mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)\equiv I(\mathbf{x},t)$$

and

$$\mathbf{V}'^*(\mathbf{x},t)\cdot\mathbf{V}'(\mathbf{x},t)\equiv I'(\mathbf{x},t). \tag{64}$$

In the derivation of (63) we have made use of the fact that ${}^0\langle\langle V(\mathbf{x},t)\rangle\rangle^0$ vanishes and that both ${}^0\langle\langle I(\mathbf{x},t)\rangle\rangle^0$ and $I'(\mathbf{x},t)$ are constant over the surface S. By expanding $\mathbf{V}(\mathbf{x},t)$ according to Eq. (29) we find

$$\begin{aligned}\langle\langle W\rangle\rangle &=\alpha cST\left[\frac{1}{L^3}\sum_{[\mathbf{k},s]}\sum_{[\mathbf{k}',s']}{}^0\langle\langle v_{\mathbf{k},s}{}^*v_{\mathbf{k}',s'}\rangle\rangle^0\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\exp\{i[(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}-c(k'-k)t]\}+I'(\mathbf{x},t)\right]\\ &=\alpha cST\left[\frac{1}{L^3}\sum_{[\mathbf{k},s]}{}^0\langle\langle|v_{\mathbf{k},s}|^2\rangle\rangle^0+I'(\mathbf{x},t)\right]\\ &=\alpha cST[(\mu/L^3)+I'(\mathbf{x},t)].\end{aligned} \tag{65}$$

Similarly we find that

$$\begin{aligned}\langle\langle W^2\rangle\rangle &=(\alpha cT)^2\int_S\int{}^0\langle\langle[\mathbf{V}^*(\mathbf{x},t)+\mathbf{V}'^*(\mathbf{x},t)]\cdot[\mathbf{V}(\mathbf{x},t)+\mathbf{V}'(\mathbf{x},t)]\\ &\qquad\times[\mathbf{V}^*(\mathbf{x}',t)+\mathbf{V}'^*(\mathbf{x}',t)]\cdot[\mathbf{V}(\mathbf{x}',t)+\mathbf{V}'(\mathbf{x}',t)]\rangle\rangle^0d^2xd^2x'\\ &=(\alpha cT)^2\left\{\int_S\int{}^0\langle\langle I(\mathbf{x},t)I(\mathbf{x}',t)\rangle\rangle^0d^2xd^2x'+S^2I'^2(\mathbf{x},t)+2S^2(\mu/L^3)I'(\mathbf{x},t)\right.\\ &\qquad\left.+\int_S\int[\mathbf{V}'^*(\mathbf{x},t)\cdot{}^0\langle\langle\mathbf{V}(\mathbf{x},t)\mathbf{V}^*(\mathbf{x}',t)\rangle\rangle^0\cdot V'(\mathbf{x}',t)+\text{c.c.}]d^2xd^2x'\right\}.\end{aligned} \tag{66}$$

The two integrals in this expression are evaluated in the Appendix. We finally obtain

$$\langle\langle W^2\rangle\rangle=(\alpha cTS)^2\left[\frac{\mu^2}{L^6}+\frac{2\mu}{L^6}+\frac{2\mu\beta\Delta\nu}{L^3cS}+I'^2(\mathbf{x},t)+\frac{2\mu}{L^3}I'(\mathbf{x},t)+\frac{4I'(\mathbf{x},t)\Delta\nu}{cS}\right], \tag{67}$$

where β is a constant having the order of magnitude unity, and $\Delta\nu$ is the optical bandwidth corresponding to the set of modes $[\mathbf{k},s]$ to which the counter responds. Hence

$$\langle\langle(\Delta W)^2\rangle\rangle=(\alpha cTS)^2\left[\frac{2\mu}{L^6}+2\beta\frac{\mu\Delta\nu}{L^3cS}+4I'(\mathbf{x},t)\frac{\Delta\nu}{cS}\right], \tag{68}$$

and from (54) and (55),

$$\langle(\Delta n)^2\rangle_n=(\alpha cTS)\left[\frac{\mu}{L^3}(1+2\alpha cTS/L^3+2\beta\alpha T\Delta\nu)+I'(\mathbf{x},t)(1+4\alpha T\Delta\nu)\right]. \tag{69}$$

Since $T\Delta\nu\ll 1$ by hypothesis, and $cTS\ll L^3$, we finally arrive at

$$\begin{aligned}\langle(\Delta n)^2\rangle_n &\approx(\alpha cTS)[(\mu/L^3)+I'(\mathbf{x},t)]\\ &=\langle n\rangle_n.\end{aligned} \tag{70}$$

It is interesting to note that this is the same formula as that holding for the counts registered by a photo-electric detector which is located in a similar coherent field.

As a second example we consider a radiation field from a thermal source in the form of an unpolarized, plane beam falling normally on the quantum counter. For such a field $p(\{v_{\mathbf{k},s}\})$ has already been shown to be of the form[5,9]

$$p(\{v_{\mathbf{k},s}\})=\prod_{\{\mathbf{k},s\}}\frac{1}{\pi(1+q_{\mathbf{k},s})}\exp\left[\frac{-|v_{\mathbf{k},s}|^2}{1+q_{\mathbf{k},s}}\right], \tag{71}$$

where $q_{\mathbf{k},s}$ is the average photon occupation number of the $\mathbf{k},s$ mode of the incident field. This result also follows directly from Eq. (38) when the well-known Gaussian form of $\phi(\{v_{\mathbf{k},s}\})$ is substituted. Once again we suppose that

$q_{\mathbf{k},s}$ is nonzero only for the modes belonging to the set $[\mathbf{k},s]$ to which the quantum counter responds, and that the counting interval T is sufficiently short that $T\ll 1/\Delta\nu$.

It follows from (71), (52), and (29) that

$$\begin{aligned}\langle\langle W\rangle\rangle &= \alpha cT\int_S \langle\langle \mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)\rangle\rangle d^2x \\ &= \frac{\alpha cT}{L^3}\sum_{[\mathbf{k},s]}\sum_{[\mathbf{k}',s']}\langle\langle v_{\mathbf{k},s}{}^* v_{\mathbf{k}',s'}\rangle\rangle \boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\int_S \exp\{i[(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}-c(k'-k)t]\}d^2x \\ &= (\alpha cTS/L^3)\sum_{[\mathbf{k},s]}[1+q_{\mathbf{k},s}], \\ &= \alpha cTS(\mu/L^3)(1+\delta), \end{aligned} \tag{72}$$

where we have put $\sum_{[\mathbf{k},s]} q_{\mathbf{k},s}=\mu\delta$. The parameter δ plays the role of a degeneracy parameter and is a measure of the average photon occupation number per mode.

Similarly we find that

$$\begin{aligned}\langle\langle W^2\rangle\rangle &= \frac{(\alpha cT)^2}{L^6}\iint_S \langle\langle \mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)\mathbf{V}^*(\mathbf{x}',t)\cdot\mathbf{V}(\mathbf{x}',t)\rangle\rangle d^2x d^2x' \\ &= \frac{(\alpha cT)^2}{L^6}\sum_{[\mathbf{k},s]}\sum_{[\mathbf{k}',s']}\sum_{[\mathbf{k}'',s'']}\sum_{[\mathbf{k}''',s''']}\langle\langle v_{\mathbf{k},s}{}^* v_{\mathbf{k}',s'} v_{\mathbf{k}'',s''}{}^* v_{\mathbf{k}''',s'''}\rangle\rangle(\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}) \\ &\quad\times(\boldsymbol{\varepsilon}_{\mathbf{k}'',s''}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}''',s'''})\iint_S \exp\{i[(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}+(\mathbf{k}'''-\mathbf{k}'')\cdot\mathbf{x}'-c(k'-k+k'''-k'')t]\}d^2xd^2x',\end{aligned}$$

and with the help of (71), that

$$\begin{aligned}\langle\langle W^2\rangle\rangle &= \frac{(\alpha cTS)^2}{L^6}\Bigg\{\sum_{[\mathbf{k},s]\neq[\mathbf{k}',s']}\sum \langle\langle |v_{\mathbf{k},s}|^2\rangle\rangle\langle\langle |v_{\mathbf{k}',s'}|^2\rangle\rangle \\ &\quad\times\left[1+|\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}|^2\frac{1}{S^2}\iint_S \exp[i(\mathbf{k}'-\mathbf{k})\cdot(\mathbf{x}-\mathbf{x}')]d^2xd^2x'\right]+\sum_{[\mathbf{k},s]}\langle\langle |v_{\mathbf{k},s}|^4\rangle\rangle\Bigg\} \\ &= \frac{(\alpha cTS)^2}{L^6}\Bigg\{\sum_{[\mathbf{k},s]}\sum_{[\mathbf{k}',s']}(1+q_{\mathbf{k},s})(1+q_{\mathbf{k}',s'})\left[1+|\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}|^2\frac{1}{S^2}\iint_S \exp\{i(\mathbf{k}'-\mathbf{k})\cdot(\mathbf{x}-\mathbf{x}')\}d^2xd^2x'\right]\Bigg\}.\end{aligned} \tag{73}$$

The double summation in the last equation is evaluated in the Appendix. With the help of the result obtained there and Eq. (72) we find

$$\langle\langle(\Delta W)^2\rangle\rangle=\alpha cTS(\mu/L^3)[2\alpha T\Delta\nu(\beta+\beta'\delta)+\tfrac{1}{2}\alpha cTS(\mu/L^3)\delta^2], \tag{74}$$

where β and β' are constants of order of magnitude unity. From Eq. (55) together with (72) and (74) we finally arrive at

$$\langle(\Delta n)^2\rangle_n=\alpha cTS(\mu/L^3)[(1+\delta)+2\alpha T\Delta\nu(\beta+\beta'\delta)+\tfrac{1}{2}\alpha cTS(\mu/L^3)\delta^2]. \tag{75}$$

Since $\alpha T\Delta\nu\ll 1$, the second term within the square brackets is always small compared with the first and can be neglected.

Two limiting forms of Eq. (75) are of interest. When the incident light beam is nondegenerate and $\delta\ll 1$, the first term within the square brackets in Eq. (75) becomes the dominant term, and in view of (72) and (54),

$$\langle(\Delta n)^2\rangle_n\approx\langle n\rangle_n. \tag{76}$$

In the degenerate limit $\delta\gg 1$, the last term within the square brackets in Eq. (75) becomes important, and we have

$$\begin{aligned}\langle(\Delta n)^2\rangle_n &\approx \alpha cTS(\mu/L^3)\delta[1+\tfrac{1}{2}\alpha cTS(\mu/L^3)\delta] \\ &\approx \langle n\rangle_n[1+\tfrac{1}{2}\langle n\rangle_n].\end{aligned} \tag{77}$$

This formula is exactly the same as that obeyed by the counts registered by a photoelectric detector which is illuminated by a plane beam of unpolarized thermal light.[2,22,23] Once again we note that the formulas for quantum counters and photoelectric detectors tend to coincide in the classical limit.

8. CONCLUSIONS

We have seen that a photon counting device functioning by the stimulated emission—rather than absorption—of photons is feasible in principle, and that correlations measured with a number of such counters correspond to expectation values of antinormally ordered complex field operators. We have expressed the antinormally ordered correlations in a form which shows clearly that they furnish a useful description of the field only when the average photon occupation number per mode is greater than unity. When this is not so, measurements carried out with the quantum counter are dominated by spontaneous emission. We have derived expressions for the fluctuations of the counts registered by a quantum counter, and shown that these expressions have a formal similarity to corresponding expressions for a photoelectric detector. The results of measurements performed with a quantum counter and with a photoelectric detector have been shown to coincide in the classical limit of an intense field. The discussion shows that the quantum counter in practice has disadvantages over the photodetector as a tool for the investigation of fields. The analysis was undertaken mainly with a view to understanding the differences in behavior.

APPENDIX: EVALUATION OF TERMS IN EQS. (66) AND (73)

We shall now evaluate some of the terms in Eq. (66). Remembering that $\mathbf{V}'(\mathbf{x},t)$ is constant over the surface S, we find with the help of the expansion (29b),

$$Q_3 \equiv \iint_S [\mathbf{V}'^*(\mathbf{x},t)\cdot {}^0\langle\langle \mathbf{V}(\mathbf{x},t)\mathbf{V}^*(\mathbf{x}',t)\rangle\rangle^0 \cdot \mathbf{V}'(\mathbf{x}',t)+\text{c.c.}]d^2x d^2x'$$

$$=\frac{1}{L^3}|\mathbf{V}'(\mathbf{x},t)|^2 \sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]} (\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1})(\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}^*\cdot\boldsymbol{\varepsilon}_0)\,{}^0\langle\langle v_{\mathbf{k}_1,s_1}v_{\mathbf{k}_2,s_2}^*\rangle\rangle^0$$

$$\times\iint_S \exp\{i[(\mathbf{k}_1\cdot\mathbf{x}-\mathbf{k}_2\cdot\mathbf{x}')-c(k_1-k_2)t]\}d^2xd^2x'+\text{c.c.},$$

$$=\frac{2}{L^3}|\mathbf{V}'(\mathbf{x},t)|^2\sum_{[\mathbf{k},s]}|\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}|^2\iint_S \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]d^2xdx',$$

$$=\frac{2S^2}{L^3}I'(\mathbf{x},t)\sum_{[\mathbf{k},s]}|\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}|^2\left(\frac{\sin\frac{1}{2}k_xl_x}{\frac{1}{2}k_xl_x}\right)^2\left(\frac{\sin\frac{1}{2}k_yl_y}{\frac{1}{2}k_yl_y}\right)^2, \tag{A1}$$

where we have made use of the fact that

$$ {}^0\langle\langle v_{\mathbf{k}_1,s_1}v_{\mathbf{k}_2,s_2}^*\rangle\rangle^0 = {}^0\langle\langle |v_{\mathbf{k}_1,s_1}|^2\rangle\rangle^0 \delta_{\mathbf{k}_1,\mathbf{k}_2}{}^3\delta_{s_1,s_2},$$
$$=\delta_{\mathbf{k}_1,\mathbf{k}_2}{}^3\delta_{s_1,s_2}. \tag{A2}$$

$I'(\mathbf{x},t)$ is the instantaneous light intensity of the incident beam at any point $\mathbf{x}$ on the surface of the quantum counter, $\boldsymbol{\varepsilon}_0$ is the unit vector in the direction of $\mathbf{V}'(\mathbf{x},t)$ defined by $\boldsymbol{\varepsilon}_0=\mathbf{V}'(\mathbf{x},t)/|\mathbf{V}'(\mathbf{x},t)|$, and l_x, l_y are the perpendicular linear dimensions of the surface S. Since l_x, l_y will normally be very large compared with a typical wavelength corresponding to the set $[\mathbf{k},s]$, it follows from the form of Eq. (A1) that $|k_x|$, $|k_y|\ll|k|$ for all terms of the summation which contribute significantly to the sum. Hence

$$k_z\approx\pm|k|, \tag{A3}$$

and the vector $\mathbf{k}$ may be taken to point approximately in the same (or opposite) direction as the incident beam. It then follows that

$$\sum_{[s]}|\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}|^2=1. \tag{A4}$$

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[22] H. Hurwitz, J. Opt. Soc. Am. **35**, 525 (1945).
[23] E. Wolf, Proc. Phys. Soc. (London) **76**, 424 (1960).

With the help of (A4) and the rule

$$\frac{1}{L^3}\sum_{[\mathbf{k}]}\rightarrow\frac{1}{(2\pi)^3}\int_{[\mathbf{k}]}d^3k$$

we can now rewrite (A1) in the form

$$Q_3=\frac{2S^2}{(2\pi)^3}I'(\mathbf{x},t)\int_{[\mathbf{k}]}\left(\frac{\sin\frac{1}{2}k_xl_x}{\frac{1}{2}k_xl_x}\right)^2\left(\frac{\sin\frac{1}{2}k_yl_y}{\frac{1}{2}k_yl_y}\right)^2d^3k,$$

and, in view of (A3)

$$=\frac{2S^2}{(2\pi)^3}I'(\mathbf{x},t)\frac{(2\pi)^2}{l_xl_y}\frac{4\pi\Delta\nu}{c}$$

$$=4SI'(\mathbf{x},t)\Delta\nu/c, \tag{A5}$$

where $\Delta\nu$ is the optical bandwidth to which the quantum counter responds.

Consider now the first integral in Eq. (66). Again with the help of the expansion (29b) we find

$$Q_1\equiv\int\int_S {}^0\langle\langle I(\mathbf{x},t)I(\mathbf{x}',t)\rangle\rangle^0 d^2xd^2x'$$

$$=\frac{1}{L^6}\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}\sum_{[\mathbf{k}_3,s_3]}\sum_{[\mathbf{k}_4,s_4]}{}^0\langle\langle v_{\mathbf{k}_1,s_1}{}^*v_{\mathbf{k}_2,s_2}v_{\mathbf{k}_3,s_3}{}^*v_{\mathbf{k}_4,s_4}\rangle\rangle^0(\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2})$$

$$\times(\boldsymbol{\varepsilon}_{\mathbf{k}_3,s_3}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_4,s_4})\int\int_S\exp[i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{x}+(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}'-c(k_2-k_1+k_4-k_3)t]d^2xd^2x'$$

$$=\frac{1}{L^6}\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}{}^0\langle\langle|v_{\mathbf{k}_1,s_1}|^2|v_{\mathbf{k}_2,s_2}|^2\rangle\rangle^0\left[S^2+|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2\int\int_S\exp(i(\mathbf{k}_2-\mathbf{k}_1)\cdot(\mathbf{x}-\mathbf{x}'))d^2xd^2x'\right]$$

$$=\frac{S^2}{L^6}\sum_{[\mathbf{k}_1,s_1]\neq[\mathbf{k}_2,s_2]}\sum{}^0\langle\langle|v_{\mathbf{k}_1,s_1}|^2\rangle\rangle^0\,{}^0\langle\langle|v_{\mathbf{k}_2,s_2}|^2\rangle\rangle^0$$

$$\times\left\{1+|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2\left[\frac{\sin\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}{\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}\right]^2\left[\frac{\sin\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}{\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}\right]^2\right\}+\frac{2S^2}{L^6}\sum_{[\mathbf{k},s]}{}^0\langle\langle|v_{\mathbf{k},s}|^4\rangle\rangle^0$$

$$=\frac{S^2}{L^6}\left\{\mu^2+2\mu+\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2\left[\frac{\sin\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}{\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}\right]^2\left[\frac{\sin\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}{\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}\right]^2\right\}, \tag{A6}$$

where we have again made use of (A2) together with

$${}^0\langle\langle|v_{\mathbf{k},s}|^4\rangle\rangle^0=2. \tag{A7}$$

To evaluate the double summation we first note that, from geometry,

$$\sum_{s_1}\sum_{s_2}|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2=1+\cos^2\theta_{\mathbf{k}_1,\mathbf{k}_2}, \tag{A8}$$

where $\theta_{\mathbf{k}_1,\mathbf{k}_2}$ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. Since the factors involving sin*c* functions ensure that $k_2{}^x\approx k_1{}^x$ and $k_2{}^y\approx k_1{}^y$, it follows, when the allowed range Δk of k belonging to $[k]$ is small, that

$$k_2{}^z\approx\pm k_1{}^z,\quad\text{to within}\quad\delta k^z\approx-k^z+[(k^z)^2+2k\Delta k]^{1/2}. \tag{A9}$$

The factor $(1+\cos^2\theta_{\mathbf{k}_1,\mathbf{k}_2})$ is therefore approximately equal to 2 or $2(1-[k^z/k]^2)$. By putting $\mathbf{k}_2-\mathbf{k}_1=\boldsymbol{\xi}$, and converting one summation to an integral, we can write approximately

$$\frac{S^2}{L^6}\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2\left[\frac{\sin\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}{\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}\right]^2\left[\frac{\sin\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}{\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}\right]^2$$

$$\approx\frac{S^2}{(2\pi L)^3}\sum_{[\mathbf{k}_1]}\int(1+\cos^2\theta_{\mathbf{k}_1,\mathbf{k}_2})\left(\frac{\sin\frac{1}{2}\xi^x l_x}{\frac{1}{2}\xi^x l_x}\right)^2\left(\frac{\sin\frac{1}{2}\xi^y l_y}{\frac{1}{2}\xi^y l_y}\right)^2 d^2\xi\delta k^z$$

$$\approx\frac{S^2}{(2\pi L)^3}\sum_{[\mathbf{k}]}2\frac{(2\pi)^2}{l_x l_y}2\Delta k\beta$$

$$=2\beta\mu S\Delta\nu/cL^3, \tag{A10}$$

where β is a number of order of magnitude unity. We shall not evaluate β explicitly since the term involving β is negligible compared with other terms in Eq. (66). With the help of Eq. (A10), Eq. (A6) becomes

$$Q_1\approx S^2\left[\frac{\mu^2}{L^6}+\frac{2\mu}{L^6}+2\beta\frac{\mu}{L^3}\frac{\Delta\nu}{cS}\right], \tag{A11}$$

and from (66), (A5), and (A11) we finally arrive at

$$\langle\langle W^2\rangle\rangle\approx(\alpha cTS)^2\left[\frac{\mu^2}{L^6}+\frac{2\mu}{L^6}+\frac{2\beta\mu\Delta\nu}{L^3cS}+I'^2(\mathbf{x},t)+\frac{2\mu}{L^3}I'(\mathbf{x},t)+4I'(\mathbf{x},t)\frac{\Delta\nu}{cS}\right]. \tag{A12}$$

Consider now the double summation in Eq. (73), which we may write in the form

$$\langle\langle W^2\rangle\rangle\approx\frac{(\alpha cTS)^2}{L^6}\Bigg\{\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}(1+q_{\mathbf{k}_1,s_1})(1+q_{\mathbf{k}_2,s_2})$$
$$+\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2,s_2]}|\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}|^2\left[\frac{\sin\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}{\frac{1}{2}(k_2{}^x-k_1{}^x)l_x}\right]^2\left[\frac{\sin\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}{\frac{1}{2}(k_2{}^y-k_1{}^y)l_y}\right]^2+\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2]}q_{\mathbf{k}_1,s_1}q_{\mathbf{k}_2,s_2}$$
$$+2\sum_{[\mathbf{k}_1,s_1]}\sum_{[\mathbf{k}_2]}q_{\mathbf{k}_1,s_1}\left[\frac{\sin\frac{1}{2}k_2{}^x l_x}{\frac{1}{2}k_2{}^x l_x}\right]^2\left[\frac{\sin\frac{1}{2}k_2{}^y l_y}{\frac{1}{2}k_2{}^y l_y}\right]^2\Bigg\},$$

$$\approx\frac{(\alpha cTS)^2}{L^6}\left\{\mu^2(1+\delta)^2+T_2+\tfrac{1}{2}(\mu\delta)^2+2\mu\delta\left(\frac{L}{2\pi}\right)^3\int_{[\mathbf{k}]}\left[\frac{\sin\frac{1}{2}k^x l_x}{\frac{1}{2}k^x l_x}\right]^2\left[\frac{\sin\frac{1}{2}k^y l_y}{\frac{1}{2}k^y l_y}\right]^2 d^3k\right\}, \tag{A13}$$

where we have made use of the fact that the incident beam is plane, so that $q_{\mathbf{k},s}$ vanishes unless $\mathbf{k}$ points in the z-direction, and that the light is unpolarized, so that $q_{\mathbf{k},s_1}=q_{\mathbf{k},s_2}$. The term T_2 is the same as that which was evaluated in Eq. (A10), apart from a constant factor.

The value of the integral in Eq. (A13) is readily estimated. Since k has to lie within a range $\Delta k=2\pi\Delta\nu/c$, and since k^x and k^y are very small if the integrand is not to vanish, we have, as in the derivation of (A10),

$$\int_{[\mathbf{k}]}\left[\frac{\sin\frac{1}{2}k^x l_x}{\frac{1}{2}k^x l_x}\right]^2\left[\frac{\sin\frac{1}{2}k^y l_y}{\frac{1}{2}k^y l_y}\right]d^3k\approx\frac{(2\pi)^2}{l_x l_y}\beta'\Delta k$$
$$\approx(2\pi)^3\beta'\Delta\nu/cS, \tag{A14}$$

where β' is a number having order of magnitude unity. With the help of (A14) and (A10), (A13) becomes

$$\langle\langle W^2\rangle\rangle\approx(\alpha cTS)^2\left\{\frac{\mu^2}{L^6}(1+\delta)^2+\frac{1}{2}\frac{\mu^2}{L^6}\delta^2+2\beta\frac{\mu}{L^3}\frac{\Delta\nu}{cS}+2\beta'\frac{\mu}{L^3}\delta\frac{\Delta\nu}{cS}\right\}, \tag{A15}$$

and from (72) and (A15),

$$\langle\langle(\Delta W)^2\rangle\rangle\approx\alpha cTS(\mu/L^3)[2\alpha T\Delta\nu(\beta+\beta'\delta)+\tfrac{1}{2}(\mu/L^3)\alpha cTS\delta^2]. \tag{A16}$$