PHYSICAL REVIEW VOLUME 152, NUMBER 1 2 DECEMBER 1966

LC-Time Behavior of Weak Superconducting Loops

ERNEST E. H. SHIN* AND BRIAN B. SCHWARTZ
National Magnet Laboratory,† Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 15 July 1966)

We use Anderson's operator representation to obtain the radiative properties of a superconducting loop with a single Josephson junction. We treat n, the number of displaced pairs, and φ, the relative phase across the junction, as canonically conjugate operators. Since n is related to the voltage across the junction by $V=2en/\hbar$ and φ is related to the current in the loop via the fluxoid quantization, the Hamiltonian for the system, $H=\frac{1}{2}CV^2+\frac{1}{2}LI^2-E_J\cos\varphi$, can be reduced to operator form. By applying the Hamiltonian formalism in the usual manner, the equations of motion are obtained. In the limit of small oscillation ($\langle\varphi^2\rangle<1$), the Hamiltonian can be expanded and yields a solution which corresponds to a resonant frequency

$$\tilde{\omega}_{LC}=\left[\left(\frac{1}{LC}\right)+\frac{(2e)^2E_J}{\hbar C}\right]^{1/2}.$$

This mode corresponds to an "LC" oscillation of the loop modified by the Josephson junction. For physical situations, the frequency of the oscillation can be adjusted to be as high as 10^{12} sec^{-1}, with a purity of 1 part in 10^5 and at a power output of 10^{-13} W. Under appropriate conditions, in addition to the modified LC resonance, we obtain solutions corresponding to the familiar ac Josephson radiation as well as the metastable flux states of a loop with a weak link.

I. INTRODUCTION

IN this paper, we report a theoretical study on the radiative properties of a superconducting loop with a Josephson junction.[1] The study shows that by self-consistently including the inductance L and capacitance C of the loop, and the properties of the junction, the system can have radiative properties quite different from that usually associated with an isolated Josephson junction. In particular we obtain a solution corresponding to a modified LC oscillation of the loop. Physically, this resonant mode is possible only when the inductance is small enough to allow a harmonic-oscillator approximation for the true Hamiltonian. Since the frequency of the radiation can be as high as 10^{12} sec^{-1}, it may prove useful in obtaining radiation in the experimentally difficult *Zwischenwellen* region.

In Sec. II we introduce the Hamiltonian for the loop with a weak link. By using Anderson's operator formalism we are able to rewrite the Hamiltonian in terms of canonically conjugate operators and obtain the equation of motion for the operators. In Sec. III we present solutions which represent two limiting situations and correspond to whether or not we can make a harmonic-oscillator-type approximation for the Hamiltonian. For small oscillations of the system we find a modified LC frequency. For large oscillations, the system behavior is similar to the usual ac Josephson situation. In Sec. IV we calculate the power and purity of the modified LC radiation. In Sec. V discussion is presented concerning maximizing the parameters so as to enhance the ability of the system to oscillate with the modified LC frequency.

II. THEORY–OPERATOR FORMALISM

The system we wish to investigate is represented in Fig. 1 and consists of a superconducting loop of inductance L interrupted by a single Josephson junction of capacitance C and junction coupling energy E_J. To study the radiative properties of the loop, we make use of Anderson's operator representation[2] in which we treat n, the number of displaced pairs, and $\tilde{\varphi}$, the relative phase across the junction, as canonically conjugate operators such that $[\tilde{\varphi},n]=i$. In the operator representation, the Hamiltonian for the loop is given by $H=\frac{1}{2}CV^2+\frac{1}{2}LI^2-E_J\cos\tilde{\varphi}$ where $-E_J\cos\tilde{\varphi}$ represents the Josephson phase coupling energy, $V=(2en/C)$ is the voltage drop across the junction in operator form, and I is the current in the loop. For a thick loop, the phase continuity condition implies $\tilde{\varphi}+\varphi=2m\pi(m=0,\pm1,\pm2,\cdots)$ where φ is the phase associated with the magnetic flux enclosed within the loop and is, therefore, related to I by the relation[3]

$$\varphi=\frac{2\pi LI}{\Phi_0}=-\frac{2e}{\hbar}\oint \mathbf{A}\cdot d\mathbf{l}, \tag{1}$$

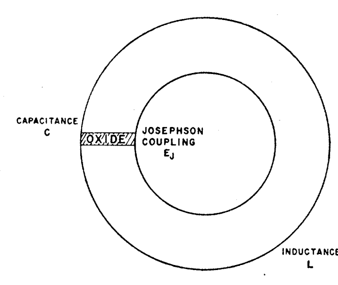

FIG. 1. A superconducting loop of inductance L interrupted by a Josephson junction of capacitance C with coupling energy E_J.

---

* Present address: Department of Physics, University of Miami, Coral Gables, Florida.

† Supported by the U. S. Air Force Office of Scientific Research.

[1] B. B. Schwartz and E. E. H. Shin, Bull. Am. Phys. Soc. 11, 191 (1966).

[2] P. W. Anderson, in *Lectures on the Many-Body Problem*, edited by E. Caianello (Academic Press Inc., New York, 1964), pp. 113–135.

[3] For a junction thickness $\Delta x(\approx10^{-7}$ cm), which is very small compared with the loop dimension, we are safe in neglecting the line integral across Δx. Except in Sec. IV and the caption of Fig. 2, we have let $c=1$.

where LI is the flux in the loop and Φ_0 is the flux quantum $h/2e$. Substituting for I and V the operators φ and n, the Hamiltonian obtained can be put into the form

$$H=\frac{2e^2}{C}n^2+\frac{\hbar^2}{8e^2L}\varphi^2-E_J\cos\varphi,$$
$$=H_{LC}-E_J\cos\varphi, \tag{2}$$

and H_{LC} can be written as

$$H_{LC}=\tfrac{1}{2}\hbar\omega_{LC}(N^2+\Phi^2), \tag{3}$$

where $\omega_{LC}=1/(LC)^{1/2}$ is the usual resonance frequency and

$$\Phi=\varphi/\rho_0,\quad N=\rho_0 N,\quad [\Phi,N]=i \tag{4}$$

with

$$\rho_0=\left[\frac{(2e)^2}{\hbar}\left(\frac{L}{C}\right)^{1/2}\right]^{1/2}. \tag{5}$$

By applying the Hamiltonian formalism in the usual manner, the equations of motion obtained are

$$\frac{d\varphi}{dt}=\frac{[\varphi,H]}{i\hbar}=\frac{4e^2n}{\hbar C}=\frac{2eV}{\hbar}, \tag{6a}$$

$$\frac{dn}{dt}=\frac{[n,H]}{i\hbar}=\frac{-\hbar}{(2e)^2L}\varphi-\frac{E_J}{\hbar}\sin\varphi \tag{6b}$$

The first equation is the same as in the usual ac Josephson effect.[4] The second equation, however, is new because we have included the inductance of the loop. Equation (6b) reduces the usual dc Josephson equation

$$J_J=(2eE_J/\hbar)\sin\varphi$$

when the inductance L is large. In addition, if we ask for the steady-state solutions of Eqs. (6a) and (6b), we obtain the conditions

$$(E_J/\hbar)\sin\varphi=[-\hbar/(2e)^2L]\varphi,\quad V=0; \tag{7}$$

which are the equations for the metastable current-carrying states of the loop with a single Josephson or "weak" link.[5] The exact time-dependent solutions to Eqs. (6a) and (6b) are difficult to obtain because of the nonlinear coupling term $(E_J/\hbar)\sin\varphi$. In the usual ac effect, harmonics and subharmonics occur in substantial proportion because of the contribution of higher order terms of φ in the expansion of $\sin\varphi$.[6] Equations (6a) and (6b) are quite general equations of motion for the loop with a weak link and contain, under appropriate conditions, all the previously discussed solutions. (In the presence of a magnetic field at the junction, terms involving the gradient of the phase must be included in the Hamiltonian. This field can be important in determining the resonant modes of oscillation of the junction treated as a cavity.[7])

III. APPROXIMATE SOLUTIONS OF THE OPERATOR EQUATIONS

Equations (6a) and (6b) are a set of highly nonlinear operator equations whose solutions are quite difficult to obtain. From the Hamiltonian equation (2), however, we see that the contribution of these higher order nonlinear terms depends critically on whether or not the operator $(E_J/\hbar)\cos\varphi$ can be replaced by the approximate operator

$$(E_J/\hbar)(1-\tfrac{1}{2}\varphi^2),$$

which is valid for small oscillation in phase. In the calculation which follows, we assume we can make such an approximation. After obtaining the new Hamiltonian and equations of motion we will return to ask: under what conditions was the expansion of the operator $\cos\varphi$ valid? Replacing the operator $\cos\varphi$ by $(1-\frac{1}{2}\varphi^2)$, the new Hamiltonian can then be written

$$H\simeq\tilde{H}_{LC}=\tfrac{1}{2}\hbar\tilde{\omega}_{LC}(\tilde{N}^2+\tilde{\Phi}^2), \tag{8}$$

where

$$\tilde{\omega}_{LC}=[\omega_{LC}^2+(2e)^2E_J/\hbar^2C]^{1/2},$$
$$\tilde{\Phi}=\varphi/\tilde{\rho}_0,\quad \tilde{N}=\tilde{\rho}_0N,\quad [\tilde{\Phi},\tilde{N}]=i,$$

with

$$\tilde{\rho}_0=\left[\frac{(2e)^2/C}{(\hbar^2/(2e)^2L)+E_J}\right]^{1/4}.$$

Equation (8) is a good approximation when $\langle\varphi^2\rangle<1$. We can simply determine the "root-mean-square" value of φ, which we denote by $\varphi_{\rm rms}=\langle\varphi^2\rangle^{1/2}$, by realizing that the average "potential" or "kinetic" energy of a harmonic oscillator is just half the energy of the system, so that

$$\tfrac{1}{2}\hbar\tilde{\omega}_{LC}\langle\tilde{\Phi}^2\rangle=\tfrac{1}{2}\epsilon_l=\tfrac{1}{2}\hbar\tilde{\omega}_{LC}(l+\tfrac{1}{2})$$

and

$$\varphi_{\rm rms}=\tilde{\rho}_0\langle\tilde{\Phi}^2\rangle^{1/2}=\tilde{\rho}_0(l+\tfrac{1}{2})^{1/2}=\tilde{\rho}_l<1. \tag{9}$$

Equation (9) gives the restriction for which we expect the harmonic approximation to be a good one. Thus, whenever Eq. (9) is satisfied, the system obeys a harmonic equation and oscillates at a modified LC frequency

$$\tilde{\omega}_{LC}=(\omega_{LC}^2+(2e)^2E_J/C\hbar)^{1/2}.$$

It is important that one should not confuse the expansion in powers of φ with treating the coupling energy as a perturbation.

To enhance the possibility of observing the modified LC oscillation, the parameter $\tilde{\rho}_0$ should be as small as possible so as to justify the small-oscillation expansion even when l is large, viz. $\tilde{\rho}_l<1$ even if $l\gg1$. Physically

[4] B. D. Josephson, Phys. Letters **1**, 251 (1962).
[5] A. M. Goldman, P. J. Kreisman, and D. J. Scalapino, Phys. Rev. Letters **11**, 495 (1964); B. B. Schwartz (unpublished); J. Mercereau (unpublished).
[6] D. N. Langenberg, D. J. Scalapino, B. N. Taylor, and R. E. Eck, Phys. Rev. Letters **11**, 294 (1965).
[7] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Phys. Rev. Letters **13**, 15 (1964).

realizable values for C and L are 10^2 cm and 10^{-5} cm (cgs units), respectively which, for a junction coupling energy E_J on the order of a few electron volts, results in a value for $\tilde{\rho}_0 \approx 10^{-2}$ (see Fig. 2). These same values of L and C produce an LC frequency $\omega_{LC} \approx 10^{12}$ sec^{-1} which for most physical situations ($E_J \lesssim 5$ eV) is larger than Anderson's term

$$((2e)^2 E_J / C\hbar)^{1/2}$$

in the modified frequency. Therefore, one is able to excite the system to a state $l = 10^2$, and still remain in the small oscillation limit $\tilde{\rho}_l \approx \tilde{\rho}_0 \ell^{1/2} = \frac{1}{10}$. The transitions from states ($l \to l-1$) will give rise to dipole radiation at the modified frequency. The magnitude and purity of this radiation is discussed in Sec. IV.

The oscillations of a Josephson junction in an open circuit ($L = \infty$) was considered earlier by Anderson.[2] In that case $\tilde{\rho}_0$ is typically on the order of 1 or greater so that even the zero point motion of the system violates the requirement of small excursion in φ. Since both the capacitance and E_J scale linearly with the junction area, the value for $\tilde{\rho}_0$ cannot be decreased. Only by including the inductance term can the value of $\tilde{\rho}_0$ be adjusted to be less than one.

The situation opposite to that discussed above is the limit $\tilde{\rho}_l \gg 1$. In this case higher harmonics cannot be neglected and a solution of the full operator Eq. (6b) is required. Differentiating Eq. (6a) and inserting the

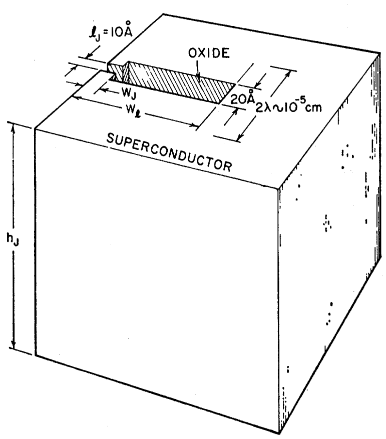

FIG. 2. The figure indicates schematically, but not to scale, the optimum specimen in which the modified LC resonance will be significant. The inductance L is given by $4\pi A_l / h_J$, where A_l is the area of the loop and h_J is the height of the loop, which is also the length of the junction. The area of the loop A_l is approximately $2\lambda w_l$, where 2λ is twice the penetration depth and w_l is the width of the loop. The capacitance C is given by $1/4\pi\ (A_J / l_J)$, where A_J is the surface area of the junction and l_J is the thickness of the oxide. The area of the junction A_J is equal to $w_J h_J$, where w_J is the width of the junction and h_J is the length of the junction. Thus, $\omega_{LC} = c/(LC)^{1/2}$ and in terms of the dimensions above is equal to $c(l_J / 2\lambda w_l w_J)^{1/2}$. For $2\lambda \sim 10^{-5}$ cm, $w_l = 10^{-2}$ cm, $w_J = 10^{-3}$ cm, and $l_J = 10^{-7}$ cm, $\omega_{LC} \approx 10^{12}$ sec^{-1}. For these same values of L and C, with h_J equal to its maximum effective value 10^{-1} cm (approximately twice Josephson penetration depth $2\lambda_J$), $\rho_0 \approx 10^{-2}$. We have assumed E_J is on the order of a few electron volts.

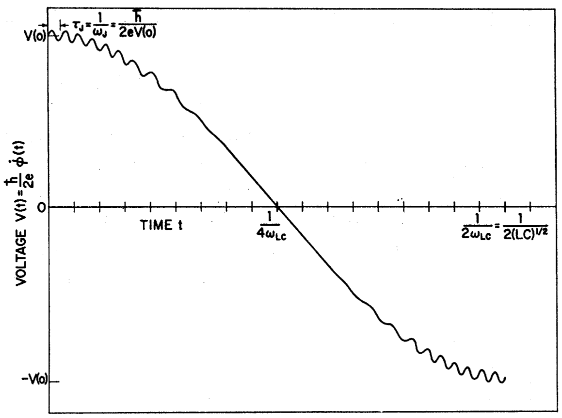

FIG. 3. A schematic representation of a half-cycle in the time behavior of the loop when $\tilde{\rho}_l \gg 1$. Note there are two main frequencies: $\omega_J = 2eV/\hbar$ and $\omega_{LC} = 1/(LC)^{1/2}$.

result into Eq. (6b) we have

$$d^2\varphi/dt^2 + \hbar^2 \varphi / LC + [E_J (2e)^2 / C] \sin\varphi = 0. \quad (10)$$

Equation (10) corresponds to the equation of motion for a pendulum in a gravitational field suspended from the center of a coil spring.[8] If Eq. (10) is treated classically, then for times such that $(\hbar^2/LC)\varphi(t)$ is small compared to $E_J(2e)^2/C$, the behavior is precisely that for the usual ac Josephson effect. In this situation, the system oscillates at a frequency determined by the instantaneous voltage across the junction according to $2eV(t)/\hbar$. For times such that $(\hbar^2/LC)\hbar(t)$ is greater than $E_J(2e)^2/C$, Eq. (10) reduces to the motion for the usual LC oscillation of the loop. A sketch of the motion is presented in Fig. 3.

The restriction on the magnitude of $\tilde{\rho}_l$ has a very physical meaning. One can determine the ratio of the maximum allowed Josephson frequency to the LC frequency $\tilde{\omega}_{LC}$, or what turns out to be equivalent, the ratio of the maximum enclosed flux to the flux quantum. Since the mean-square value of the number and phase, or equivalently voltage and current can be related to the energy of the system, we have

$$\tfrac{1}{2}C\langle V^2\rangle = \tfrac{1}{2}L\langle I^2\rangle + \tfrac{1}{2}E_J\langle\varphi^2\rangle = \tfrac{1}{2}\tilde{\omega}_{LC}(l+\tfrac{1}{2}). \quad (11)$$

Using Eqs. (11) to obtain the ratios we get the remarkable result

$$\frac{2e[\langle V^2\rangle]^{1/2}}{\hbar\tilde{\omega}_{LC}} = \frac{2\pi L[\langle I^2\rangle]^{1/2}}{\Phi_0} = \tilde{\rho}_l. \quad (12)$$

Equation (12) is by no means accidental and may be understood as follows: The Josephson junction prefers to oscillate at its own resonance frequency while the loop has its own LC frequency. If $\tilde{\rho}_l \gg 1$, then $\omega_J \gg \omega_{LC}$, and the pairs oscillate many times across the junction

[8] A unit-length pendulum with mass $C\hbar^2/4e^2$, coil spring constant $k^2 = \hbar^2/4e^2L$ in a gravitation field with $g = 4E_Je^2/C\hbar^2$ has the same equation of motion as the loop with a weak link.

by the Josephson mechanism before the LC oscillation has a chance to commence. Equivalently from Eq. (12), when $\tilde{\rho}_l \gg 1$ the system encloses many flux quanta and oscillates by converting one flux quantum of field into potential energy in the form of voltage across the capacitor or vice versa. On the other hand if $\tilde{\rho}_l < 1$, then $\omega_J < \omega_{LC}$, and the system oscillates at the modified LC frequency $\tilde{\omega}_{LC}$. In terms of the enclosed flux argument, for $\tilde{\rho}_l < 1$, the system encloses but a fraction of a flux quantum (actually 0 fluxoid) and therefore Josephson oscillation with the concomitant change of the fluxoid is impossible.

The conditions on $\tilde{\rho}_l$ can be understood quite well by considering the analogy with the pendulum[8] in a gravitational field. When $\tilde{\rho}_l \gg 1$, the pendulum has a large amount of total energy compared with the possible gravitational energy. As a result, the pendulum rotates through many angles of 2π before the coiling of the spring becomes effective so as to transform some of the kinetic energy of the moving pendulum (potential energy of the capacitor) into potential energy of the spring (kinetic energy due to current). These rotations of the pendulum through 2π correspond to the usual ac Josephson effect. When $\tilde{\rho}_l \ll 1$ we have the situation of small oscillation of the pendulum system. We can then make the harmonic approximation and obtain a modified frequency which is related to the sum of the effective coil spring and gravitational constants.

IV. POWER RADIATED

To calculate the power output of the spontaneous radiation, we carry the operator formalism still further by introducing the oscillator–radiation-field interaction of the form

$$H_{\text{rad}} = \frac{1}{c}\boldsymbol{\mu}\cdot\mathbf{A}, \tag{13}$$

where we assume that $\mathbf{A}$ interacts with the oscillating current only at the junction and we define the effective dipole moment operator $\boldsymbol{\mu}$ to be

$$\boldsymbol{\mu} = (2e)n(\Delta x)_{\text{eff}} \tag{14}$$

with $(\Delta x)_{\text{eff}} (\geq \Delta x)$ as the effective displacement (in cm) of the dipole and

$$\dot{\boldsymbol{\mu}} = (2e)(\Delta x)_{\text{eff}}\frac{dn}{dt} = (\Delta x)_{\text{eff}} I(t).$$

For a radiation field of frequency ω, with unit vector of polarization $\hat{e}$, $\mathbf{A}$ is given by

$$\mathbf{A} = \hat{e}(2\pi\hbar c^2/\omega)^{1/2} a_\omega^+ \exp(-i(\mathbf{k}\cdot\mathbf{r}-\omega t)), \tag{15}$$

where a_ω^+ is the photon creation operator and $k=\omega/c$. For the frequencies and junction thicknesses of practical interest, we are safe in using the dipole approximation $\exp(i\mathbf{k}\cdot\mathbf{r}) \simeq 1$. The radiative transition probability $\gamma(l)$ of state l is given by

$$\gamma(l) = \frac{2\pi}{\hbar}\sum_{l'<l}\int_0^\infty d(\hbar\omega)\rho(\omega)|\langle l',1_\omega| \times H_{\text{rad}}|l,0_\omega\rangle|^2 \delta(\epsilon_l - \epsilon_{l'} - \hbar\omega),$$

where $\rho(\omega)$ is the density of final states for the photons

$$\rho(\omega)d(\hbar\omega) = 4\pi(\hbar\omega)^2 d(\hbar\omega)/(2\pi\hbar c)^3. \tag{16}$$

If we choose $1 \ll l < l_c$ and $\tilde{\rho}_0 l^{1/2} \leq 10^{-1} (\tilde{\rho}_0 l_c = 1)$, then

$$\gamma(l) \approx \gamma(l \to (l-1)) \simeq \frac{l(2e)^2}{\hbar\tilde{\rho}_0 2c^3}\tilde{\omega}_{LC}^3 |(\Delta x)_{\text{eff}}|^2 \tag{17}$$

where we have used

$$|\langle l-1|(dn/dt)|l\rangle| = \tilde{\omega}_{LC}|\langle (l-1)|n|l\rangle| = l^{1/2}\tilde{\omega}_{LC}.$$

For the relative content of the higher harmonics, it is sufficient to calculate the ratio $\gamma(l \to l-3)/\gamma(l \to l-1)$ which for $\tilde{\rho}_0 l^{1/2} \leq 10^{-1}$ becomes

$$\gamma(l \to l-3)/\gamma(l \to l-1) \approx [1/(3!)^2 (\tilde{\rho}_0 l^{1/2})^4 \leq 10^{-5}, \tag{18}$$

which implies that the intensity of radiation at frequency $\omega = 3\tilde{\omega}_{LC}$ is only one part in 10^5 of the fundamental frequency $\tilde{\omega}_{LC}$. This is to be contrasted with the usual ac Josephson radiation in which harmonics abound by virtue of the high degree of nonlinearity inherent in the coupling.

Taking $(\Delta x)_{\text{eff}} = (\Delta x)$ of the junction would imply that the charges are polarized at the oxide-superconductor interface. However, since the displacement field may extend into the superconductor to a distance on the order of the penetration depth $\lambda (\simeq 10^{-5}$ cm), we can only say that $(\Delta x)_{\text{eff}}$ takes a value somewhere in the range $\Delta x \leq (\Delta x)_{\text{eff}} \leq (2\lambda + \Delta x)$ or more precisely 10^{-7} to 10^{-5} cm. For $\tilde{\omega}_{LC}$ typically on the order of 10^{12} sec^{-1}, $l \approx 10^2$, $\tilde{\rho}_0 \approx 10^{-2}$, and $(\Delta x)_{\text{eff}} \approx 10^{-6}$ cm, we obtain $\gamma(l) \simeq 10^{-13}$ W.

V. CONCLUSIONS

Since the power is relatively small, the modified LC radiation does not have any special advantage over the usual ac Josephson radiation insofar as power output is concerned. It does, however, have a clear advantage over the latter in its coherence properties as given by Eq. (18). The power output cannot be increased much further by changing the system to higher l or by increasing $\tilde{\omega}_{LC}$. The coherence requirement $(\tilde{\rho}_0 l^{1/2} < 1)$ and the condition[9] $I_{\max} < I_c$ of the junction restricts the maximum possible l, and practical geometry restricts the value of $\tilde{\omega}_{LC}$.

The optimum specimen in which to observe the modified LC resonance is schematically presented in Fig. 2. The requirement of small L (to make $\rho_0 \ll 1$) leads to a practical shape for the loop which looks very

[9] For the parameters of interest $[\langle I^2\rangle]^{1/2}$ is on the order of $10^4(l+\frac{1}{2})^{1/2}$ statamps which for $l \approx 100$ is 10^5 statamps. That is below the critical current of the junction which is typically 10^7 statamperes.

much like a Josephson junction of variable oxide thickness connected by a short. Thus, even "normal" junctions, (which usually have a nonuniform oxide thickness and many times contain shorts) may display the modified *LC* resonance.[10]

[10] D. N. Langenberg (private communication).

Perhaps the best way to excite or "energyze" the superconducting loop, in order to force it to oscillate, would be to place it in an external field and then quickly turn off the field. The decaying field inside the loop produces a voltage, which in turn will stimulate the resonance oscillations of the system. This process, of course, can be repeated many times per second.

PHYSICAL REVIEW VOLUME 152, NUMBER 1 2 DECEMBER 1966

Macroscopic Field Equations for Metals in Equilibrium*

B. D. JOSEPHSON
Department of Physics, University of Illinois, Urbana, Illinois
and
Royal Society Mond Laboratory, Cambridge University, Cambridge, England
(Received 9 June 1966)

The usual derivation of Maxwell's equations for magnetic materials rests on the assumption that the sources of magnetic field within the material can be split up into a magnetization density **M** and a current density **j**. In metals the same electrons (the conduction electrons) contribute both to **M** and to **j**, and one is forced to consider the question of what one means by **M** and what one means by **j**. In this paper we answer the question for systems in equilibrium, using a thermodynamic approach. The separation of sources of magnetic field into **M** and **j** is to a large extent arbitrary, but can be done in such a way that **M** is uniquely related to the local magnetic field and **j** is zero for a normal metal in equilibrium, while in the mixed state of a superconductor it satisfies the force-balance equation $(\mathbf{j}\times\mathbf{B})/c+\mathbf{P}=0$, **P** being the pinning force. The stress tensor for a magnetic system is derived from first principles (not assuming the field equations), and used to obtain the force-balance equation by an alternative method. Finally, two-dimensional systems such as superconducting thin films and surface sheaths are examined by similar methods.

I. INTRODUCTION

THE magnetic properties of substances in equilibrium are governed completely, as far as a macroscopic description is concerned, by Maxwell's equations

$$\mathrm{div}\mathbf{B}=0, \tag{1}$$

$$\mathrm{curl}\mathbf{H}=4\pi\mathbf{j}/c, \tag{2}$$

together with the corresponding boundary conditions, the equations relating **B** and **H**, and the condition that in equilibrium the current density **j** is zero. Maxwell's equations are not regarded as basic; they can be obtained from the assumption that the field at any point is the sum of three contributions, the field applied to the system (which obeys the free space Maxwell's equations), a contribution from a transport current with density **j**, and a contribution from a distribution of magnetic dipoles with density **M**. One is led naturally[1] to the existence of two fields **B** and **H**, satisfying (1) and (2), and related to each other by the equation

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}. \tag{3}$$

The use of Maxwell's equations, then, involves a number of assumptions: that the sources of magnetic field in a material can be divided in a definite way into currents and magnetic dipoles, that the currents are zero in equilibrium (even in the presence of a magnetic field), and that the magnetization density is a definite function of the magnetic field. These assumptions are plausible for an insulator, where the sources of the magnetic field can be considered as localized on particular atomic sites. It is not at all clear, though, why they should be applicable to metals. In this case part of the magnetism (the Landau diamagnetism) is due to the conduction electrons, and these are the same electrons as participate in the conduction process when an electric field is applied. None of the assumptions mentioned above is obviously justified. One other point is particularly worth mentioning: The **j** occurring in Maxwell's equations is not in general equal to the local average of the microscopic current density.[2]

The main purpose of the present paper is to discuss

* Work supported in part by the Office of Army Research under Contract No. DA-31-124-ARO(D)-114.

[1] C. A. Coulson, *Electricity* (Interscience Publishers, Inc., New York, 1961), Chaps. 6 and 7.

[2] To see this, note that if we ignore spin paramagnetism, the part of the magnetic field outside a metal due to the conduction electrons can be written in the form $(1/c)\mathrm{curl}\int(\mathbf{j}'/r)dV$, where $\mathbf{j}'$ is the current density with fluctuations on an atomic scale averaged out. Since the Landau diamagnetism of the conduction electrons does influence the magnetic field outside the metal, **j** must certainly be nonzero somewhere (in fact it is nonzero in a layer near the surface of width of the order of the cyclotron radius). The **j** occurring in Maxwell's equations, however, is everywhere zero in equilibrium, so that **j** and $\mathbf{j}'$ must be different.