N/D Effective-Range Theory with $SU(3)$-Symmetric Short-Range Forces*

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(Received 6 July 1966)

A simple N/D representation is written down in which nearby singularities (long-range contributions) are calculated explicitly from single-particle-exchange graphs in the conventional way. The remaining singularities (short-range forces) are not ignored, but are approximated in terms of two "effective-range" parameters. These are taken to be $SU(3)$-symmetric, an assumption which is closely related to the idea that internal symmetries should manifest themselves at high energies and momentum transfers. For instance, we can relate the paramaters in the $I=1, J=1$ $\pi\pi$-$K\bar{K}$ problem to the corresponding ones in the $I=0, J=1$ $K\bar{K}$ problem. Thus if we know the mass and width of the ρ meson, say from experiment, we can calculate the corresponding quantities for the ϕ meson. A simplified version of this calculation gives values in fairly good agreement with experiment.

I. INTRODUCTION

MOST of the difficulties in interpreting $SU(3)$ symmetry[1,2] in strong interactions arise from the fact that it is badly broken. In fact, many of its successes have involved certain very specific additional assumptions. The prime example is octet dominance, which was used in deriving the Gell-Mann–Okubo mass formula.[1,3]

It has been proposed that unitary symmetry may, however, manifest itself at high energies and momentum transfers,[1] i.e., in interactions involving small distances. Unfortunately, it has not proven possible to check on this directly. On the other hand, interactions at small distances probably have a very important effect in giving rise to bound states and resonances. Of course, long-range forces are also important. Indeed, it has been argued by Dashen and Frautschi[4] that $SU(3)$-breaking may be due primarily to just such forces. They argued further that this is particularly likely to be true if we are dealing with bound states rather than elementary particles, and used this in studying the perturbations on $SU(3)$-symmetric bootstraps.

In most bootstrap calculations, only long-range forces are considered,[5] since these are the only ones which can be handled directly with present-day dispersion techniques. Thus the left-hand cut is assumed to arise from single-particle exchange graphs, while two-body unitarity is imposed on the right-hand cut, usually through the N/D method. This procedure generally leads to resonance widths which are much broader than the observed values.[5] However, distant singularities can always be represented by a few constants through an effective-range approximation. This is most easily seen in terms of the analogy between the singularities in the complex plane and charge distributions in electrostatics.[6] As long as we are interested in a small region, the effect of any set of singularities (or charges), if sufficiently far away, can be replaced by a pole (or point charge) at infinity. If we want to increase our accuracy, we can add multipoles or replace our singularities by a small number of simpler singularities (such as poles) at an appropriately chosen finite distance. This is because far-away singularities can be expected to produce only smooth variations, which can always be reproduced quite accurately through a small number of parameters.

Although the inclusion of distant singularities through an effective-range approximation should increase the reliability of our N/D equations, it generally leads to the introduction of as many extra parameters as we are trying to explain in the first place. It is here that an assumption that distant singularities obey a symmetry is particularly helpful, since such a symmetry can be used to reduce considerably the number of independent parameters.[7] In fact, as we shall see, it should enable us to calculate all the masses and coupling constants of the members of an $SU(3)$ multiplet in terms of only two or three constants. The symmetry breaking is achieved, not by first assuming exact symmetry and then adding a perturbation, but simply by not assuming the symmetry in the first place as far as long-range effects (nearby singularities) are concerned.

In Sec. II we set up an N/D representation which incorporates the features discussed above. Two-body unitarity is exactly satisfied, while the nearby left-hand cut can be calculated from one-particle exchange graphs.

* This work was supported in part by the National Science Foundation.

[1] M. Gell-Mann, CTSL-20, 1961 (unpublished); reprinted in M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964), p. 11.

[2] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961); reprinted in Gell-Mann and Ne'eman, Ref. 1, p. 58.

[3] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962); reprinted in Gell-Mann and Ne'eman, Ref. 1, p. 66.

[4] R. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1192 (1964); **137**, B1318 (1965); 1331 (1965).

[5] The earliest example is due to F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); **7**, 268 (1961). For a recent example see, for instance, J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965), to which the reader is also referred for references to other bootstrap calculations.

[6] See, for instance, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), Chap. 1.

[7] Actually, all we have to assume is that the *effect* of distant singularities in the region of interest is *as if* they were $SU(3)$ symmetric. This is a very weak assumption, as can be seen in our electrostatic analogy. Here symmetry breaking manifests itself at worst as a dipole term, which is always small if the singularities are sufficiently far away.

These usually lead to divergences, so a cutoff is introduced. Such a cutoff, of course, is a way of representing at least part of the distant left-hand cut, and so will be treated as one of our effective-range parameters. A second parameter, which represents inelastic effects as well as the remaining left-hand cut, is introduced by adding a linear term to the D function.

In Sec. III, the above representation is applied to the $I=1$, $J=1$ $\pi\pi$-$K\bar{K}$ problem. By retaining a certain amount of residual $SU(3)$ symmetry, it is possible to reduce this problem to the $\pi\pi$ problem with inelasticity. A rough approximation to ρ exchange is taken for the nearby left-hand cut, and our two effective-range parameters are adjusted so as to give the correct mass and width of the ρ resonance in the direct channel. They are then related to the corresponding parameters in the $I=0$, $J=1$ $K\bar{K}$ problem, making possible a calculation of the mass and width of the ϕ meson, which occurs as a resonance in this state.

II. N/D EFFECTIVE-RANGE APPROXIMATION

Suppose, for the sake of being definite, we consider the P-wave scattering of the PS (pseudoscalar) mesons π, K, and η in the octet state. This breaks up into the two-channel $I=1$ $\pi\pi$-$K\bar{K}$ and $I=\frac{1}{2}$ πK-ηK problems, and the one-channel $I=0$ $K\bar{K}$ problem. The invariant amplitude for the process $j\to i$ has the form

$$A_{ij}=s^{1/2}(S_{ij}-\delta_{ij})/(4iq_i^{3/2}q_j^{3/2}), \tag{1}$$

where s equals w^2, w is the total c.m. energy, S is the S matrix, which is unitary, q_i is the c.m. three-momentum for the ith channel. Thus the matrix A made up of these elements satisfies the unitarity relation

$$\operatorname{Im}A^{-1}(s)=-\rho(s), \tag{2}$$

with

$$\rho_{ij}(s)=(2q_i^3/s^{1/2})\theta(q_i^2)\delta_{ij}. \tag{3}$$

Equation (2) fails at high energies, where other channels become important.

If we knew the discontinuity $\operatorname{Im}A(s)$ across the left-hand cut we could find the amplitude by the N/D method,[8] which gives

$$A(s)=N(s)D^{-1}(s), \tag{4}$$

with

$$N(s)=\int_{-\infty}^{s_L} ds'\frac{\operatorname{Im}A(s')D(s')}{s'-s}, \tag{5}$$

and

$$D(s)=1+\frac{s-s_0}{\pi}\int_{s_t}^{\infty} ds'\frac{\operatorname{Im}A^{-1}(s')N(s')}{(s'-s_0)(s'-s)}, \tag{6}$$

where s_0 is some subtraction point, and s_L and s_t mark the start of the left- and right-hand cuts, respectively. The expressions (5) and (6) give the correct $\operatorname{Im}A(s)$ on the left and $\operatorname{Im}A^{-1}(s)$ on the right. The latter can be obtained from Eq. (2) if we assume two-body unitarity.

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

Normally, the left-hand cut is approximated by the contribution of one-particle exchange graphs. Such graphs should give a reasonable approximation to the nearby part of the cut but often give rise to a divergence unless some kind of cutoff is imposed. This would happen if we exchanged vector mesons, for instance. We therefore put in a cutoff at $s=s_c$, so that Eq. (5) becomes

$$N(s)=\int_{s_c}^{s_L} ds'\frac{\operatorname{Im}A(s')D(s')}{s'-s}. \tag{7}$$

In fact, if we treat s_c as an adjustable parameter, it can be used to represent at least part of the left cut. This is because, by the arguments of Sec. I, we should always be capable of approximating such singularities by some adjustable simpler singularity, in this case, by whatever one gets from one-particle exchange. The actual strength of this simpler singularity can be adjusted by varying s_c.

The normal procedure for representing any additional singularities would be to add one or more poles to Eq. (7). This would introduce at least two extra parameters, which is more than we can handle. But actually, even if we use Eq. (7) we can still get an exact expression for the amplitude if we modify Eq. (6) to read

$$D(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty} ds'\frac{\rho(s')N(s')}{(s'-s_0)(s'-s)} +\frac{s-s_0}{\pi}\left\{\int_{s_I}^{\infty} ds'\frac{[\operatorname{Im}A^{-1}(s')+\rho(s')]N(s')}{(s'-s_0)(s'-s)} +\int_{-\infty}^{s_c} ds'\frac{\operatorname{Im}A^{-1}(s')N(s')}{(s'-s_0)(s'-s)}\right\}. \tag{8}$$

We have used Eq. (2) for $s<s_I$, where $s=s_I$ is the point at which other channels begin to become important. Of course, we have no way of calculating the last two integrals in Eq. (8). Since, however, we are only interested in low energies and since these integrals represent high-energy effects, we shall approximate them by a constant matrix C. This corresponds to approximating the integrals by a (Castillejo-Dalitz-Dyson) pole at infinity. Equation (8) then becomes

$$D(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty} ds'\frac{\rho(s')N(s')}{(s'-s_0)(s'-s)}+C(s-s_0). \tag{9}$$

Equation (9) no longer has the same asymptotic behavior as the original D function. This should not make much difference, however, since we are only interested in the low-energy region.

So far, we have not assumed any kind of symmetry. Suppose now that $SU(3)$ holds exactly, with the masses of the PS mesons equal to each other. Then the amplitude for all the PS scattering processes that we are considering can be written in terms of a single function $f(s)$;

$$A(s)=Mf(s), \qquad (10)$$

where M is a purely numerical matrix whose elements can be calculated from $SU(3)$ Clebsch-Gordan coefficients. Of course, s_c must then be taken to be the same for every process, something we have been effectively assuming all along.

Suppose we now break the symmetry for all nearby singularities but preserve it for the far-away cuts. The only way of doing this with Eqs. (7) and (9) is to keep the same s_c for all channels and to require that C have the form[9]

$$C=\kappa M^{-1}\lim_{s\to\infty} sN(s), \qquad (11)$$

where κ is a pure number. This is because s_c and C represent the distant singularities within our representation. The last term in Eq. (11) has to be put in because we are assuming that it is the amplitude which should obey $SU(3)$ for large s, and not some purely auxiliary quantity like the D function. In fact, we now have an asymptotic behavior

$$A^{-1}(s)\to(\alpha s+\beta\ln s)M^{-1}+O(1), \qquad (12)$$

where α and β are numbers. In other words, the leading singularities are indeed $SU(3)$-symmetric at infinity.

We now have an N/D representation which depends on the two parameters s_c and κ.[10] Suppose we are interested in the vector mesons, which occur as resonances in the channels we have been considering. Now a resonance occurs at $s=s_R$ if

$$\det D(s_R)=0. \qquad (13)$$

The residue of the corresponding pole, i.e., the reduced width, is

$$\gamma=-N(s_R)\,\mathrm{cof}D(s_R)/(d/ds)[\det D(s)]_{s=s_R}, \qquad (14)$$

where $\mathrm{cof}D(s_R)$ means the cofactor matrix of $D(s_R)$. We can adjust s_c and κ to give the experimental mass and width of one of the resonances, say the ρ. Equations (7), (9), (11), (13), and (14) are then capable of predicting the masses and widths of the others. The expressions for $\mathrm{Im}A(s)$ in Eq. (7) can be evaluated from vector-meson exchange, which should be the dominant contribution to the nearby left-hand cut. Of course, within our limited framework, experimental values have to be used for the masses of the pseudoscalar mesons.

[9] We are assuming that $\mathrm{Im}A^{-1}(s)$ obeys $SU(3)$ for large s. Equation (11) is equivalent to approximating it with an $SU(3)$-symmetric delta function at infinity. We could not have done this unless s_c were the same for all channels.

[10] In PS-B scattering (B=baryon octet), we would have had three parameters, since M would have also contained the D- to F-ratio.

III. SIMPLIFIED CALCULATION OF THE ϕ IN TERMS OF THE ρ

We now illustrate the above method by looking at the $I=1$ $\pi\pi$-$K\bar{K}$ problem. Instead of doing a full two-channel calculation, however, we shall take advantage of two features which are peculiar to this particular case. The first is that the isospin crossing matrix element β_{11} is the same for the $\pi\pi$ problem with ρ exchange as it is for the PS-PS problem with vector exchange in the pure $SU(3)$ limit with degenerate masses ($\beta_{11}=\frac{1}{2}$). The second is that the $\pi\pi$ and $K\bar{K}$ thresholds are widely separated. Suppose we write the unitarity relation for the $\pi\pi\to\pi\pi$ amplitude

$$\mathrm{Im}A_{11}=\rho_{11}|A_{11}|^2+\rho_{22}|A_{12}|^2, \qquad (15)$$

which can be rewritten as

$$\mathrm{Im}A_{11}^{-1}=-\rho_{11}-(|A_{12}|^2/|A_{11}|^2)\rho_{22}. \qquad (16)$$

If $SU(3)$ were exact, we would have had $A_{11}=\sqrt{2}A_{12}$, so that Eq. (16) would have been

$$\mathrm{Im}A_{11}^{-1}=-(\rho_{11}+\tfrac{1}{2}\rho_{22}). \qquad (17)$$

We shall assume that Eq. (16) is approximately valid even if the symmetry is broken. It is certainly exact below the $K\bar{K}$ threshold because of the θ function in the definition (3). We can then write

$$A_{11}(s)=N/D, \qquad (18)$$

where N and D would be again given by Eqs. (7) and (9) but with $\rho=(\rho_{11}+\frac{1}{2}\rho_{22})$. Equation (11) is also the same, but with $M=\frac{2}{3}$. For $\mathrm{Im}A_{11}$ in Eq. (7), we shall take an approximation to the contribution of ρ exchange. It can then be shown that our N/D equations give the same result as the multichannel equations in the limit of exact $SU(3)$.[11]

In dealing with ρ exchange we exploit the fact that its contribution to A_{11} is roughly constant up to quite high energies.[12] This is unchanged by the introduction of a cutoff, provided that s_c is sufficiently large. On the other hand, the cutoff gives rise to an s^{-1} behavior at infinity. One way of approximating this graph would therefore be to replace it with a pole adjusted so as to give the correct threshold. Specifically, we take an expression of the form

$$\mathrm{Im}A_{11}(s)=\pi b\lambda\delta(s+\lambda), \qquad (19)$$

which gives a contribution to the amplitude of

$$B_{11}(s)=b\lambda/(\lambda+s), \qquad (20)$$

so that, with sufficiently large λ, b is the threshold value given by the ρ-exchange graph. Of course, Eq. (20) may

[11] This is made possible only by the accidental equality of crossing-matrix elements mentioned in the first paragraph. We could not have done the same thing in the $I=\frac{1}{2}$ πK-ηK problem, for instance.

[12] See, for instance, L. A. P. Balázs, Phys. Rev. **134**, B1315 (1964).

lead to a completely meaningless result at high energies; this does not matter within an effective-range approach, however. The position $s=-\lambda$ of our pole plays exactly the same role as the cutoff and will be taken to be the same in all channels, just like s_c. With the approximation (19) we can solve our N/D equations exactly to obtain

$$N(s)=b\lambda/(\lambda+s)\,, \tag{21}$$

$$D(s)=1-\frac{s+\lambda}{\pi}b\lambda\int_{s_t}^{\infty}ds'\frac{\rho_{11}(s')+\frac{1}{2}\rho_{22}(s')}{(s'+\lambda)^2(s'-s)}+\tfrac{3}{2}\kappa b\lambda(s+\lambda)\,. \tag{22}$$

The requirement that this reproduce the experimental mass and width of the ρ[13] gives $\ln|\lambda|=17.62$ and $\pi\kappa=1-649\lambda^{-1}$.

We next turn to the $I=0$ $K\bar{K}$ state. This is a one-channel problem, so we can use Eqs. (7), (9), and (11) directly. With the same normalization as in the $\pi\pi$ problem, we would have $M=1$ in this case. The nearby left-hand cut is taken from ρ and ϕ exchange. We shall approximate their contribution by a pole adjusted so as to give the correct threshold behavior, just as we did for the $\pi\pi$ problem. Now in a more complete calculation all the necessary parameters could be calculated self-consistently. To simplify matters, however, the ρKK and ϕKK coupling constants were calculated from the experimental $\rho\pi\pi$ coupling, assuming unbroken $SU(3)$. Of course, for λ and κ we take the values obtained in the preceding paragraph.

The main complication in the $I=0$ $K\bar{K}$ problem arises from the possibility of ϕ-ω mixing.[1,14] Two different models were considered:

(a). The simplest possibility would be to assume that R symmetry[1] is meaningful, at least for mesons.[15] Then there is no ϕ-ω mixing, and the ϕ is the eighth member of the octet with no coupling to $\pi\rho$, while the ω is a singlet with zero coupling to the $K\bar{K}$ channel. In other words, the ω does not come into our calculation at all. The experimental value was taken for the mass of the ϕ exchanged in the crossed channel. Our equations then give an output ϕ in the direct channel with mass$=1040$ MeV and reduced width $\gamma_{\phi KK}=0.92$.

(b). If ϕ-ω mixing does occur, then both the ϕ and the ω can be exchanged in the crossed channel. Instead of putting both in explicitly, however, we shall simulate their combined effect by means of a single effective vector meson acting like the $I=0$ member of the octet. For its mass m_8 we use the Gell-Mann–Okubo mass formula.[1,3]

$$m_8^2=\tfrac{1}{3}(4m_{K^*}^2-m_\rho^2)\,. \tag{23}$$

This mass lies between the ϕ and ω masses; indeed, it was this fact which first motivated the study of ϕ-ω mixing.[14] Our effective meson is the one which would occur if there were no ϕ-ω mixing and the mass formula were satisfied exactly. It can therefore be thought of as some kind of average of the ϕ and ω.

With ϕ-ω mixing, we might expect the $\pi\rho$ channel to couple strongly to our $K\bar{K}$ state. Its inclusion would complicate the problem enormously. It was therefore dropped, in the hope that since the $\phi\rightarrow\rho\pi$ decay is known to be very weak experimentally,[13] the $\pi\rho$ channel may be unimportant for calculating the ϕ resonance, even though it may be important in calculating other effects. Our single-channel $I=0$ $K\bar{K}$ representation then gives an output ϕ with mass$=1017$ MeV and $\gamma_{\phi KK}=0.91$.

Experimentally,[13] we have $\phi_{\text{mass}}=1020$ MeV and $\gamma_{\phi KK}=0.79\pm0.15$, which corresponds to a full width $=3.1\pm0.6$ MeV; if we had used unbroken $SU(3)$, the value of $\gamma_{\phi KK}$ calculated from the experimental $\rho\pi\pi$ coupling constant would have been $\gamma_{\phi KK}=1.02$. Thus model (b) seems to be favored, although the numerous approximations we have made preclude any definite conclusions.

IV. CONCLUSION

We have seen how it is possible to correlate the properties of the various members of an $SU(3)$ multiplet even when the symmetry is badly broken. This is done by using the familiar techniques of dispersion theory to treat nearby singularities and assuming $SU(3)$ for the more distant ones, which are represented in terms of a pair of effective-range constants. We can then calculate the masses and couplings of the members of our multiplet in terms of as many parameters as would have occurred in the unbroken symmetry.

In the example of the preceding sections, for instance, we considered the members of the vector-meson octet (V_8), which occur as bound systems in PS-PS scattering. Assuming the PS masses to be known, it is then possible to calculate all the V_8 parameters in terms of only two constants. In a more complete calculation we could try to calculate the PS masses at the same time by looking at PS-V_8 scattering. This is technically a much more complicated problem. Not only do we have spin complications, but in $\pi\rho$ scattering, for instance, the exchange of the π in the crossed channel gives rise to a cut in the physical region, so that the N/D formalism has to be modified. Of course, since we do not have vertex symmetry, we would actually be over-determining many coupling constants which also occur in PS-PS scattering. We could either take advantage of this to simplify our calculation, or use any discrepancies between the parameters as a measure of the accuracy of our calculations.

Similar calculations can be done for the baryon octet (B_8) and decimet (B_{10}) in PS-B_8 scattering.

[13] We are taking mass$=765$ MeV and full width$=105$ MeV for the ρ. See A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[14] J. J. Sakurai, Phys. Rev. Letters **9**, 472 (1962); reprinted in Gell-Mann and Ne'eman, Ref. 1, p. 108.

[15] J. Bronzan and F. E. Low, Phys. Rev. Letters **12**, 522 (1964).

The static model[16] would be the simplest framework for dealing with this problem, since the nearby left-hand cut can then be approximated quite naturally by a small number of poles. To remove the divergences which arise in the D function, we can introduce a pole and a dipole at some point on the left, with residues adjusted so as to have the N function fall off sufficiently rapidly at infinity. This position would play essentially the same role as the cutoffs Λ or λ in PS-PS scattering, and can therefore be taken at the same point in all channels. A second effective-range parameter can again be introduced by adding a linear term to the D function.

If we tried to do a relativistic calculation of PS-B_8 scattering, the left-hand cut would no longer be restricted to the real axis in the plane of W, the natural variable in this problem.[17] This makes it difficult to impose a cutoff. One way of getting around this is to calculate the discontinuity in a variable such as q^2 for which the singularities are restricted to the real axis. One could then write a dispersion relation to calculate the full contribution B of the left-hand cut, which would have a cutoff at some value of this variable. The corresponding value of W could be taken to be $SU(3)$ symmetric. Once we have $B(W)$, we can solve the Uretsky form[18] of the N/D equations.

One of the difficulties often encountered in a problem such as PS-B_8 scattering is that it is not possible to avoid kinematical singularities, at least if one wishes to have reasonable behavior at infinity. This does not give any trouble in an effective-range method, however, since we can always take the kinematical singularity at the position of the cutoff. It then becomes part of the faraway singularities which we are parametrizing anyway.

Finally, it should be possible to extend our approach to include weak and electromagnetic interactions. Here it is convenient to treat all nonstrong effects as perturbations. We could, for instance, combine the above effective-range formalism with the Dashen-Frautschi perturbation technique.[4] Since these perturbations break various internal symmetries which are assumed to be valid at small distances, it should be meaningful to include only their contribution to the nearby singularities, as we have seen.

[16] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1571 (1956); **101**, 1579 (1956).
[17] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).
[18] J. Uretsky, Phys. Rev. **123**, 1459 (1961).

ACKNOWLEDGMENTS

The author wishes to thank Professor E. S. Abers for reading the manuscript, Dr. I. Grodsky for helpful conversations, and Professor N. Byers, Professor S. C. Frautschi, Professor J. S. Ball, Professor J. M. Cornwall, Professor R. Norton, and Professor M. Parkinson for discussions.

APPENDIX

In setting up our N/D representation, we have attempted to treat explicitly as many singularities as can be handled simply with present-day dispersion techniques. It would be interesting to see what would happen if we left some out. We shall therefore consider a representation in which only the elastic right-hand cut is treated as a nearby singularity. All remaining singularities are approximated by a distant pole, whose position and residue are assumed to be $SU(3)$-symmetric. In other words, we use Eqs. (4), (5), and (6) but with

$$\mathrm{Im}A(s)=\pi k M\delta(s+\eta)\,, \tag{A1}$$

where k is a number and M is the same matrix as in Sec. II. We can now solve Eqs. (5) and (6) exactly, and fix k and η by requiring our N/D equations to give the experimental mass and width of the ρ. We can then use them to predict the masses and widths of the other members of the multiplet.

In practice, the same sort of approximations were made as in Sec. III. In the $I=1$ $\pi\pi$-$K\bar{K}$ problem, N/D equations were written down for $A_{11}(s)$ instead of $A(s)$, and ρ was replaced by $(\rho_{11}+\frac{1}{2}\rho_{22})$. We then used Eq. (A1) but with $M=\frac{2}{3}$. The requirement that the resulting equations reproduce the experimental mass and width of the ρ [13] gives $\ln|\eta|=16.60$ and $\pi k^{-1}=1+29.0\eta^{-1}$.

We next turn to the $I=0$ $K\bar{K}$ state. Being a one-channel problem, we can use Eqs. (5), (6), and (A1), where now $M=1$. Since we are assuming $SU(3)$ for our distant pole, we naturally take for η and k the values we obtained in the preceding paragraph. Assuming again that the $\pi\rho$ channel does not couple strongly to our $K\bar{K}$ state, we obtain an output ϕ with mass$=1210$ MeV and $\gamma_{\phi KK}=0.99$. Although $\gamma_{\phi KK}$ is not too different from the values obtained in Sec. III, the mass is considerably larger. This suggests that it may not be safe to leave out an explicit consideration of the nearby left-hand cut, except perhaps for obtaining a first crude approximation.