much like a Josephson junction of variable oxide thickness connected by a short. Thus, even "normal" junctions, (which usually have a nonuniform oxide thickness and many times contain shorts) may display the modified LC resonance.[10]

[10] D. N. Langenberg (private communication).

Perhaps the best way to excite or "energyze" the superconducting loop, in order to force it to oscillate, would be to place it in an external field and then quickly turn off the field. The decaying field inside the loop produces a voltage, which in turn will stimulate the resonance oscillations of the system. This process, of course, can be repeated many times per second.

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Macroscopic Field Equations for Metals in Equilibrium*

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The usual derivation of Maxwell's equations for magnetic materials rests on the assumption that the sources of magnetic field within the material can be split up into a magnetization density $\mathbf{M}$ and a current density $\mathbf{j}$. In metals the same electrons (the conduction electrons) contribute both to $\mathbf{M}$ and to $\mathbf{j}$, and one is forced to consider the question of what one means by $\mathbf{M}$ and what one means by $\mathbf{j}$. In this paper we answer the question for systems in equilibrium, using a thermodynamic approach. The separation of sources of magnetic field into $\mathbf{M}$ and $\mathbf{j}$ is to a large extent arbitrary, but can be done in such a way that $\mathbf{M}$ is uniquely related to the local magnetic field and $\mathbf{j}$ is zero for a normal metal in equilibrium, while in the mixed state of a superconductor it satisfies the force-balance equation $(\mathbf{j}\times\mathbf{B})/c+\mathbf{P}=0$, $\mathbf{P}$ being the pinning force. The stress tensor for a magnetic system is derived from first principles (not assuming the field equations), and used to obtain the force-balance equation by an alternative method. Finally, two-dimensional systems such as superconducting thin films and surface sheaths are examined by similar methods.

I. INTRODUCTION

THE magnetic properties of substances in equilibrium are governed completely, as far as a macroscopic description is concerned, by Maxwell's equations

$$\mathrm{div}\mathbf{B}=0, \tag{1}$$

$$\mathrm{curl}\mathbf{H}=4\pi\mathbf{j}/c, \tag{2}$$

together with the corresponding boundary conditions, the equations relating $\mathbf{B}$ and $\mathbf{H}$, and the condition that in equilibrium the current density $\mathbf{j}$ is zero. Maxwell's equations are not regarded as basic; they can be obtained from the assumption that the field at any point is the sum of three contributions, the field applied to the system (which obeys the free space Maxwell's equations), a contribution from a transport current with density $\mathbf{j}$, and a contribution from a distribution of magnetic dipoles with density $\mathbf{M}$. One is led naturally[1] to the existence of two fields $\mathbf{B}$ and $\mathbf{H}$, satisfying (1) and (2), and related to each other by the equation

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}. \tag{3}$$

The use of Maxwell's equations, then, involves a number of assumptions: that the sources of magnetic field in a material can be divided in a definite way into currents and magnetic dipoles, that the currents are zero in equilibrium (even in the presence of a magnetic field), and that the magnetization density is a definite function of the magnetic field. These assumptions are plausible for an insulator, where the sources of the magnetic field can be considered as localized on particular atomic sites. It is not at all clear, though, why they should be applicable to metals. In this case part of the magnetism (the Landau diamagnetism) is due to the conduction electrons, and these are the same electrons as participate in the conduction process when an electric field is applied. None of the assumptions mentioned above is obviously justified. One other point is particularly worth mentioning: The $\mathbf{j}$ occurring in Maxwell's equations is not in general equal to the local average of the microscopic current density.[2]

The main purpose of the present paper is to discuss

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[1] C. A. Coulson, *Electricity* (Interscience Publishers, Inc., New York, 1961), Chaps. 6 and 7.

[2] To see this, note that if we ignore spin paramagnetism, the part of the magnetic field outside a metal due to the conduction electrons can be written in the form $(1/c)\mathrm{curl}\int(\mathbf{j}'/r)dV$, where $\mathbf{j}'$ is the current density with fluctuations on an atomic scale averaged out. Since the Landau diamagnetism of the conduction electrons does influence the magnetic field outside the metal, $\mathbf{j}$ must certainly be nonzero somewhere (in fact it is nonzero in a layer near the surface of width of the order of the cyclotron radius). The $\mathbf{j}$ occurring in Maxwell's equations, however, is everywhere zero in equilibrium, so that $\mathbf{j}$ and $\mathbf{j}'$ must be different.

how the quantities **B**, **H**, **M**, and **j** are to be defined in a metal. We shall show that they can be defined in such a way that Maxwell's equations remain valid. In addition, for a normal metal the **j** we define is zero in equilibrium. For the mixed state of a superconductor the existence of metastable states requires that the treatment be modified and we obtain instead the usual "force-balance" equation[3,4] $(\mathbf{j}\times\mathbf{B})/c+\mathbf{P}=0$, where **P** is the pinning force. Our derivation makes no explicit reference, however, to the concept of forces acting on flux lines, and we believe it to be of more general validity than the usual derivation.

In Sec. II we define the quantities **B**, **H**, **M**, and **j**, and in Sec. III we illustrate the method for deriving equilibrium field equations by thermodynamic means, with an example from hydrodynamics. In Sec. IV–VI we apply these methods to normal metals and to the mixed state of a superconductor, and in Sec. VII we derive the force-balance equation by an alternative method using the stress-tensor concept. Finally in Sec. VIII we apply the methods developed previously to thin films and the superconducting surface sheath.

II. DEFINITIONS OF MACROSCOPIC QUANTITIES

In view of the difficulty in defining **j** and **M** satisfactorily in a metal, let us begin by considering what quantities relevant to a macroscopic theory can be defined. The best starting point is the magnetic induction **B**. This quantity can be considered as some suitable macroscopic average of the microscopic magnetic field. Either a volume average, or a surface average obtained by dividing the flux through a closed curve by the area of the curve, is suitable for this purpose. This definition of **B** can be shown to be consistent with the usual definition of **B**[1] valid for insulators. The fact that the microscopic magnetic field is divergence free implies that the macroscopic average **B** is also divergence free, so that (1) holds.[5]

Now let us consider the other quantities to be defined. First let us suppose that we have arrived at some prescription for defining **H**. Equations (2) and (3) can then be used to define **j** and **M**. By defining a surface current

$$\mathbf{j}^s=(c/4\pi)(\mathbf{n}\times[\mathbf{H}]), \tag{4}$$

where $[\mathbf{H}]$ is the discontinuity in **H** in crossing the surface in the direction of **n**, and requiring that **H** be equal to **B** in free space one can ensure that even though **j** as defined may not be related to the microscopic current density the total current flowing through a wire is correctly given by the integral of **j** over any cross section plus the integral of $\mathbf{j}^s$ over the boundary of the cross section. Similarly the total magnetic moment of a specimen is correctly given in terms of **j** and **M** by the usual formula (if **j** is zero the magnetic moment is simply the volume integral of **M**).[6]

We now return to the definition of **H**. Our choice is governed by two considerations: (i) that **H** be well-defined and single-valued, and (ii) that the resulting equations for **j** and $\mathbf{j}^s$ in equilibrium should have a simple character. For the pure superconducting state ($\mathbf{B}=0$) the first consideration restricts us essentially to choosing $\mathbf{H}=0$. In other cases **H** will be defined in terms of the thermodynamic functions of the system. Consider a homogeneous specimen of such a shape that application of a suitable magnetic field produces a *uniform* magnetic field in its interior. Since the interior of the specimen is in the same state over the whole volume, it will contribute to the Helmholtz free energy of the system an amount f per unit volume, where f will depend only on **B**, the composition of the specimen, and quantities such as temperature and elastic strain. We shall indicate only the dependence on **B** explicitly in the following equations.

We define **H** to be 4π times the gradient of the free-energy density with respect to **B**, keeping the composition, temperature, elastic strain, etc. fixed.

$$\mathbf{H}=4\pi\nabla_{\mathbf{B}}f. \tag{5}$$

Hence **H**, like f, is a function of **B** which depends on the material under consideration. For the vacuum, f is $B^2/8\pi$, so that $\mathbf{H}=\mathbf{B}$ as required.

We have so far defined **H** only for homogeneous systems. However, in inhomogeneous systems where the spatial variations are slow we may define the value of **H** at any point by making it the same function of the value of **B** at the point as would obtain for a homogeneous system having the same composition as that present at the point under consideration. This definition is not valid when the composition varies appreciably over distances such as those associated with the microscopic structure (in particular the situation of pinning in the mixed state of a superconductor), and this situation requires separate consideration (Sec. VI).

III. HYDRODYNAMIC ILLUSTRATION

The method by which field equations may be obtained from thermodynamic considerations may be illustrated by the derivation of the hydrodynamic equations for a fluid in a gravitational field. Let ϕ be the gravitational potential, and $f(\rho)$ the Helmholtz free energy of the fluid per unit volume when the density is

[3] C. J. Gorter, Phys. Letters **1**, 69 (1962); P. W. Anderson, Phys. Rev. Letters **9**, 309 (1962).

[4] J. Friedel, P. G. de Gennes, and J. Matricon, Appl. Phys. Letters **2**, 119 (1963).

[5] Whenever **B** is referred to subsequently we shall mean the macroscopic **B** just defined rather than the true microscopic **B** which varies rapidly with position.

[6] These statements follow from the fact that the *microscopic* Maxwell's equations allow the total current and magnetization to be expressed in terms of integrals involving only magnetic fields *outside* the metal. If in free space all the fields considered are identical, then the same manipulations applied to the *macroscopic* equations allow these integrals to be related to the **j** and **M** defined by the macroscopic equations.

ρ (in zero gravitational potential). If the fluid is confined to the interior of a rigid container enclosing a volume V, its free energy is

$$F=\int_V \{f(\rho)+\rho\phi\}dV, \tag{6}$$

while its mass is

$$M=\int_V \rho dV. \tag{7}$$

Since the walls of the container keep the mass of the system constant but do no work, the equilibrium condition is that δF is zero for processes occurring at constant temperature, subject to the constraint $\delta M=0$. The constraint may be taken into account by using a Lagrange multiplier μ_0, giving the result

$$\delta\int_V \{f(\rho)+\rho\phi-\rho\mu_0\}dV=0,$$

so that

$$\frac{\partial}{\partial\rho}\{f(\rho)+\rho\phi-\rho\mu_0\}=0,$$

i.e.,

$$\mu+\phi=\mu_0=\text{const}, \tag{8}$$

where $\mu=\partial f/\partial\rho$ is the chemical potential per unit mass (for $\phi=0$). The hydrodynamic equation in its usual form,

$$-\rho\nabla\phi-\nabla p=0, \tag{9}$$

may be recovered by taking the gradient of (8), multiplying by ρ, and using the result $\rho d\mu=dp$.

The variational method may be applied to more complicated systems, such as a gas in equlibrium with its solution in a liquid. The foregoing analysis still holds, with ρ for the liquid phase now being the concentration of the dissolved component rather than the total density. In this case (8) implies not only the field equation (9) but also a boundary condition, namely the equality of the relevant chemical potential on the two sides of the boundary between the gas and the solution. This system is, as will be seen, closely related to a magnetic system consisting of partly magnetic material and partly vacuum (the magnetic material playing the role of a "solvent" for the magnetic field).

The basic assumption of the above approach to field equations is the existence of a free energy of the type (6), i.e., a volume integral of a free-energy density. In real systems the free energy cannot be regarded as infinitely localizable, and we should therefore consider the modifications, if any, introduced by using a nonlocal free-energy expression. The field equations will clearly become nonlocal, but the nonlocality will have little effect if the fields are slowly varying. At first sight the boundary conditions will be seriously modified, since the boundary is a region of rapid spatial variations. However, the changes in the free energy caused by the nonlocality are important only close to the boundary, while in a macroscopic system the dominant contribution to the free energy comes from the regions away from the boundary. Therefore the only result of the nonlocality is to cause the discontinuous changes at a boundary to become smeared out, and the boundary conditions remain unchanged, if they are taken to refer to the values of the field variables outside the boundary region. This result is a general feature of systems described by a variational principle.

IV. NORMAL MAGNETIC SYSTEMS

For a special type of field geometry, there exists a very close analogy between magnetic systems and the type of system discussed in the previous section, as was pointed out by Friedel *et al.*[4] The geometry is one in which the magnetic field is everywhere in the same direction (say the z direction) and the system has translational invariance in that direction. There is then only one nonvanishing field component B_z, and this may be considered as the "density of magnetic field," analogous to the gas density ρ in the previous section. The analogy is seen to be deeper when we consider what corresponds to the rigid container for the magnetic field. The important requirement for the equilibrium condition used above to be valid is that the container should do no work. A suitable container for magnetic field consists of a hollow cylinder made of a perfect conductor. Such a cylinder maintains the flux through it constant. This constraint is the exact analog of the constraint in the fluid case, i.e., that the mass of fluid is fixed, since if we multiply the flux through the cylinder by its length we obtain $\int B_z dV$, which is just the quantity obtained from (7) by replacing ρ by B_z.

We now suppose the free energy of the system to be given by an expression analogous to (6) (with ϕ put equal to zero)

$$F=\int_V f(B_z)dV. \tag{10}$$

No term involving the current density is included in the expression for F, since the current density is effectively a gradient of the magnetic field. Omitting terms in the current density in (10) is therefore equivalent to not including terms involving the density gradient in (6), and is justified in dealing with a macroscopic system. In the case of the mixed state, (10) ignores the lattice structure of the flux lines and assumes that only their density is relevant in determining the free-energy density. Numerical calculations indicate that this is a good approximation in practice. A more general formulation of the theory, which permits the effects of lattice structure in the flux lines to be taken into account, will be given in Sec. VII, but for the moment we shall assume (10) to be valid.

Because of the exact analogy between the two types

of system under consideration, the equilibrium condition for the magnetic system may be obtained immediately from that previously derived for the fluid system by replacing ρ by B_z (and putting ϕ equal to zero). The quantity analogous to $\mu=\partial f/\partial\rho$ is seen to be what we have defined to be $H_z/4\pi$ [Eq. (5)]. The equilibrium condition is therefore that H_z is a constant.[7] Since in this geometry H_x and H_y are zero, this statement is equivalent to the assertion that $\mathbf{j}$ is zero.

It is of interest to note the magnetic analog to pressure. Since for a fluid $p=\rho\mu-f$, the magnetic pressure is $B_zH_z/4\pi-f$. This result is in agreement with that obtained from the general stress tensor derived in Sec. VII.

Now let us consider the general case of arbitrary field configurations. In this case we must take into account explicitly the condition $\mathrm{div}\mathbf{B}=0$, which was automatically satisfied with the special geometry considered above. This constraint on the way $\mathbf{B}$ can vary is most simply taken into account by writing the variational condition in terms of $\delta\mathbf{A}$ rather than $\delta\mathbf{B}$, $\mathbf{A}$ being the vector potential. For general field configurations, it is not convenient to regard the field as being produced by constraints which do no work. Instead, we suppose the field applied to the system to be produced by a current distribution $\mathbf{J}(\mathbf{r})$. The work done against induced emf's by the sources of the currents if the field changes is given by the well-known formula

$$dW=(1/c)\int(\mathbf{J}\cdot\delta\mathbf{A})dV. \tag{11}$$

We may suppose the source currents to be flowing in free space, so that the macroscopic form of Maxwell's equations is not assumed in deriving this result.

Assuming for F the form

$$F=\int f(\mathbf{B})dV \tag{12}$$

and using the definition of $\mathbf{H}$, we may write

$$\delta F=(1/4\pi)\int(\mathbf{H}\cdot\delta\mathbf{B})dV=(1/4\pi)\int(\mathbf{H}\cdot\mathrm{curl}\delta\mathbf{A})dV,$$

which, using definitions (2) and (4), may be transformed into the form

$$(1/c)\left\{\int(\mathbf{j}\cdot\delta\mathbf{A})dV+\int(\mathbf{j}^s\cdot\delta\mathbf{A})dS\right\}.$$

The thermodynamic condition for equilibrium, $\delta F=dW$, thus reads

$$(1/c)\int(\mathbf{J}\cdot\delta\mathbf{A})dV=(1/c)\left\{\int(\mathbf{j}\cdot\delta\mathbf{A})dV+\int(\mathbf{j}^s\cdot\delta\mathbf{A})dS\right\}. \tag{13}$$

Since δA is arbitrary, (13) implies the field equation

$$\mathbf{j}=0, \tag{14}$$

the source current J being assumed to be zero in the region under consideration, and the boundary condition

$$\mathbf{j}^s=0. \tag{15}$$

V. SUPERCONDUCTORS

The conclusions of the previous section are true only for normal metals. For the pure superconducting state they are not valid quite simply because $\nabla_{\mathbf{B}}f$ is not well defined for $\mathbf{B}=0$ (the dominant term in f for small $\mathbf{B}$ being proportional to $|\mathbf{B}|$). The behavior on a macroscopic scale of the pure superconducting state is determined completely by the equation $\mathbf{B}=0$ for the interior, and the boundary condition that the normal component of $\mathbf{B}$ at an interface is zero. The definition given for the pure superconducting state, $\mathbf{H}=0$, does not give any extra information by itself, and was given simply in order that the surface current density should be a well-defined quantity.

The analysis of the preceding section breaks down in the mixed state for a different reason, namely that the structure of the mixed state[8] permits metastable states to be formed. This possibility is quite independent of the existence of pinning centers, and its cause may be visualized as follows. Consider a mixed-state wire in which the flux lines have a helical geometry like the strands of a rope. This corresponds essentially to a situation where the wire is carrying a current in a longitudinal magnetic field. The free energy of the system can be lowered by a process corresponding to the untwisting of the strands of the rope. If the flux lines are held in fixed positions at the ends of the wire this untwisting process cannot occur without cutting the flux lines and rejoining them, which cannot happen physically on account of the large activation energy involved.[9] The clamping action at the ends of the wire may come about in two ways. One possibility is to connect the ends to a current source, in which case the flux

[7] J. Friedel *et al.* (Ref. 4) assumed for the equilibrium condition (in the absence of pinning) the constancy of the magnetic pressure. This is correct for a homogeneous system, but if the composition varies slowly in space there is an effective force exerted on the magnetic field by the magnetic medium, and the magnetic pressure is not constant. The present derivation of the result that the effective chemical potential is constant remains valid even for inhomogeneous systems. It is likewise true that the magnetic chemical potential is continuous at a metal-vacuum interface, though again the magnetic pressure is not continuous.

[8] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

[9] It is possible that this process may actually occur if the critical current is exceeded in the longitudinal-field case, where the exact mechanism determining the critical current density has not yet been elucidated. The flux lines would then become entangled around each other, and the magnetic properties of the specimen might as a result become permanently altered as long as the specimen remained in the mixed state.

lines adjust themselves so that the current through the wire is equal to the applied current. Alternatively, the ends may be joined together to form a loop of wire. In this case the process of joining the flux lines together stops them unwinding and one has a persistent ring current. Clearly, metastability in the mixed state occurs because of the existence of the structure associated with the flux lines, and cannot occur in the normal state, where the concept of a flux line is only a picture to aid the imagination.

A second type of metastability occurs in the mixed state and is caused by inhomogeneities on a similar scale to that of the flux lines. We shall defer consideration of this possibility to the following section.

Metastability of the first type affects the equilibrium condition in the following way. The equation $\delta F = dW$ is now true not for all changes $\delta\mathbf{A}$, but only for those which correspond to continuous displacements of the flux lines without cutting them. Let $\xi(\mathbf{r})$ be a vector field specifying the displacement of the flux lines as a function of position (only the component of ξ normal to the flux lines having any physical significance), and let C be any closed curve lying entirely inside the superconductor. If a number of flux lines cross C, the change in the flux through C is equal to this number multiplied by the flux quantum. Using flux quantization again to relate the flux line density to $\mathbf{B}$, we can write this result in the form

$$\oint_C \delta\mathbf{A}\cdot d\mathbf{s} = \delta \oint_C \mathbf{A}\cdot d\mathbf{s} = \oint_C (\xi\times\mathbf{B})\cdot d\mathbf{s},$$

The integral of the quantity $\delta\mathbf{A}-(\xi\times\mathbf{B})$ therefore vanishes when taken round any closed curve, so that $\delta\mathbf{A}-\xi\times\mathbf{B}$ must be the gradient of a scalar, which may be chosen to be zero by means of a suitable gauge transformation. Consequently, inside the superconductor $\delta\mathbf{A}$ can be taken to be of the form $\xi\times\mathbf{B}$, while outside it is arbitrary. Equation (13) must therefore be replaced by

$$(1/c)\int(\mathbf{J}\cdot\delta\mathbf{A})dV = (1/c)\Big[\int(\mathbf{j}\cdot\delta\mathbf{A})dV + \int\{\mathbf{j}\cdot(\xi\times\mathbf{B})\}dV + \int\{\mathbf{j}^s\cdot(\xi\times\mathbf{B})\}dS\Big], \quad (16)$$

where the integrals are taken over the region outside the superconductor, the inside of the superconductor, and the surfaces of discontinuity respectively. The field equations resulting from the application of the variational principle for the superconductors are

$$(\xi\times\mathbf{B})\cdot\mathbf{j} = 0, \quad \text{all } \xi$$

$$(\xi\times\mathbf{B})\cdot\mathbf{j}^s = 0, \quad \text{all } \xi$$

which imply

$$\mathbf{j}\times\mathbf{B} = 0, \quad (17)$$

$$\mathbf{j}^s\times\mathbf{B} = 0. \quad (18)$$

These equations replace (14) and (15). Equation (18) implies that there is no surface current unless $\mathbf{B}$ is parallel to the surface of the specimen. In this case the surface barrier[10] tends to prevent the flux lines crossing the surface. As far as the application of the variational principle is concerned, the surface barrier acts as an additional constraint, requiring that only displacements whose normal component vanishes at the surface should be considered. In this case $(\xi\times\mathbf{B})\cdot\mathbf{j}^s$ is equal to zero even if $\mathbf{j}^s$ is not parallel to $\mathbf{B}$. The surface currents possible in this situation give rise to the hysteresis observed in surface-barrier experiments,[10] since the values of $\mathbf{B}$ inside and outside the specimen are no longer uniquely related. The critical value of the surface current is determined by microscopic considerations.[11]

VI. EFFECTS OF PINNING

We shall treat the inhomogeneities which can pin flux lines as a perturbation. Specifically, we assume that the free energy of the system is the sum of a term F_0 given by (12) and an inhomogeneity term F_1. F_1 cannot be written in terms of $\mathbf{B}$ only as F_0 can, since it depends on the way in which the positions of the individual flux lines are related to the positions of the pinning centers. We may define a pinning force $\mathbf{P}$ by the relation

$$\delta F_1 = -\int(\mathbf{P}\cdot\xi)dV. \quad (19)$$

That is, $\mathbf{P}$ is related to the change in F_1 when a flux line is displaced by a small amount, for the particular configuration of flux lines under consideration. $\mathbf{P}$ is to be regarded as some macroscopic average which makes (19) true when ξ is slowly varying in space.

The effect of incorporating the term δF_1 in the expression for δF is easily seen to be that of replacing (17) by

$$(1/c)(\mathbf{j}\times\mathbf{B})+\mathbf{P} = 0. \quad (20)$$

The angular dependence of the critical current density in a fixed field[12] follows from (20) on assuming the existence of a maximum pinning force P_c for a given density of flux lines. For $\mathbf{j}\|\mathbf{B}$, (20) predicts an infinite critical current density. Clearly some form of instability other than with respect to the motion of flux lines must limit the current density in this case (such as that discussed in footnote 9).

[10] C. P. Bean and J. D. Livingston, Phys. Rev. Letters **12**, 14 (1964); M. A. R. LeBlanc and D. J. Griffiths, Phys. Letters **21**, 150 (1966).

[11] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), p. 79; J. G. Park, Phys. Rev. Letters **15**, 353 (1965).

[12] G. D. Cody, G. W. Cullen, and J. P. McEvoy, Jr., Rev. Mod. Phys. **36**, 95 (1964).

VII. STRESS-TENSOR FORMULATION

The procedure adopted in the preceding two sections is probably the simplest way to justify the relation (20) rigorously. Equation (20) is normally regarded in a quite different light, as expressing the equilibrium of a flux line under the influence of two forces, the first term on the left-hand side being the Lorentz force and the second the pinning force. Identification of $(1/c)(\mathbf{j}\times\mathbf{B})$ as the Lorentz force is not immediately justifiable since $\mathbf{j}$ and $\mathbf{B}$ are both averages of microscopic quantities, the former being an average of a particularly obscure type. In this section we shall rederive (20) by a method which shows it more explicitly to be a force-balance equation. The method is based on the stress-tensor concept, and can be generalized to allow for the possibility of shearing stresses in the lattice of flux lines. We shall not consider the latter case further here, beyond remarking that the force-balance equation certainly needs modification in this situation. What we shall find in the simplified case where the lattice structure is neglected is that the identification of the terms on the left-hand side of (20) as Lorentz and pinning forces is correct if the only inhomogeneities are the rapid fluctuations whose effect is contained in F_1, but that otherwise modifications in the interpretation are needed, though (20) is still correct. The modification is of slightly more than academic interest in the case of the pinning force, since the mutual force between the electromagnetic field and the lattice is what determines the elastic stresses in the lattice.

Stress tensors may be defined by a general procedure for systems whose field equations can be derived from a variational principle.[13] We assume that we are dealing with systems for which the concept of a displacement of the system is well defined. In the present case we must consider two types of displacement, a displacement of the crystal lattice $\boldsymbol{\xi}^l$, having the usual meaning, and a displacement of the magnetic field $\boldsymbol{\xi}^m$, defined by the previously derived relationship $\delta\mathbf{A}=\boldsymbol{\xi}^m\times\mathbf{B}$. The electrons are assumed to follow the lattice to preserve charge neutrality.[14] Since the displacements have a well-defined effect on the field variables, an expression can be derived for the change in free-energy density in terms of the displacements, of the form

$$\delta f=\sum_\alpha\left(f_i{}^\alpha\xi_i+f_{ij}{}^\alpha\frac{\partial\xi_i{}^\alpha}{\partial x_j}\right) \tag{21}$$

($\alpha=l$ or m, and the summation convention applies to coordinate indices). The condition $\delta\int f dV=0$ then gives rise to the field equations

$$\partial f_{ij}{}^\alpha/\partial x_j=f_i{}^\alpha. \tag{22}$$

The quantity $(f_i{}^l+f_i{}^m)\xi_i$ represents the change in f resulting when the lattice and magnetic field are both given a uniform displacement ξ_i, and is therefore equal to $-(\partial f/\partial x_i)\xi_i$. Therefore

$$\frac{\partial}{\partial x_j}(\sum_\alpha f_{ij}{}^\alpha)=-\frac{\partial f}{\partial x_i}. \tag{23}$$

Hence if we define

$$S_{ij}=f\delta_{ij}+\sum_\alpha f_{ij}{}^\alpha, \tag{24}$$

then

$$\partial S_{ij}/\partial x_j=0. \tag{25}$$

S_{ij} may therefore be identified with the stress-tensor of the system. To obtain force equations for the individual components of the system we write f as f^l+f^m, where f^l is the free-energy density when $\mathbf{B}=0$ and f^m is the magnetic free-energy density. Since f^l does not involve $\mathbf{B}$ explicitly, either f^m or f may be used in Eq. (5) defining $\mathbf{H}$. We may now separate S_{ij} into two parts $S_{ij}{}^l$, $S_{ij}{}^m$, where

$$S_{ij}{}^\alpha=f^\alpha\delta_{ij}+f_{ij}{}^\alpha. \tag{26}$$

From (22) we have

$$\partial S_{ij}{}^\alpha/\partial x_j+P_i{}^\alpha=0, \tag{27}$$

where

$$P_i{}^\alpha=-\partial f^\alpha/\partial x_i-f_i{}^\alpha, \tag{28}$$

and from (23),

$$\sum_\alpha P_i{}^\alpha=0. \tag{29}$$

It is clear that boundary conditions related to (27) can also be derived.

It only remains to derive the stress tensors and to interpret $P_i{}^\alpha$. $S_{ij}{}^l$ is simply the ordinary elastic stress tensor for zero magnetic field. To find the magnetic stress tensor, we note that $\delta f^m=(1/4\pi)(\mathbf{H}\cdot\delta\mathbf{B})$ and $\delta\mathbf{A}=\boldsymbol{\xi}^m\times\mathbf{B}$ imply

$$\delta f^m=(1/4\pi)\mathbf{H}\cdot\mathrm{curl}(\boldsymbol{\xi}^m\times\mathbf{B}).$$

$f_{ij}{}^m$ may then be found from the definition (21) and substituted into (26), giving

$$S_{ij}{}^m=\delta_{ij}(f_0{}^m-\mathbf{H}\cdot\mathbf{B}/4\pi)+H_iB_j/4\pi. \tag{30}$$

If we ignore pinning, $P_i{}^m$ is equal to minus the spatial gradient of the magnetic free-energy density at constant magnetic field. The effect of pinning is to add to $\mathbf{P}^m$ the quantity we have defined as $\mathbf{P}$. Hence

$$\begin{aligned}\mathbf{P}^m&=\mathbf{P}-(\nabla_\mathrm{r})_\mathbf{B}f_0{}^m,\\ \mathbf{P}^l&=-\mathbf{P}^m=-\mathbf{P}+(\nabla_\mathrm{r})_\mathbf{B}f_0{}^m.\end{aligned} \tag{31}$$

The derivative on the last term of these equations is to be taken regarding $f_0{}^m$ as a function of $\mathbf{B}$ and $\mathbf{r}$ only.

[13] In most applications the action is the object which is stationary, rather than the free energy as in the present instance.

[14] One can also consider displacements in which the electrons move relative to the crystal lattice and the field. The force-balance equation resulting reduces to the result $\nabla\mu=0$, where μ is the chemical potential for electrons.

The elastic force equation has the usual form, with $\mathbf{P}^l$ acting as a body force. The magnetic force equation reduces to Eq. (20), since the true Lorentz force comes out to be

$$(P_L)_i=\frac{\partial S_{ij}{}^m}{\partial x_i}=\frac{1}{4\pi}(\mathrm{curl}\mathbf{H}\times\mathbf{B})_i+\left(\frac{\partial f_0{}^m}{\partial x_i}\right)_{\mathbf{B}},$$

i.e.,

$$\mathbf{P}_L=(\mathbf{j}\times\mathbf{B})/c+(\nabla_r)_{\mathbf{B}}f_0{}^m. \tag{32}$$

The simple form of the force-balance equation is seen to be the result of the cancellation of two terms $\pm(\nabla_r)_{\mathbf{B}}f_0{}^m$.

VIII. TWO-DIMENSIONAL SYSTEMS

Techniques similar to those described above may be used to treat systems which are effectively two-dimensional. Examples include the superconducting surface sheath formed when a type-II superconductor is placed in a field somewhat greater than that required to destroy superconductivity in the bulk,[15] and superconducting thin films. Unless the applied field is exactly in the plane of the sheath or film, a "mixed state," in which the surface contains "flux spots," i.e., points analogous to flux lines in bulk systems, where the superconducting order parameter is zero, occurs. Field equations can be found for systems in this "mixed-surface state" by requiring the condition $\delta F=dW$ to hold for changes in the system brought about by displacements of the flux spots. Here we shall indicate briefly the results obtained by this analysis.

In the first place, it is necessary to redefine the surface current density, to take into account the effect of a surface free-energy density f^s associated with the sheath or film. If we define the surface magnetic field H^s to be 4π times the derivative of f^s with respect to the normal component of $\mathbf{B}$, then the modified surface-current density is

$$j^s=(c/4\pi)\mathbf{n}\times\{[\mathbf{H}]-\nabla H^s\}. \tag{33}$$

A surface pinning force $\mathbf{P}^s$ may be defined in a manner analogous to the definition of the bulk pinning force. The field equation resulting from the application of the variational principle is

$$(1/c)(\mathbf{j}^s\times\mathbf{B})_t+\mathbf{P}^s=0, \tag{34}$$

where t denotes the tangential component.

If there is a barrier present tending to prevent flux spots crossing the edge of the film or sheath, a current may flow round the edge, similar to the surface current produced by the surface barrier in bulk samples.

One interesting system to which the ideas of this section may be applied is that of magnetically coupled thin films.[16] The magnetic coupling between the two films gives rise to a new type of pinning force, which modifies the critical currents of the films, as observed experimentally.

IX. CONCLUSIONS

We have shown that macroscopic field equations for metals in equilibrium may be obtained from a thermodynamic approach. These equations have the same form as Maxwell's equations, provided that the appropriate nonlinear relationship between $\mathbf{B}$ and $\mathbf{H}$ is used. The only variable occurring in the equations which has a simple physical interpretation is the average field $\mathbf{B}$, and $\mathbf{H}$, $\mathbf{j}$, and $\mathbf{M}$ appear only as auxiliary mathematical quantities. However, any quantity which can be measured in a macroscopic experiment, such as the magnetization of a specimen or the total current carried by a wire, can be computed correctly by regarding $\mathbf{j}$ and $\mathbf{M}$ as the true current density and magnetization respectively, as was pointed out in Sec. II.

Normal metals and superconductors in the mixed state are treated on the same footing, the differences in behavior between the two being clearly seen to be the consequence of the flux-line structure present in the mixed state.

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[15] D. Saint-James and P. G. de Gennes, Phys. Letters **7**, 306 (1963).

[16] I. Giaever, Phys. Rev. Letters **15**, 825 (1965); **16**, 466 (1966).