distance from it. It is best seen at the highest speeds when the velocity gradient is greater. Its rate of travel depends on Ω." The similarity with Pellam's results (a) and (b) (nearly 40 years later) is unmistakable.

The stopping experiments are complicated by inherent instabilities. Comparing Figs. 1 and 2, both helium and H_2O respond more rapidly to stopping rotation than to starting and the departure from two-dimensional theory is greater for stopping. One expects that the response to stopping will appear to be faster when measuring v^2 with a disk fixed in the nonrotating laboratory frame even if the results for v are identical when compared in appropriate frames. The observed differences, however, are too large to account for in this way. According to McLeod, the large differences in H_2O are ". . . due to the break-up of the regular motion, owing to instability at the fixed outer wall. Except with the high speed and the large cylinder, the motion on starting, on the other hand, appears to the eye to be without appreciable irregularity." For a long, slender cylinder, one would expect the stopping instability to be primarily of Rayleigh[7] type, while for a shorter cylinder instability of the endwall boundary layer may dominate. Since the helium II and H_2O data for the largest Re are nearly identical, it would seem unnecessary to speculate that liquid helium II acquires any special rigidity in rotation not ordinarily possessed by classical liquids.

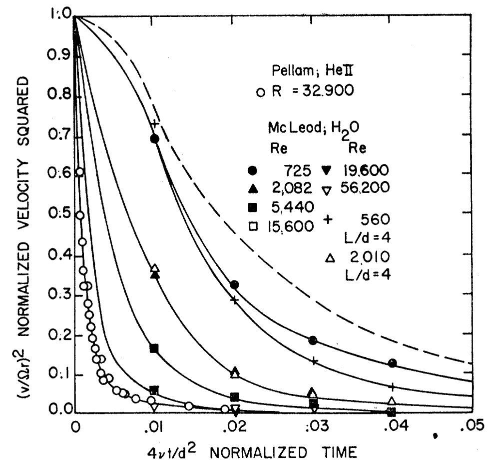

FIG. 2. Square of fluid velocity versus time at $r/d=0.375$ in suddenly decelerated cylindrical containers. Velocity normalized with equilibrium value. Time normalized with characteristic viscous diffusion time. For H_2O data, $L/d=2$ except where noted. Dashed line is two-dimensional theory for $L/d=\infty$.

The author would like to thank Professor J. R. Pellam, who kindly provided additional information about his experiment in order to make the present comparison possible.

[7] J. W. Strutt, Lord Rayleigh, Sci. Papers **6**, 447 (1920).

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Equilibrium Distribution of Rectilinear Vortices in a Rotating Container*

ALEXANDER L. FETTER
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
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A system of rectilinear vortices in an arbitrary multiply connected domain rotating with angular velocity Ω is studied with Lin's general formalism. In the limit of many vortices, the equilibrium distribution is shown to be a uniform vortex density $n=2\Omega/\kappa$ where κ is the circulation about each vortex. If the inner boundaries are specified by a set of contours C_α, each enclosing an area A_α, then the equilibrium value of the circulation Γ_α about C_α is given by $\Gamma_\alpha=2\Omega A_\alpha$. In equilibrium, the fluid rotates as a solid body with angular momentum $L=I\Omega$ and energy $E=\frac{1}{2}I\Omega^2$, where I is the moment of inertia.

I. INTRODUCTION

THE behavior of rotating liquid He II has frequently been studied with the model of a classical inviscid fluid containing rectilinear vortices.[1] The only important manifestation of the quantum nature of He II is the appearance of quantized circulation,[2,3] so that the circulation κ about each vortex is given by $\kappa=h/m$, where h is Planck's constant and m is the mass of a helium atom. The simplest experimental situation is a rotating cylinder of radius R, where a great many vortices are present for reasonable angular velocities ($\Omega\gg h/mR^2\approx1.6\times10^{-4}$ rad/sec for $R\approx1$ cm). It is

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[1] See, for example, H. E. Hall, Advan. Phys. **9**, 89 (1960); or W. F. Vinen, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. III, p. 1.

[2] L. Onsager, Nuovo Cimento **6**, Suppl. II, 249 (1949).

[3] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

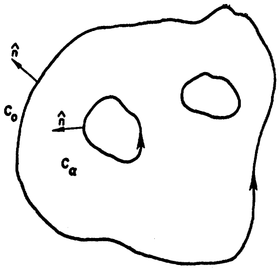

FIG. 1. The geometry of a multiply connected domain, showing the sense of the contours and the outward normal vectors.

generally assumed that the equilibrium configuration of a large number of vortices in a cylinder is a uniform density $n=2\Omega/\kappa=2m\Omega/h$.[3] This configuration has also been shown to minimize the free energy for a cylindrical container.[4]

Other more complicated geometries are also of interest in connection with rotating He II.[5] In particular, several recent theoretical papers[6] have examined the distribution of vortices in an annular region bounded by concentric cylinders ($R_1<r<R_2$). These studies predict that the fluid forms two regions: an inner irrotational vortex-free region ($R_1<r<r_a$) and an outer region ($r_a<r<R_2$) filled with a uniform density of vortices $n=2\Omega/\kappa$. In these calculations, it is assumed without comment that the energy E and angular momentum L of the system are composed of additive contributions from the mean fluid velocity and from the vortices.[7] This assumption is by no means obvious, however, and must be tested by a calculation based on first principles. For this reason, the present paper contains an exact treatment of rectilinear vortices in an arbitrary multiply connected container bounded externally by a contour C_0 and internally by a set of contours $\{C_\alpha\}$. The equilibrium vortex density depends on the angular velocity Ω. In the limit of many vortices, the equilibrium distribution is shown to be a uniform density $n=2\Omega/\kappa$ filling the whole container; the equilibrium circulation Γ_α about each internal contour is given by $\Gamma_\alpha=2\Omega A_\alpha$, where A_α is the area enclosed by C_α. The corresponding angular momentum and energy are precisely the values for solid-body rotation: $L_{eq}=I\Omega$ and $E_{eq}=\frac{1}{2}I\Omega^2$, where I is the classical moment of inertia.

The basic formalism (due to Lin[8]) is reviewed in Sec. II and applied in Sec. III to the special case of a simply connected region. Section IV treats the general multiply connected domain.

II. RECTILINEAR VORTICES IN TWO DIMENSIONS

The fundamental problem considered here is the two-dimensional motion of an incompressible fluid bounded externally by a contour C_0 and internally by a set of contours $\{C_\alpha\}$. This question has been studied in great detail by Lin,[8] whose notation will be used throughout. When the container rotates with angular velocity Ω, the fluid must move with the walls; this provides the necessary boundary condition

$$\mathbf{v}=\mathbf{\Omega}\times\mathbf{r} \quad \text{for} \quad \mathbf{r} \text{ on } C_0,\ \{C_\alpha\}, \tag{1}$$

where $\mathbf{r}$ is measured from the axis of rotation $\mathbf{\Omega}$. The equilibrium configuration for the rotating system is obtained by minimizing the free energy $F=E-\Omega L$, where E and L are the total energy and angular momentum.[9]

The motion of the fluid is most simply described with the stream function $\psi(\mathbf{r})$; the fluid velocity can be computed directly from ψ with the equations

$$\begin{aligned} v_x&=-\partial\psi/\partial y,\\ v_y&=\partial\psi/\partial x. \end{aligned} \tag{2}$$

In the present problem, the motion arises from three different sources: the vortices, the rotation of the walls, and the circulation about the internal boundaries. The stream function satisfies a linear differential equation, and each of the above contributions to ψ may be considered separately.

In the absence of vortices, ψ is a harmonic function and obeys Laplace's equation

$$\nabla^2\psi=0. \tag{3}$$

The rotation of the walls requires that

$$\psi(\mathbf{r})=\tfrac{1}{2}\Omega r^2 \quad \text{for} \quad \mathbf{r} \text{ on } C_0,\ \{C_\alpha\}, \tag{4}$$

which is equivalent to Eq. (1). It is convenient to separate the irrotational stream function into two terms

$$\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r}). \tag{5}$$

Here, the first term ψ_Ω represents the effect of the rotating boundaries and has the following properties:

$$\nabla^2\psi_\Omega(\mathbf{r})=0, \tag{6}$$

$$\psi_\Omega(\mathbf{r})=\tfrac{1}{2}\Omega r^2 \quad \text{for} \quad \mathbf{r} \text{ on } C_0,\ \{C_\alpha\}, \tag{7}$$

$$\oint_{C_\alpha} ds\,\partial\psi_\Omega/\partial n=0, \tag{8}$$

where the contour integral is taken in the positive sense and $\partial/\partial n$ represents the normal derivative in the

[4] A. L. Fetter, Phys. Rev. **138**, A429 (1965).

[5] See, for example, P. J. Bendt, Phys. Rev. **127**, 1441 (1962); H. A. Snyder, Phys. Fluids **6**, 755 (1963).

[6] P. J. Bendt and T. A. Oliphant, Phys. Rev. Letters **6**, 213 (1961); M. P. Kemoklidze and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **46**, 1677 (1964) [English transl.: Soviet Phys.—JETP **19**, 1134 (1964)].

[7] This assumption was introduced by H. E. Hall, Advan. Phys. **9**, 89 (1960), in a calculation of the equilibrium distribution of vortices in a cylinder. See also I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), Sec. 16.

[8] C. C. Lin, *On the Motion of Vortices in Two Dimensions* (University of Toronto Press, Toronto, Canada, 1943); an abbreviated but more accessible account is C. C. Lin, Proc. Natl. Acad. Sci. U. S. **27**, 570 (1941); **27**, 575 (1941).

[9] See, for example, L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 72.

outward direction (Fig. 1). The second term ψ_0 in Eq. (5) arises from the presence of circulation $\{\Gamma_\alpha\}$ about the inner boundaries $\{C_\alpha\}$; ψ_0 is defined by the equations

$$\nabla^2\psi_0(\mathbf{r})=0; \tag{9}$$

$$\psi_0(\mathbf{r})=\psi_{0\alpha}(=\text{const}),\ \text{for } \mathbf{r} \text{ on } C_\alpha; \tag{10}$$

$$\oint_{C_\alpha} ds\partial\psi_0/\partial n=\Gamma_\alpha; \tag{11}$$

$$\psi_0(\mathbf{r})=0,\ \text{for } \mathbf{r} \text{ on } C_0. \tag{12}$$

Equations (6) and (9) together ensure that the motion is irrotational while Eqs. (8) and (11) fix the correct circulation about each inner boundary. Since the fluid velocity is defined only through derivatives of the stream function, the physical motion [Eq. (1)] is unaffected by the addition of a constant [Eq. (10)] to the boundary condition (7). In the special case that the set $\{\Gamma_\alpha\}$ vanishes, $\psi_0(\mathbf{r})$ is identically equal to zero. Furthermore, $\psi_\Omega(\mathbf{r})$ vanishes for a stationary system ($\Omega=0$). It can be shown that ψ_Ω and ψ_0 are uniquely defined by the above set of equations.[8,10]

Our discussion must be modified when vortices are present in the fluid. The additional contribution to the stream function is expressible in terms of a unique Green's function $G(\mathbf{r},\mathbf{r}_0)$ that satisfies the following conditions[8]:

$$G(\mathbf{r},\mathbf{r}_0)=G(\mathbf{r}_0,\mathbf{r}); \tag{13}$$

$$\nabla^2G(\mathbf{r},\mathbf{r}_0)=\delta(\mathbf{r}-\mathbf{r}_0); \tag{14}$$

$$G(\mathbf{r},\mathbf{r}_0)=G_\alpha(\mathbf{r}_0)\ \text{for } \mathbf{r} \text{ on } C_\alpha; \tag{15}$$

$$\oint_{C_\alpha} ds\partial G/\partial n=0; \tag{16}$$

$$G(\mathbf{r},\mathbf{r}_0)=0\ \text{for } \mathbf{r} \text{ on } C_0. \tag{17}$$

As $\mathbf{r}\rightarrow\mathbf{r}_0$, the Green's function becomes singular like $(2\pi)^{-1}\ln|\mathbf{r}-\mathbf{r}_0|$, and the auxiliary function

$$g(\mathbf{r},\mathbf{r}_0)=G(\mathbf{r},\mathbf{r}_0)-(2\pi)^{-1}\ln|\mathbf{r}-\mathbf{r}_0| \tag{18}$$

satisfies Laplace's equation throughout the whole fluid

$$\nabla^2g(\mathbf{r},\mathbf{r}_0)=0. \tag{19}$$

If a single vortex with circulation κ is situated at $\mathbf{r}_0$, the additional stream function is given by

$$\kappa G(\mathbf{r},\mathbf{r}_0), \tag{20}$$

which must be added to Eq. (5). More generally, for a system of identical vortices with circulation κ at the points $\{\mathbf{r}_k\}$ combined with circulation $\{\Gamma_\alpha\}$ about $\{C_\alpha\}$, the total stream function is given by

$$\psi(\mathbf{r})=\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})+\kappa\sum_k G(\mathbf{r},\mathbf{r}_k). \tag{21}$$

[10] See, for example, S. G. Mikhlin, *Integral Equations* (The Macmillan Company, New York, 1964), 2nd ed., pp. 157–159.

Equation (21) describes irrotational flow except at the vortex cores:

$$|\text{curl}\mathbf{v}|=\nabla^2\psi=\kappa\sum_k\delta(\mathbf{r}-\mathbf{r}_k). \tag{22}$$

It is straightforward to verify that Eq. (21) actually satisfies the boundary conditions and therefore represents the correct solution to the boundary-value problem. In principle, the explicit form of G, ψ_Ω, and ψ_0 may be found for any particular geometry; in practice, the actual calculation may be difficult. Such detailed knowledge is unnecessary for the present purpose, however, and it is sufficient to work directly with Eq. (21).

In the rotating system, the free energy F is given by

$$F=E-\Omega L, \tag{23}$$

and we must calculate the angular momentum L and energy E of the fluid. These are easily found by integrating over the allowed area

$$L=\rho\int d^2r(xv_y-yv_x), \tag{24}$$

$$E=\tfrac{1}{2}\rho\int d^2rv^2=\tfrac{1}{2}\rho\int d^2r(v_x^2+v_y^2), \tag{25}$$

where ρ is the fluid density. Equation (24) may be rewritten in terms of the stream function ψ,

$$\begin{aligned}L&=\rho\int d^2r[x(\partial\psi/\partial x)+y(\partial\psi/\partial y)]\\&=\rho\int d^2r[\partial(x\psi)/\partial x+\partial(y\psi)/\partial y]-2\rho\int d^2r\psi\\&=\rho\oint_{C_0} ds(\mathbf{r}\cdot\hat{n})\psi-\rho\sum_\alpha\oint_{C_\alpha} ds(\mathbf{r}\cdot\hat{n})\psi-2\rho\int d^2r\psi.\end{aligned} \tag{26}$$

Green's theorem[11] has been used to obtain the last line of Eq. (26), where the contours are all taken in the positive sense and $\hat{n}$ is a unit vector in the normal (outward) direction (Fig. 1).

The line integrals in Eq. (26) may be simplified since ψ assumes specified values on the boundaries

$$\begin{aligned}&\rho\oint_{C_0} ds(\mathbf{r}\cdot\hat{n})\psi-\rho\sum_\alpha\oint_{C_\alpha} ds(\mathbf{r}\cdot\hat{n})\psi\\&=\tfrac{1}{2}\rho\Omega\left[\oint_{C_0} ds(\mathbf{r}\cdot\hat{n})r^2-\sum_\alpha\oint_{C_\alpha} ds(\mathbf{r}\cdot\hat{n})r^2\right]\\&\qquad-\rho\sum_\alpha[\psi_{0\alpha}+\kappa\sum_k G_\alpha(\mathbf{r}_k)]\oint_C ds(\mathbf{r}\cdot\hat{n}),\end{aligned} \tag{27}$$

[11] See, for example, T. M. Apostol, *Mathematical Analysis* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1957), pp. 283–292.

where Eqs. (7), (10), (12), (15), and (17) have been used. An elementary calculation with Green's theorem shows that

$$\rho\left[\oint_{C_0} ds(\mathbf{r}\cdot\hat{n})r^2-\sum_\alpha \oint_{C_\alpha} ds(\mathbf{r}\cdot\hat{n})r^2\right] =4\rho\int d^2r r^2=4I\,, \quad (28)$$

which defines the moment of inertia I, and that

$$\oint_{C_\alpha} ds(\mathbf{r}\cdot\hat{n})=2A_\alpha\,, \quad (29)$$

where A_α is the area enclosed by C_α. Hence the angular momentum reduces to

$$L=2I\Omega-2\rho\sum_\alpha \psi_{0\alpha}A_\alpha-2\rho\kappa\sum_\alpha\sum_k G_\alpha(\mathbf{r}_k)A_\alpha -2\rho\int d^2r[\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})]-2\rho\kappa\sum_k\int d^2rG(\mathbf{r},\mathbf{r}_k)\,. \quad (30)$$

Here and subsequently, sums over Greek subscripts refer to the contours while sums over Latin subscripts refer to the vortices.

The energy may be calculated in a similar manner:

$$\begin{aligned}E&=\tfrac{1}{2}\rho\int d^2r[v_y(\partial\psi/\partial x)-v_x(\partial\psi/\partial y)]\\&=\tfrac{1}{2}\rho\int d^2r[\partial(\psi v_y)/\partial x-\partial(\psi v_x)/\partial y]-\tfrac{1}{2}\rho\int d^2r\psi|\mathrm{curl}\mathbf{v}|\\&=\tfrac{1}{2}\rho\oint_{C_0} d\mathbf{s}\cdot\mathbf{v}\psi-\tfrac{1}{2}\rho\sum_\alpha\oint_{C_\alpha} d\mathbf{s}\cdot\mathbf{v}\psi-\tfrac{1}{2}\rho\int d^2r\psi|\mathrm{curl}\mathbf{v}|\,, \end{aligned}\quad (31)$$

where Green's theorem has again been used to obtain the last line. The contour integrals may be rewritten as in Eq. (27):

$$\begin{aligned}\tfrac{1}{2}\rho\oint_{C_0} d\mathbf{s}\cdot\mathbf{v}\psi-\tfrac{1}{2}\rho\sum_\alpha\oint_{C_\alpha} d\mathbf{s}\cdot\mathbf{v}\psi&=\tfrac{1}{4}\rho\Omega\left[\oint_{C_0} d\mathbf{s}\cdot\mathbf{v}r^2-\sum_\alpha\oint_{C_\alpha} d\mathbf{s}\cdot\mathbf{v}r^2\right]-\tfrac{1}{2}\rho\sum_\alpha[\psi_{0\alpha}+\kappa\sum_k G_\alpha(\mathbf{r}_k)]\oint_{C_\alpha} d\mathbf{s}\cdot\mathbf{v}\\&=\tfrac{1}{2}\rho\Omega\int d^2r[(xv_y-yv_x)+\tfrac{1}{2}r^2|\mathrm{curl}\mathbf{v}|]-\tfrac{1}{2}\rho\sum_\alpha[\psi_{0\alpha}+\kappa\sum_k G_\alpha(\mathbf{r}_k)]\Gamma_\alpha\,.\end{aligned}\quad (32)$$

Since the motion is irrotational except at the vortex cores, $|\mathrm{curl}\mathbf{v}|$ is nonzero only at a finite number of points, where it becomes singular. Substitution of Eqs. (22) and (24) into Eq. (32) yields

$$\tfrac{1}{2}\Omega L+\tfrac{1}{4}\rho\Omega\sum_k r_k^2-\tfrac{1}{2}\rho\sum_\alpha\psi_{0\alpha}\Gamma_\alpha -\tfrac{1}{2}\rho\kappa\sum_\alpha\sum_k G_\alpha(\mathbf{r}_k)\Gamma_\alpha\,. \quad (33)$$

The last term of Eq. (31) may be treated in a similar manner

$$\int d^2r\psi|\mathrm{curl}\mathbf{v}|\approx\kappa\sum_j\psi_{(j)}\,, \quad (34)$$

where $\psi_{(j)}$ is the nonsingular part of the stream function evaluated at $\mathbf{r}_j$,

$$\psi_{(j)}=\lim_{\mathbf{r}\to\mathbf{r}_j}[\psi(\mathbf{r})-(2\pi)^{-1}\ln|\mathbf{r}-\mathbf{r}_j|]\,. \quad (35)$$

In principle, Eq. (34) should also contain the small model-dependent contributions from the vortex cores. These self-energy terms may be omitted here since they are negligible in the limit of many vortices and do not affect the conclusions of this paper. Equations (18) and (21) show that

$$\psi_{(j)}=\psi_\Omega(\mathbf{r}_j)+\psi_0(\mathbf{r}_j)+\kappa\sum_{k(\neq j)}G(\mathbf{r}_k,\mathbf{r}_j)+\kappa g(\mathbf{r}_j,\mathbf{r}_j)\,, \quad (36)$$

and Eq. (31) may be rewritten as

$$\begin{aligned}E=\tfrac{1}{2}L\Omega+\tfrac{1}{4}\rho\kappa\Omega\sum_k r_k^2-\tfrac{1}{2}\rho\sum_\alpha\psi_{0\alpha}\Gamma_\alpha&\\-\tfrac{1}{2}\rho\kappa\sum_\alpha\Gamma_\alpha\sum_k G_\alpha(\mathbf{r}_k)&\\-\tfrac{1}{2}\rho\kappa\sum_k[\psi_\Omega(\mathbf{r}_k)+\psi_0(\mathbf{r}_k)]&\\-\tfrac{1}{2}\rho\kappa^2{\sum_{jk}}' G(\mathbf{r}_j,\mathbf{r}_k)-\tfrac{1}{2}\rho\kappa^2\sum_k g(\mathbf{r}_k,\mathbf{r}_k)\,,&\end{aligned}\quad (37)$$

where the primed sum is over j and k separately, omitting the terms $j=k$.

Equations (30) and (37) are exact expressions for the angular momentum and energy of a rotating fluid moving with prescribed circulation about inner boundaries and containing a system of rectilinear vortices. Although the results were obtained by integrating over the volume of the fluid, the final formulas depend explicitly on the coordinates of the vortices and may be interpreted as the angular momentum and energy of the vortices themselves. In particular, the quantity $-\rho\kappa^2 G(\mathbf{r}_j,\mathbf{r}_k)$ represents the interaction energy of a pair of vortices situated at the points $\mathbf{r}_j$ and $\mathbf{r}_k$. The Green's function G serves both as the stream function at $\mathbf{r}$ due to a vortex at $\mathbf{r}'$ and as the interaction energy between two vortices at $\mathbf{r}$ and $\mathbf{r}'$. This dual role is familiar in electrostatics, where r^{-1} is simultaneously the solution of Laplace's equation for a point charge and the interaction energy between two charges. In a similar manner, the total energy of a system of charged particles can be expressed either as an integral of the electrostatic field energy over all space [as in Eq. (25)] or as the sum of the interaction energy between all pairs of particles [as in Eq. (37)].

III. SIMPLY CONNECTED DOMAIN

In order to simplify the calculation of the equilibrium fluid configuration, we shall first consider the special case of a simply connected domain, where $\psi_0(\mathbf{r})$ vanishes identically. The energy and angular momentum assume an especially simple form

$$E=\tfrac{1}{2}L\Omega+\tfrac{1}{4}\rho\kappa\Omega\sum_k r_k^2-\tfrac{1}{2}\rho\kappa\sum_k \psi_\Omega(\mathbf{r}_k)-\tfrac{1}{2}\rho\kappa^2\sum_{jk}{}' G(\mathbf{r}_j,\mathbf{r}_k)-\tfrac{1}{2}\rho\kappa^2\sum_k g(\mathbf{r}_k,\mathbf{r}_k)\,, \quad (38)$$

$$L=2I\Omega-2\rho\int d^2r\psi_\Omega(\mathbf{r})-2\rho\kappa\sum_k\int d^2rG(\mathbf{r},\mathbf{r}_k)\,. \quad (39)$$

Suppose that the total number of vortices N becomes large, so that the sums over separate vortices may be approximated by integrations over a smoothed vortex density $n(\mathbf{r})$. The last term in Eq. (28) is negligible in the limit $N\to\infty$, and the free energy reduces to

$$F=\tfrac{1}{4}\rho\kappa\Omega\int d^2rn(\mathbf{r})r^2-\tfrac{1}{2}\rho\kappa\int d^2r\psi_\Omega(\mathbf{r})n(\mathbf{r})-\tfrac{1}{2}\rho\kappa^2\int\int d^2rd^2r'G(\mathbf{r},\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')-I\Omega^2+\rho\Omega\int d^2r\psi_\Omega(\mathbf{r})+\rho\kappa\Omega\int\int d^2rd^2r'G(\mathbf{r},\mathbf{r}')n(\mathbf{r}')\,. \quad (40)$$

The equilibrium configuration is obtained by minimizing the free energy with respect to the vortex density, which leads to the simple condition

$$\delta F/\delta n(\mathbf{r})=\tfrac{1}{4}\rho\kappa\Omega r^2-\tfrac{1}{2}\rho\kappa\psi_\Omega(\mathbf{r})-\rho\kappa^2\int d^2r'G(\mathbf{r},\mathbf{r}')n(\mathbf{r}')+\rho\kappa\Omega\int d^2r'G(\mathbf{r},\mathbf{r}')=0\,. \quad (41)$$

Equation (41) is an integral equation for $n(\mathbf{r})$,[12] and the solution may be obtained by applying the Laplacian operator

$$\tfrac{1}{4}\Omega\nabla^2r^2-\tfrac{1}{2}\nabla^2\psi_\Omega(\mathbf{r})-\kappa\nabla^2\int d^2r'G(\mathbf{r},\mathbf{r}')n(\mathbf{r}')+\Omega\nabla^2\int d^2r'G(\mathbf{r},\mathbf{r}')=0\,. \quad (42)$$

Since $\psi_\Omega(\mathbf{r})$ is harmonic and $\nabla^2G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$, the unique solution to Eq. (41) is

$$n(\mathbf{r})=2\Omega/\kappa(=\text{const})\,, \quad (43)$$

throughout the whole simply connected domain. The vortices are uniformly distributed with density $2\Omega/\kappa$, and the equilibrium condition (41) then reduces to

$$[\psi_\Omega(\mathbf{r})]_{eq}=\tfrac{1}{2}\Omega r^2-2\Omega\int d^2r'G(\mathbf{r},\mathbf{r}')\,. \quad (44)$$

The total stream function (21) may be greatly simplified, and we find

$$[\psi(\mathbf{r})]_{eq}=\tfrac{1}{2}\Omega r^2\,. \quad (45)$$

The mean vorticity $\nabla^2\psi=2\Omega$ is just the value associated with solid-body rotation.

IV. MULTIPLY CONNECTED DOMAIN

The treatment of Sec. III must now be generalized in order to deal with a multiply connected domain since the circulation Γ_α must be considered as a variational parameter. Equation (21) shows that the total stream function is composed of distinct contributions from the vortices, from the rotating walls, and from the prescribed circulation. Changes in Γ_α affect only the irrotational stream function ψ_0, which we shall examine in more detail. As mentioned in Sec. II, ψ_0 is uniquely defined by specifying the value of the circulation Γ_α about each contour C_α, along with the condition that ψ_0 vanish on C_0. The constant boundary value $\psi_{0\beta}$ of $\psi_0(\mathbf{r})$ on C_β thus depends on the complete set of constants $\{\Gamma_\alpha\}$. It is not difficult to show that $\psi_{0\beta}$ is in fact a linear homogeneous function of the circulation Γ_α,[10] so that

$$\psi_{0\beta}=\sum_\alpha\Gamma_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta)\,. \quad (46)$$

[12] The simple solution of this integral equation is possible only for a Newtonian interaction ($E\propto r_{12}^{-1}$ in three dimensions, $E\propto\ln r_{12}$ in two dimensions). A similar variation of the free energy has been used by K. Maki, Ann. Phys. (N. Y.) **34**, 363 (1965), in a study of the distribution of quantized flux lines in a thin film of type-I superconductor in a perpendicular magnetic field. In this latter two-dimensional situation, however, the energy varies as r_{12}^{-1}, so that the kernel of the corresponding integral equation is much more complicated. Another feature of the Newtonian interaction is that the fluid velocity vanishes in the interior of a ring of continuously distributed vorticity and varies as r^{-1} outside of the ring. This velocity pattern is the two-dimensional analog of the gravitational force field of a thin spherical shell.

Equation (46) implies that $\psi_{0\beta}$ may be written explicitly as

$$\psi_{0\beta}=\sum_\alpha L_{\beta\alpha}\Gamma_\alpha\,, \tag{47}$$

where $L_{\beta\alpha}$ is a constant depending only on the geometric configuration of the contours C_α and C_β. It is intuitively obvious that the constants $L_{\alpha\beta}$ are symmetric under the interchange of α and β; this result can also be shown directly by examining the total energy of the system.

It is worth noting the existence of an exact mathematical equivalence between two-dimensional irrotation flow and two-dimensional magnetostatics in the presence of current-carrying wires.[13] The velocity field $\mathbf{v}$ corresponds to the magnetic field $\mathbf{H}$, while the circulation $\Gamma_\alpha[=\oint_{C_\alpha}d\mathbf{s}\cdot\mathbf{v}]$ about C_α is equivalent to the current $i_\alpha\,[=(c/4\pi)\oint_{C_\alpha}d\mathbf{s}\cdot\mathbf{H}]$ in the cylindrical wire bounded by C_α. Furthermore, the constant $\psi_{0\alpha}$ is analogous to the total magnetic flux Φ_α linking the circuit made up of the αth wire and its return path on the outer boundary. This identification follows immediately from the usual definition of the stream function $\psi_0(\mathbf{r})$ as a measure of the total fluid flux crossing a line drawn from C_0 to $\mathbf{r}$.[14] The hydrodynamic problem of irrotational (vortex-free) flow in an arbitrary multiply connected container is therefore identical with the determination of the magnetic field in an arbitary polycore cable. Equations that are valid in one system provide a correct description of the other. In particular, the total magnetic flux Φ_α linking the contours C_α and C_0 is known to be a linear homogeneous function of the currents i_β, and the coefficients of mutual inductance between C_α and C_β are symmetric functions of α and β.[15] Separate proof of the corresponding hydrodynamic results [Eqs. (46) and (47)] is unnecessary in view of the precise equivalence between the two systems.

We shall now return to the original problem of the two-dimensional motion of a system of vortices in a multiply connected region. The expressions for the total angular momentum (30) and the total energy (37) may be substituted into Eq. (23) to obtain the exact free energy, which may be used to study the behavior of an arbitrary number of vortices in a multiply connected container. Such a general problem presents computational difficulties, however, and the present work will be restricted to the limiting case of a great many vortices. The sums over separate vortices may then be approximated by integrations over a smoothed vortex density $n(\mathbf{r})$, exactly as in Sec. III. In this limit, the self-energy is again negligible, so that the free energy becomes

$$\begin{aligned}F=&\tfrac{1}{4}\rho\kappa\Omega\int d^2r\,n(\mathbf{r})r^2-\tfrac{1}{2}\rho\sum_\alpha\psi_{0\alpha}\Gamma_\alpha-\tfrac{1}{2}\rho\kappa\sum_\alpha\Gamma_\alpha\int d^2r\,G_\alpha(\mathbf{r})n(\mathbf{r})-\tfrac{1}{2}\rho\kappa\int d^2r\,n(\mathbf{r})[\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})]\\&-\tfrac{1}{2}\rho\kappa^2\int\int d^2r\,d^2r'\,G(\mathbf{r},\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')-I\Omega^2+\rho\Omega\sum_\alpha\psi_{0\alpha}A_\alpha+\rho\kappa\Omega\sum_\alpha A_\alpha\int d^2r\,G_\alpha(\mathbf{r})n(\mathbf{r})\\&+\rho\Omega\int d^2r[\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})]+\rho\kappa\Omega\int\int d^2r\,d^2r'\,G(\mathbf{r},\mathbf{r}')n(\mathbf{r}')\,.\end{aligned}\tag{48}$$

A rotating system is in equilibrium if the free energy is a minimum; this condition requires that

$$\delta F/\delta n(\mathbf{r})=0\,, \tag{49a}$$

$$\partial F/\partial\Gamma_\beta=0\,, \tag{49b}$$

for all β. The explicit expressions corresponding to Eq. (49a) and (49b) are easily found to be

$$\tfrac{1}{4}\rho\kappa\Omega r^2-\tfrac{1}{2}\rho\kappa\sum_\alpha\Gamma_\alpha G_\alpha(\mathbf{r})-\tfrac{1}{2}\rho\kappa[\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})]-\rho\kappa^2\int d^2r'\,G(\mathbf{r},\mathbf{r}')n(\mathbf{r}')+\rho\kappa\Omega\sum_\alpha A_\alpha G_\alpha(\mathbf{r})+\rho\kappa\Omega\int d^2r'\,G(\mathbf{r},\mathbf{r}')=0\,, \tag{50a}$$

$$\begin{aligned}&-\tfrac{1}{2}\rho\psi_{0\beta}-\tfrac{1}{2}\rho\sum_\alpha\Gamma_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta)-\tfrac{1}{2}\rho\kappa\int d^2r\,G_\beta(\mathbf{r})n(\mathbf{r})-\tfrac{1}{2}\rho\kappa\int d^2r\,n(\mathbf{r})[\partial\psi_0(\mathbf{r})/\partial\Gamma_\beta]\\&\qquad+\rho\Omega\sum_\alpha A_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta)+\rho\Omega\int d^2r[\partial\psi_0(\mathbf{r})/\partial\Gamma_\beta]=0\,.\end{aligned}\tag{50b}$$

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[13] See, for example, H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., pp. 210, 217; A. L. Fetter and R. J. Donnelly, Phys. Fluids **9**, 619 (1966).

[14] H. Lamb, Ref. 13 pp. 62–63.

[15] See, for example, M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie and Son Ltd., London, 1954), 2nd ed., pp. 165–172; or L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960), pp. 131–136.

In computing Eq. (50b), it is important to remember that the variations in the vortex density and in the circulation are independent, so that $\partial G/\partial\Gamma_\alpha = \partial\psi_\Omega/\partial\Gamma_\alpha = 0$. Equation (50a) may be solved by applying the Laplacian operator; since ψ_Ω and ψ_0 represent irrotational flow and $\nabla^2 G_\alpha(\mathbf{r})$ vanishes in the interior of the domain, Eq. (50a) reduces to Eq. (42). It follows that the equilibrium vortex density $n = 2\Omega/\kappa$ is uniform and independent of the circulation about any of the inner boundaries. Equation (50a) may then be simplified to yield a relation between the equilibrium value of the irrotational stream function and the Green's function describing the uniformly distributed vortices,

$$[\psi_\Omega(\mathbf{r})+\psi_0(\mathbf{r})]_{eq} = \tfrac{1}{2}\Omega r^2 - 2\Omega \int d^2r' G(\mathbf{r},\mathbf{r}') + \sum_\alpha [2\Omega A_\alpha - \Gamma_\alpha] G_\alpha(\mathbf{r}). \quad (51)$$

The second equilibrium condition (50b) may be rewritten using the uniform density $n = 2\Omega/\kappa$:

$$-\tfrac{1}{2}\rho\psi_{0\beta} - \tfrac{1}{2}\rho \sum_\alpha \Gamma_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta) - \rho\Omega \int d^2r G_\beta(\mathbf{r}) + \rho\Omega \sum_\alpha A_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta) = 0. \quad (52)$$

The linear relation between $\psi_{0\alpha}$ and Γ_β [Eqs. (46) and (47)] remains valid in the presence of vortices, and Eq. (52) may be simplified to

$$2\psi_{0\beta} - \sum_\alpha 2\Omega A_\alpha(\partial\psi_{0\alpha}/\partial\Gamma_\beta) = -2\Omega \int d^2r G_\beta(\mathbf{r}). \quad (53)$$

If Eq. (51) is evaluated for $\mathbf{r}$ on C_β, we obtain a second expression for $\psi_{0\beta}$:

$$\psi_{0\beta} = -2\Omega \int d^2r' G_\beta(\mathbf{r}') + \sum_\alpha [2\Omega A_\alpha - \Gamma_\alpha] G_{\alpha\beta}, \quad (54)$$

where the constants $G_{\alpha\beta}$ are given by

$$G_{\alpha\beta} = \lim_{\mathbf{r}\to C_\beta} G_\alpha(\mathbf{r}) = \lim_{\mathbf{r}\to C_\alpha} G_\beta(\mathbf{r}) = G_{\beta\alpha}. \quad (55)$$

Equations (53) and (54) are consistent if

$$[\Gamma_\alpha]_{eq} = 2\Omega A_\alpha, \quad (56)$$

$$L_{\alpha\beta} = G_{\alpha\beta}, \quad (57)$$

which fixes the equilibrium circulation and identifies the mutual-coupling coefficients.

It is now possible to find the total stream function $[\psi(\mathbf{r})]_{eq}$ describing the equilibrium state of a large number of vortices. A combination of Eqs. (21), (51), and (56) leads to

$$[\psi(\mathbf{r})]_{eq} = \tfrac{1}{2}\Omega r^2, \quad (58)$$

which is precisely the stream function for fluid rotating as a solid body with

$$\mathbf{v}(\mathbf{r}) = \mathbf{\Omega} \times \mathbf{r}. \quad (59)$$

Detailed evaluation of Eqs. (30), (37), and (48) yields the equilibrium values

$$L_{eq} = I\Omega, \quad (60)$$

$$E_{eq} = \tfrac{1}{2}I\Omega^2, \quad (61)$$

$$F_{eq} = -\tfrac{1}{2}I\Omega^2, \quad (62)$$

which also follow immediately from Eq. (59).

This paper has shown that the equilibrium configuration of an assembly of identical rectilinear vortices in an arbitrary multiply connected region is a uniform density $n = 2\Omega/\kappa$. Hence the mean vorticity in the fluid is 2Ω, which is identical with the value for solid-body rotation. The equilibrium circulation about each of the inner boundaries C_α is $2\Omega A_\alpha$, where A_α is the area enclosed by C_α. This circulation is precisely the value that would occur if vortices filled the interior of C_α uniformly with density $2\Omega/\kappa$. It follows that the circulation about *any* contour lying wholly in the fluid is given by 2Ω times the area enclosed by the contour. Although the fluid remains irrotational at every point except at the vortex cores, the flow pattern is indistinguishable on a macroscopic scale from a uniform rotation, in which $|\text{curl}\,\mathbf{v}| = 2\Omega$. It must be emphasized that these conclusions are valid only in the limit of many vortices, when the discrete structure may be approximated by a continuous distribution. The very interesting experimental question of the critical angular velocity for the appearance of vortices in a given container requires an explicit calculation of the Green's function for that geometry. The particular case of an annular region has been studied in detail and will be presented in a separate paper.

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