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# Faddeev Equations with Inelastic Processes

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Starting from the general Lippmann-Schwinger equation, we study the effect of inelastic processes on the scattering of three-body states. We show that it is possible to incorporate the inelastic effects into Faddeev-type equations in two different ways. The first approach is a straightforward generalization of the single-channel Faddeev equations to the multichannel form. However, we have to introduce the concepts of *position* and *particle* labeling in order to obtain a meaningful generalization. The second approach, which is derived from an extension of the concept of a complex potential, yields single-channel Faddeev equations with a modified input. The essential difference between this input and the input for the elastic case is the presence of a completely connected term. The two approaches are shown to be equivalent in their common region of validity. Under the resonance approximation, both approaches yield one-dimensional equations. The structure of the completely connected term is investigated for certain specific models and further simplifications on it are obtained. We also observe that the concept of the inelasticity parameter in the two-body case does not seem to have a natural generalization.

## I. INTRODUCTION

RECENTLY, the successful work of Faddeev[1] and others[2,3] in constructing a mathematically correct theory for the nonrelativistic scattering of three-particle systems has generated a considerable interest[4] in the three-body problem. While a relativistic generalization of Faddeev's work now seems in sight,[5] some serious attempts have already been made[4,6] in applying Faddeev equations to strong interaction dynamics at low and intermediate energies. On the other hand, a physical situation may involve several other nearby states which may exercise considerable influence and should, therefore, be included for any meaningful comparison with the experiments. Also, some recent attempts (see, for example, Ref. 6) in this direction already seem to indicate that these states may play a considerable role in the energy regions of interest. It is to this question that we focus our attention in this paper.

Our starting point is the multichannel multibody Lippmann-Schwinger equation.[7] Drawing the analogy with two-particle scattering, one can adopt two basically different approaches to handle the inelastic states. In the two-body case, inelastic scattering can either be handled by the multichannel approach using, for example, a set of matrix $N/D$ equations[8]; or by using a modified single-channel formalism involving some function expressing the effect of inelasticity.[9] The second procedure is more potent in that, unlike the first procedure, it is not limited to only two-body inelastic states.

In Sec. II, we follow the first procedure and discuss inelastic scattering within the framework of a set of $N$-coupled 3-body channels, and we obtain a generalization of the Faddeev equations. Since the multichannel Faddeev equations are obtained from a set of matrix Lippmann-Schwinger equations, an apparent ambiguity may arise in the decomposition of the potential matrix $V$ into three parts analogous to the single-channel decomposition. However, this can be easily resolved by a careful definition of particle and "position" labels.

In Sec. III, we consider the second approach, where we allow for the possibility of multibody states. Starting from the multichannel multibody Lippmann-Schwinger equations, we obtain a single-channel three-body Lippmann-Schwinger equation with a modified potential. From this it is then shown that we can obtain the single-channel modified Faddeev equations which describe the scattering of the three-body state in the presence of any general inelastic process. An important new feature, however, now emerges in the structure of the modified Faddeev equations. The analog to the two-body amplitude consists here of not only the disconnected term from the off-shell inelastic two-body amplitude, but also of a completely connected term. The effect of the inelastic states cannot, therefore, be completely determined through modifications of the disconnected input two-body amplitudes alone.

In Sec. IV, we show that it is also possible to obtain the single-channel modified Faddeev equations starting

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from the $N$-coupled channel Faddeev equations obtained in Sec. II, thus showing the equivalence of the two procedures in this common region of validity. In Sec. V, we carry out a partial-wave analysis of both the coupled-channel Faddeev equations and the single-channel modified equations. Section VI shows that the resonance approximation yields one-dimensional equations in both cases. Section VII considers some possible further simplifying approximations, and Sec. VIII consists of a short resume and discussion.

Throughout the present discussion we shall generally ignore considerations of relativistic generalization,[10] spin, and statistics. These complications are not really relevant to the central object of the present discussion, and can be added later to any given problem.

## II. MULTICHANNEL APPROACH

Given an initial three-particle channel and the various possible production processes, the total number $N$ of distinct three-body channels is determined. We shall assume as usual that the interactions are all two-body processes, although the inclusion of three-body processes presents no additional complication. We shall also assume that the three particles in each channel are distinct (symmetrization can be accomplished afterwards as usual). We take the following generalized (off-shell) Lippmann-Schwinger[7] equation to hold:

$$T_{\mu\nu}=V_{\mu\nu}+\sum_{\alpha=1}^{N} V_{\mu\alpha}G_\alpha T_{\alpha\nu}, \quad \mu, \nu=1, \cdots, N. \quad (2.1)$$

$T_{\mu\nu}$ and $V_{\mu\nu}$ are the elements of the scattering amplitude and the potential matrices between the channels $\mu$ and $\nu$; $G_\alpha$ is the free-propagation Green's function appropriate to the channel $\alpha$, and satisfies

$$G_\alpha(Z)=[H_0{}^\alpha-Z]^{-1}, \quad \alpha=1, \cdots, N, \quad (2.2)$$

$H_0{}^\alpha$ being the free Hamiltonian for the channel $\alpha$, and $Z$ the total energy.

In order to obtain Faddeev-type equations, we must decompose the potential. If we define $U_{\alpha\beta}{}^k$ to be zero unless particle $k$ is present in both channel $\alpha$ and $\beta$, and then to be the potential for the reaction of the other two pairs of particles, then it is clear that

$$V=\sum_{k=1}^{M} U^k, \quad (2.3)$$

where $M$ is the total number of distinct particles. The equations

$$h^k=U^k+U^kGh^k, \quad k=1, \cdots, M \quad (2.4)$$

have well-defined[11] solutions, whose nonzero elements are appropriate two-body coupled scattering amplitudes. We may define

$$T^k=U^k+U^kGT, \quad k=1, \cdots, M \quad (2.5)$$

so that

$$T=\sum_{k=1}^{M} T^k \quad (2.6)$$

and

$$T^k=h^k+h^kG\sum_{l\neq k} T^l, \quad k=1, \cdots, M. \quad (2.7)$$

This is a rather large number of amplitudes and equations, and one would prefer to have just three equations, in a natural generalization of the elastic case. It appears possible to do this, using a "positional" notation. In listing the particles present in a channel, we shall do so in an order such that any given particle label occupies the same position in the list as it does in the lists for all other channels in which it appears. This procedure can run into trouble only when there are two candidates for the same position in some channel or when a particle is forced into different positions by requirements on other particles. A typical example of this is illustrated in the Appendix, and such pathological cases can generally be resolved by invoking the requirement that the particle labels in any three-body channel be different. In any case, all physical examples examined do not have this difficulty.

We now define[12]

$$V_{\mu\nu}{}^i=\sum_k U_{\mu\nu}{}^k, \quad (2.8)$$

where the sum over $k$ includes only those potentials for which the particle $k$ is in the $i$th position. We remark that the equations

$$t^i=V^i+V^iGt^i \quad (2.9)$$

have well-defined solutions (which are the appropriate sums over $h^k$, and therefore consist of coupled two-body scattering amplitudes). A simple specific example of the foregoing is given in the Appendix.

In order to obtain Faddeev equations, we proceed as usual by defining

$$T^i=V^i+V^iGT, \quad (2.10)$$

so that

$$T=\sum_{i=1}^{3} T^i \quad (2.11)$$

and

$$T^i=t^i+t^iG(T^j+T^k). \quad (2.12)$$

[10] Though our considerations are nonrelativistic, the derivations can be carried through as easily on relativistic equations such as the Bethe-Salpeter equation [Phys. Rev. 84, 1232 (1951)] since the essential structure is the same.

[11] The solutions are well defined because the kernel and the inhomogeneous term have the same disconnected structure. See Refs. 1, 2, 3.

[12] The free subscripts $\mu$ and $\nu$ are always channel indices, and independently take on the values 1 through $N$. The free superscript $i$ takes on the values 1, 2, 3; when $i$, $j$, $k$ occur together as superscripts they are assumed to be any permutation of 1, 2, 3.

The amplitude $T^i$ may be interpreted as the sum of the set of diagrams in which the particle in the $i$th position does not interact initially.

If we put the channel indices into Eq. (2.12), it becomes

$$T_{\mu\nu}{}^i = t_{\mu\nu}{}^i + \sum_{\alpha=1}^{N} t_{\mu\alpha}{}^i G_\alpha (T_{\alpha\nu}{}^j + T_{\alpha\nu}{}^k). \tag{2.13}$$

We may obtain a different form of these equations by defining

$$t_{\mu\nu} = \begin{bmatrix} t_{\mu\nu}{}^1 \\ t_{\mu\nu}{}^2 \\ t_{\mu\nu}{}^3 \end{bmatrix}, \qquad \tilde{t}_{\mu\nu} = \begin{bmatrix} t_{\mu\nu}{}^1 & 0 & 0 \\ 0 & t_{\mu\nu}{}^2 & 0 \\ 0 & 0 & t_{\mu\nu}{}^3 \end{bmatrix},$$
$$T_{\mu\nu} = \begin{bmatrix} T_{\mu\nu}{}^1 \\ T_{\mu\nu}{}^2 \\ T_{\mu\nu}{}^3 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2.14}$$

We then have, for example,

$$\begin{bmatrix} T_{1\nu} \\ T_{2\nu} \\ \vdots \\ T_{N\nu} \end{bmatrix} = \begin{bmatrix} t_{1\nu} \\ t_{2\nu} \\ \vdots \\ t_{N\nu} \end{bmatrix} + \begin{bmatrix} \tilde{t}_{11} & \tilde{t}_{12} \cdots \tilde{t}_{1N} \\ \tilde{t}_{21} & \tilde{t}_{22} \cdots \tilde{t}_{2N} \\ \vdots & \vdots \quad \vdots \\ \tilde{t}_{N1} & \tilde{t}_{N2} \cdots \tilde{t}_{NN} \end{bmatrix} \begin{bmatrix} G_1K & 0 & 0 & 0 \\ 0 & G_2K & & \\ & & \ddots & \\ 0 & 0 & & G_NK \end{bmatrix} \begin{bmatrix} T_{1\nu} \\ T_{2\nu} \\ \vdots \\ T_{N\nu} \end{bmatrix}. \tag{2.15}$$

There are a number of general comments which can be made at this point. First, as can be seen from Eq. (2.15) or Eq. (2.13), one only needs $3N$ equations to find a given amplitude, as the $T_{\mu\nu}$ for fixed $\nu$ close upon themselves. Second, when some of the particles are actually identical and the appropriate symmetrization is done, the number of amplitudes and equations will reduce from $3N$ to the number of physically distinct processes. Third, the number of equations increases very fast with the number of production mechanisms, going roughly as the cube of the number of distinct particles. Fourth, the lowest-order correction to the elastic scattering amplitude is trivially obtained from the first iteration of Eq. (2.13). Finally, this multichannel formalism is inherently restricted to inelastic channels containing exactly three particles.

## III. COMPLEX-POTENTIAL APPROACH

We now wish to approach the problem from a different point of view, one which is applicable when there are arbitrary numbers of channels containing arbitrary numbers of particles. The similar situation in the two-body problem is handled by the introduction of a complex potential (equivalently, an inelasticity parameter). We shall proceed in the same spirit from the generalized Lippmann-Schwinger equation

$$\mathcal{T} = V + V\mathcal{G}\mathcal{T}, \tag{3.1}$$

where the matrices are in a channel space not restricted to three-body states.

We shall define $P$ as the projection operator onto the three-body state of interest (note that $P$ commutes with $G$). Our desired amplitude is $P\mathcal{T}P \equiv T$, and by taking projections on Eq. (3.1) one finds

$$T = W + WGT, \tag{3.2}$$

where

$$G = P\mathcal{G}, \tag{3.3}$$

$$W = PSP, \tag{3.4}$$

$$S = V + V\mathcal{G}(1-P)S. \tag{3.5}$$

Equation (3.2) is exactly a single-channel equation with a modified potential $W$. The structure of $W$ is extremely complex, but it is clear that it can unambiguously be decomposed into its disconnected parts $W^i$ ($i=1, 2, 3$) and a completely connected part $W^0$.

We can give an explicit representation for $W^i$ by defining $P_i$ as the projection operator onto the states containing the particle $i$. Then we define[12]

$$V^i = (P_i V P_i)^i, \tag{3.6}$$

where the superscript indicates that the particle $i$ does not interact (note that this is not a positional notation). We now define

$$\tau^i = V^i + V^i \mathcal{G}(1-P)\tau^i. \tag{3.7}$$

Then

$$W^i = P\tau^i P. \tag{3.8}$$

The connected part $W^0$ is given implicitly by

$$W = W^1 + W^2 + W^3 + W^0. \tag{3.9}$$

We can proceed to obtain four Faddeev equations by defining as usual

$$\tilde{T}^\mu = W^\mu + W^\mu G T, \quad \mu = 0, 1, 2, 3, \tag{3.10}$$

and

$$t^\mu = W^\mu + W^\mu G t^\mu, \quad \mu = 0, 1, 2, 3. \tag{3.11}$$

Again, Eq. (3.11) actually has well-defined solutions. We have

$$T = \sum_{\mu=0}^{3} \tilde{T}^\mu \tag{3.12}$$

and

$$\tilde{T}^\mu = t^\mu + t^\mu G \sum_{\nu\neq\mu} \tilde{T}^\nu, \quad \mu = 0, 1, 2, 3. \tag{3.13}$$

We remark that $t^1$ is the coupled amplitude for $2+3 \to 2+3$ via all allowed intermediate states; similarly for $t^2$ and $t^3$. We note that $t^1$, $t^2$, and $t^3$ are disconnected amplitudes; $t^0$ is completely connected. The essential difference, aside from the fact that $t^i$ ($i=1, 2, 3$) are now coupled amplitudes, between Eqs. (3.13) and the elastic-scattering equations is seen to be due to the presence of an additional term (and an additional equation) arising from the presence of completely connected graphs in the potential $W$, whose origin is due solely to the existence of inelastic processes. As the coupling to the inelastic channels tends to zero, the quantities $W^0$, $t^0$, and $T^0$ all tend to zero,[13] and the elastic equations are recovered.

We would like to rewrite these equations in a form more closely analogous to the elastic case. Toward this end, we observe that $W^0$ can be decomposed into three parts, according to which pair of particles interact first (and labeled by the initially noninteracting third particle). We write

$$W^0=W^{01}+W^{02}+W^{03} \tag{3.14}$$

and

$$\tilde{T}^{0i}=W^{0i}+W^{0i}GT, \tag{3.15}$$

so that

$$\tilde{T}^0=\sum_{i=1}^{3}\tilde{T}^{0i}. \tag{3.16}$$

If we now define

$$T^i=\tilde{T}^i+\tilde{T}^{0i}, \tag{3.17}$$

then

$$T=\sum_{i=1}^{3}T^i. \tag{3.18}$$

If we substitute Eqs. (3.15)–(3.18) into Eq. (3.13), we obtain[12]

$$T^i=u^i+u^iG(T^j+T^k), \tag{3.19}$$

where

$$u^i=t^i+(1+t^iG)W^{0i}+(1+t^iG)W^{0i}Gu^i. \tag{3.20}$$

The disconnected part of $u^i$ is just $t^i$, and if we write

$$u^i=t^i+I^i, \tag{3.21}$$

then we have

$$I^i=(1+t^iG)J^i(1+Gt^i) \tag{3.22}$$

and

$$J^i=W^{0i}+W^{0i}G(1+t^iG)J^i. \tag{3.23}$$

Equations (3.19) are of exactly the same form as the elastic Faddeev equations, only with a modified input. Aside from the fact that the $t^i$ are now coupled two-body amplitudes, the inelasticity makes itself felt only in the presence of the completely connected terms $I^i$.

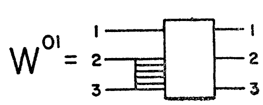


FIG. 1. Schematic representation of $W^{01}$. A square indicates that only inelastic states are involved; a vertical line between particles is a single potential interaction; the multiple horizontal lines indicate propagation of an inelastic state.

[13] If the elastic problem has a three-body interaction $V^0$, then $W^0$ goes to $V^0$.

In order to investigate $I^i$, we need to know more about $W^{0i}$. We define

$$S^i=V^i+S^i\mathcal{G}(1-P)V \tag{3.24a}$$

or equivalently

$$S^i=V^i+V^i\mathcal{G}(1-P)S, \tag{3.24b}$$

where $V^i$ is given by Eq. (3.6). Then

$$W^{0i}=P(S^i-\tau^i)P. \tag{3.25}$$

Equation (3.25) gives explicitly the representation of $W^{0i}$ in terms of the potentials. The subtraction in Eq. (3.25) is easy to understand: Between three-body states, and in particular for our state of interest, $\tau^i$ is just the disconnected part of $S^i$. Thus, $W^{0i}$ must consist of an initial scattering of particles $j$ and $k$ into an inelastic state, subsequent scatterings, at least one of which must involve particle $i$, among the inelastic states, and finally a transition back to the state 1, 2, 3 (see Fig. 1). The Eq. (3.23) for $J^i$ gives just an iteration of these diagrams, with a modified propagator (see Fig. 2).

It is clear from Eqs. (3.24) and (3.7) that $S^i$ and $\tau^i$, and hence $W^{0i}$, have branch cuts on the positive real axis of the total energy[14] arising from all the relevant inelastic channels, but not from the elastic channel. In fact, from Eqs. (3.4) and (3.5) we may obtain for the discontinuity[15] of $W$

$$\Delta W=PS\Delta\mathcal{G}(1-P)S^{\dagger}P. \tag{3.26}$$

From Eqs. (3.7) and (3.8) we can find the discontinuities of the disconnected parts of $W$:

$$\Delta W^i=P\tau^i\Delta\mathcal{G}(1-P)\tau^{i\dagger}P; \tag{3.27}$$

the discontinuity of the connected part is thus given by Eq. (3.9) as

$$\Delta W^0=\Delta W-\sum_{i=1}^{3}\Delta W^i. \tag{3.28}$$

Due to the presence of the modified elastic propagator in Eq. (3.23), $J^i$ and hence $I^i$ have an elastic branch cut as well as the inelastic cuts.

We may finally remark that it can be easily seen that $I^i$ necessarily begins with a two-body transition amplitude, since $(1+t^iG)W^{0i}$ can be written as a transition amplitude times something involving only inelastic processes, followed by a transition via a single interaction to the elastic state.

## IV. EQUIVALENCE OF THE TWO APPROACHES

In the previous sections, we have described two different approaches for the treatment of inelastic states in three-body equations. The question naturally arises

[14] In the nonrelativistic problem the energy under consideration is the kinetic energy and hence all thresholds occur at the same point, namely zero. In the relativistic case the appropriate energy is the total energy and hence all the thresholds are separated.

[15] $S^+$ is the adjoint of $S$ and $\Delta W=1/2i[W-W^+]$.

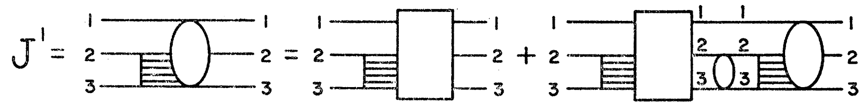


FIG. 2. Representation of $J^1$ and Eq. (3.23). The symbols have the same meaning as in Fig. 1. A circle indicates a scattering where all states may be present.

whether, when it is possible to apply the two approaches to the same situation, one obtains the same amplitudes and equations. One would expect so, since the total amplitude $T$ and the subamplitude $T^i$ have a definite physical meaning. We shall examine this question in the domain of common validity of the two approaches, namely the situation described in Sec. II.

First, let us see what the complex approach becomes in this situation. Equation (3.5) becomes, for $S$,[12]

$$S_{\mu\nu}=V_{\mu\nu}+\sum_{\alpha=2}^{N} V_{\mu\alpha}G_\alpha S_{\alpha\nu}\,, \tag{4.1}$$

where we have used the notation of Sec. II and $G_1=PG$. Equation (3.4) is now

$$W=S_{11}. \tag{4.2}$$

We may use the positional decomposition of Sec. II to define[12]

$$S_{\mu\nu}{}^i=V_{\mu\nu}{}^i+\sum_{\alpha=2}^{N} V_{\mu\alpha}{}^iG_\alpha S_{\alpha\nu} \tag{4.3}$$

and

$$\tau_{\mu\nu}{}^i=V_{\mu\nu}{}^i+\sum_{\alpha=2}^{N} V_{\mu\alpha}{}^iG_\alpha \tau_{\alpha\nu}{}^i. \tag{4.4}$$

Let us remark that $V^i$ of Eq. (3.6) is $U^i$ of Sec. II, and that the $S^i$ and $\tau^i$ defined here are closely related to, but not identical with, the $S^i$ and $\tau^i$ of Sec. III.

We have, as always,

$$S_{\mu\nu}{}^i=\tau_{\mu\nu}{}^i+\sum_{\alpha=2}^{N} \tau_{\mu\alpha}{}^iG_\alpha(S_{\alpha\nu}{}^j+S_{\alpha\nu}{}^k). \tag{4.5}$$

Since the particle and positional notations have been chosen to agree for channel 1, we have

$$W^i=\tau_{11}{}^i, \tag{4.6}$$

$$W^{0i}=\sum_{\alpha=2}^{N} \tau_{1\alpha}{}^iG_\alpha(S_{\alpha1}{}^j+S_{\alpha1}{}^k). \tag{4.7}$$

Equation (3.11) has, for its first three components,

$$t^i=\tau_{11}{}^i+\tau_{11}{}^iG_1t^i, \tag{4.8}$$

and Eq. (3.20) may be written as

$$u^i=t^i+\alpha^i+\alpha^iG_1u^i, \tag{4.9}$$

where

$$\alpha^i=(1+t^iG_1)\sum_{\beta=2}^{N} \tau_{1\beta}{}^iG_\beta(S_{\beta1}{}^i+S_{\beta1}{}^k). \tag{4.10}$$

We may gain further insight into Eqs. (4.8) and (4.10) by introducing the two-body multichannel scattering amplitudes $t_{\mu\nu}{}^i$:

$$t_{\mu\nu}{}^i=V_{\mu\nu}{}^i+\sum_{\alpha=1}^{N} V_{\mu\alpha}{}^iG_\alpha t_{\alpha\nu}{}^i. \tag{4.11}$$

Note that the summation over $\alpha$ now includes the elastic channel, and that these are the physically realizable amplitudes which appear in the multichannel Faddeev equations of Sec. II. It is straightforward to establish the following important identity:

$$\tau_{\mu\nu}{}^i=t_{\mu\nu}{}^i-t_{\mu1}{}^i(1+G_1t_{11}{}^i)^{-1}G_1t_{1\nu}{}^i\,, \tag{4.12}$$

from which it follows that the solutions of Eqs. (4.8) and (4.10) may be expressed as

$$t^i=t_{11}{}^i \tag{4.13}$$

and

$$\alpha^i=\sum_{\alpha=2}^{N} t_{1\beta}{}^iG_\beta(S_{\beta1}{}^j+S_{\beta1}{}^k). \tag{4.14}$$

This illustrates explicitly the claims made in Sec. III; namely, that $t^i$ is the full multichannel elastic-scattering amplitude and that $I^i$ begins with a physical transition amplitude. $\tau_{\mu\nu}{}^i$ and $S_{\mu\nu}{}^i$ for $\mu\neq1$ and $\nu\neq1$ can be understood as the two- and three-body amplitudes, respectively, for scattering when the coupling to channel 1 has been turned off.[16]

In summary, Eqs. (4.14), (4.13), (4.9), (3.19), and (3.18) give the results of the complex-potential approach to the problem of expressing the elastic amplitude in a situation with $N$ three-body channels.

We now wish to examine the other approach: We shall start from the multichannel Faddeev equations as given by Eqs. (2.13) or (2.15) and express them in single-channel form by elimination of the transition amplitudes. A direct reduction of Eq. (2.15) by straightforward elimination yields [in the notation of Eq. (2.14)]

$$T_{11}=\{t_{11}+\sum_{\alpha=2}^{N} \tilde{t}_{1\alpha}G_\alpha K(1-\chi)^{-1}t_{\alpha1}\}$$
$$+\{\tilde{t}_{11}+\sum_{\alpha=1}^{N} \tilde{t}_{1\alpha}G_\alpha K(1-\chi)^{-1}\tilde{t}_{\alpha1}\}G_1KT_{11}\,, \tag{4.15}$$

[16] The Eqs. (4.12) relate the full coupled-channel amplitudes $t_{\mu\nu}$ (describing the scattering of $N$-coupled channels) to the uncoupled amplitudes $\tau_{\mu\nu}$ [describing the scattering of $(N-1)$-coupled channels] for $\mu$, $\nu=2, \cdots, N$. Approximations on Eqs. (4.12 lead to the uncoupled phase method; see, for example, R. E. Kreps and P. Nath, Phys. Rev., this issue **152**, 1249 (1966) and P. Nath and G. L. Shaw, Phys. Rev. **137**, B711 (1965).

where

$$\chi=\begin{bmatrix}\tilde{t}_{22}\cdots\tilde{t}_{2N}\\ \vdots \quad \vdots\\ \tilde{t}_{N2}\cdots\tilde{t}_{NN}\end{bmatrix}\begin{bmatrix}G_2 & & 0\\ & \ddots & \\ 0 & & G_N\end{bmatrix}K. \tag{4.16}$$

The difficulty with Eq. (4.15) is that it is not in Faddeev form, and it is not at all transparent how it may be recast into such a form. This is essentially due to the fact that the matrix preceeding $G_1$ in the kernel is nondiagonal.

We shall now show that there exists a proper reduction of the multichannel Faddeev Eqs. (2.13) which brings them into the desired single-channel form. We write Eq. (2.13) with $\nu=1$:

$$T_{\mu1}{}^i=t_{\mu1}{}^i+t_{\mu1}{}^iG_1(T_{11}{}^j+T_{11}{}^k)+\sum_{\alpha=2}^{N}t_{\mu\alpha}{}^iG_\alpha(T_{\alpha1}{}^j+T_{\alpha1}{}^k). \tag{4.17}$$

We may set $\mu=1$ and rewrite Eq. (4.17) as

$$T_{11}{}^i=(1+t_{11}{}^iG_1)^{-1}t_{11}{}^i(1+G_1T_{11})+(1+t_{11}{}^iG_1)^{-1}\sum_{\alpha=2}^{N}t_{1\nu}{}^iG_\nu(T_{\nu1}{}^j+T_{\nu1}{}^k). \tag{4.18}$$

We now combine Eqs. (4.18) and (4.17), and use Eq. (4.12) to obtain

$$T_{\mu1}{}^i=\tau_{\mu1}{}^i(1+G_1T_{11})+\sum_{\alpha=2}^{N}\tau_{\mu\alpha}{}^iG_\alpha(T_{\alpha1}{}^j+T_{\alpha1}{}^k). \tag{4.19}$$

Equation (4.19) implies that

$$T_{\mu1}{}^i=S_{\mu1}{}^i(1+G_1T_{11}), \tag{4.20}$$

where $S_{\mu\nu}{}^i$ is given by Eq. (4.5). Equation (4.20) is the crucial equation in the reduction; it also allows us to identify the amplitudes of the complex and multichannel approaches as being the same, since $S_{11}{}^i=W^i+W^{0i}$ and the total amplitude is the same in both approaches. Upon substitution of Eq. (4.20) into Eq. (4.17), we obtain

$$T_{11}{}^i=t_{11}{}^i+\alpha^i+(t_{11}{}^i+\alpha^i)G_1(T_{11}{}^j+T_{11}{}^k)+\alpha^iG_1T_{11}{}^i, \tag{4.21}$$

where $\alpha^i$ is given by Eq. (4.14). Finally, Eq. (4.21) may be recast (since $\alpha^i$ is completely connected) into the Faddeev form

$$T_{11}{}^i=u^i+u^iG_1(T_{11}{}^j+T_{11}{}^k), \tag{4.22}$$

with $u^i$ given by Eq. (4.9). This is the desired result, showing that the two approaches are in fact equivalent.

## V. PARTIAL-WAVE DECOMPOSITION

The partial-wave projections of the three-body amplitudes are the physical objects of interest, especially in the investigation of resonances and bound states. Further, when the variables associated with the over-all center-of-mass motion and over-all rotation in the center of mass system are eliminated, the number of variables in the integral equations decreases from nine to three. Also, we would like to investigate whether the concept of inelasticity parameter, so useful in the description of two-body scattering, has a natural generalization to the three-body case.

We shall define our angular-momentum states in the spirit of Omnes[17] and Branson, Landshoff, and Taylor,[18] rather than by successive coupling of angular momenta.[19] In the over-all center-of-mass system, the three momenta form a triangle; if the energies are fixed, so is the shape of the triangle, and all that remains is its orientation. This can be specified by the rotation of a reference frame, fixed with respect to the triangle, relative to the space-fixed axes. We define our angular-momentum states (in the over-all center-of-mass system) by[20]

$$|JM\Lambda\omega_1\omega_2\omega_3\rangle=A\int d\alpha d(\cos\beta)d\gamma\mathfrak{D}_{\Lambda M}{}^{J*}(\alpha\beta\gamma)|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle, \tag{5.1}$$

where $\omega_i$ is the energy of the $i$th particle; $\alpha,\beta,\gamma$ are the Euler angles of the rotation, and $A$ is a normalization constant. It can easily be shown that this actually is a state of angular momentum $J$ with projection $\Lambda$ on the space-fixed $z$ axis, and projection $M$ on the body-fixed $z$ axis. We choose the normalization of the angular-momentum states such that the identity has the decomposition

$$1=\sum_{JM\Lambda}\int d\omega_1d\omega_2d\omega_3|JM\Lambda\omega_1\omega_2\omega_3\rangle\langle JM\Lambda\omega_1\omega_2\omega_3|. \tag{5.2}$$

If we choose the momentum states to be normalized by

$$\langle\mathbf{p}|\mathbf{p}'\rangle=\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \tag{5.3}$$

then

$$A=[((2J+1)/8\pi^2)m_1m_2m_3]^{1/2}. \tag{5.4}$$

Because of rotational invariance, all partial-wave matrix elements have a factor $\delta_{JJ'}\delta_{\Lambda\Lambda'}$ and no other dependence on $\Lambda$ or $\Lambda'$ which we henceforth suppress. Thus, after eliminating these factors, the partial-wave projection of Eqs. (3.19) takes on the form

$$\begin{aligned}\langle JM\omega_1\omega_2\omega_3|T^i|JM'\omega_1'\omega_2'\omega_3'\rangle&=\langle JM\omega_1\omega_2\omega_3|u^i|JM'\omega_1'\omega_2'\omega_3'\rangle+\sum_{M''}\int d\omega_1''d\omega_2''d\omega_3''\\&\times\langle JM\omega_1\omega_2\omega_3|u^i|JM''\omega_1''\omega_2''\omega_3''\rangle G(\omega_1''\omega_2''\omega_3'')\\&\times\langle JM''\omega_1''\omega_2''\omega_3''|T^j+T^k|JM'\omega_1'\omega_2'\omega_3'\rangle.\end{aligned} \tag{5.5}$$

[17] R. Omnes, Phys. Rev. 134, B1358 (1964).
[18] D. Branson, P. V. Landshoff, and J. C. Taylor, Phys. Rev. 132, 902 (1963).
[19] See, for example, A. Macfarlane, Rev. Mod. Phys. 34, 14 (1962).
[20] Our conventions on rotation matrices are those of Ref. 18. We use $\hbar=c=1$.

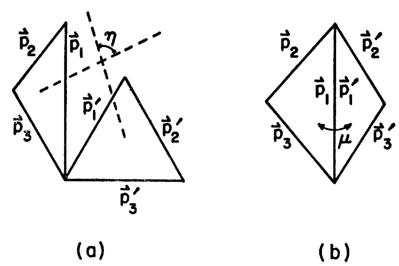


FIG. 3. (a) Relative orientation of the momentum triangles in the general connected case. The Euler angle $\eta$, for our conventions, is shown. The dashed intersecting lines are normal to their respective triangles. (b) Relative orientation of the momentum triangles for the disconnected case $\mathbf{p}_1 = \mathbf{p}_1'$. The angle $u$ is shown.

We have, after inserting a complete set of momentum states,

$$\langle JM\omega_1\omega_2\omega_3 | I^i | JM'\omega_1'\omega_2'\omega_3'\rangle$$
$$= m_1m_2m_3 \int d\xi d(\cos\eta) d\zeta \mathfrak{D}_{MM'}{}^{J*}(\xi\eta\zeta)$$
$$\times I^i(\omega_1\omega_2\omega_3, \omega_1'\omega_2'\omega_3', \xi\eta\zeta)\,, \quad (5.6)$$

where $I^i(\omega,\omega',\xi\eta\zeta)$ is the center-of-mass momentum space matrix element of $I^i$. It is clear that this matrix element depends only on the shape of the initial and final triangle, and their relative orientation [see Fig. 3(a)]. This relative orientation is specified by the Euler angles giving the rotation of the primed body-fixed system as seen from the unprimed body-fixed system. In the case of the matrix elements of $t^i$, because of the $\delta$-function in the momentum of particle $i$ inside the integration, there are $\delta$-functions in two of the angles, and the rotation is restricted to be about the axis $\mathbf{p}^i$ [see Fig. 3(b)]. If, for example, we choose the body-fixed $z$ axis along $\hat{p}_1 \times \hat{p}_2$, and the $x$ axis along $\mathbf{p}_1$, then we have as usual[17]

$$\langle JM\omega_1\omega_2\omega_3 | t^1 | JM'\omega_1'\omega_2'\omega_3'\rangle$$
$$= \frac{m_2m_3}{p_1}\delta(\omega_1-\omega_1') \sum_{\nu=-J}^{J} \Delta_{M\nu}{}^{J*}\Delta_{M'\nu}{}^{J} \int_0^{2\pi} du$$
$$\times e^{-i\nu u} t^1(\omega_1\omega_2\omega_3, \omega_1'\omega_2'\omega_3', u)\,, \quad (5.7)$$

where

$$\Delta_{M\nu}{}^{J} = \mathfrak{D}_{M\nu}{}^{J}(0,\pi/2,0)\,, \quad (5.8)$$

$u$ is the angle of rotation about $\mathbf{p}_1$, and $t^1(\omega_1\omega_2\omega_3, \omega_1'\omega_2'\omega_3', u)$ is the two-body (off-shell) scattering amplitude of particles 2 and 3 expressed in terms of the three-body center-of-mass variables. The combination of $\Delta$'s and the exponential is just the $\mathfrak{D}$ function for a rotation about the $x$ axis. The angular-momentum matrix element of $t^2$ will have additional phases due to the fact that the rotation will not be about the $x$ axis, but about $\mathbf{p}_2$.[21]

[21] Specifically, for a rotation $R$ of angle $\theta$ about $\mathbf{p}_2$, we have
$$D_{MM'}{}^{J*}(R) = e^{-iM\phi_{12}+iM'\phi_{12}'} \textstyle\sum_\nu \Delta_{M}{}^{J*}\Delta_{M'\nu}{}^{J} e^{-i\nu\theta},$$
where $\phi_{12}$ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. See also Ref. 6.

Thus, in the complex approach, the essential additional complication over the elastic case comes in the fact that the contribution from $I^i$, as expressed in Eq. (5.6), does not have the simple structure of Eq. (5.7). In the multichannel approach, the connectedness structure is still the same as in the elastic case, namely that of Eq. (5.7), but there are now present channel indices and sums over then, and kinematic modifications.

In order to examine whether a natural generalization of the inelasticity parameter exists, let us write down the partial-wave projection of the three-body multichannel discontinuity equation in the limit as the energy approaches the positive real axis from above:

$$\mathrm{Im}\langle M\omega_1\omega_2\omega_3 | T_{11} | M\omega_1'\omega_2'\omega_3'\rangle = \sum_{\nu=1}^{N} \sum_{M''=-J}^{J} \int d\omega_1'' d\omega_2'' d\omega_3''$$
$$\times \langle M\omega_1\omega_2\omega_3 | T_{1\nu} | M''\omega_1''\omega_2''\omega_3''\rangle \delta(\omega_1''+\omega_2''+\omega_3''-E)$$
$$\times \langle M\omega_1'\omega_2'\omega_3' | T_{1\nu} | M''\omega_1''\omega_2''\omega_3''\rangle^*. \quad (5.9)$$

We have set $M'=M$ and have suppressed $J$ and $\Lambda$. Note that the integration region is finite, because of the triangle restriction. In the multichannel two-body problem, the analogous equation has only one integration (and no sum over $M''$), which can be done using the $\delta$-function from the discontinuity of the propagator. One can then bound the inelastic contribution to the sum, independent of the functional form of the elastic contribution. Specifically, we have (in an obvious notation, with $\rho_\nu$ the phase-space factor)

$$\sum_{\nu=2}^{N} \rho_\nu(E) |\langle k | T_{1\nu} | k_\nu\rangle|^2$$
$$= \mathrm{Im}\langle k | T_{11} | k\rangle - \rho_1(E) |\langle k | T_{11} | k\rangle|^2 \leqslant \frac{1}{4\rho_1(E)}. \quad (5.10)$$

The inequality holds independent of any assumed functional dependence of the elastic amplitude, and hence we may write

$$\mathrm{Im}\langle k | T_{11} | k\rangle = \rho_1(E) |\langle k | T_{11} | k\rangle|^2 + (1-\eta^2)/4\rho_1(E)\,, \quad (5.11)$$

where

$$0 \leqslant \eta(E) \leqslant 1\,. \quad (5.12)$$

The quantity $\eta$ is the two-body partial-wave inelasticity parameter.

In order to follow a similar procedure on Eq. (5.9), we must first remove the disconnected parts of the amplitude, because of their explicit $\delta$-function. We are then allowed to set $\omega_i' = \omega_i$, and to consider the equation for the on-shell ($\omega_1+\omega_2+\omega_3=E$) connected part of $\mathrm{Im}T_{11}$. The problem reduces to finding an upper bound for the quantity

$$B\equiv \mathrm{Im}\langle M\omega_1\omega_2\omega_3|T_{11}{}^c|M\omega_1\omega_2\omega_3\rangle - \int d\omega_1'd\omega_2'd\omega_3'\delta(\omega_1'+\omega_2'+\omega_3'-E) \times\{\mathrm{Im}\langle M\omega_1\omega_2\omega_3|T_{11}{}^c|M\omega_1'\omega_2'\omega_3'\rangle\}^2. \quad (5.13)$$

Unfortunately, because of the residual integrations in the second term of Eq. (5.13), a functionally independent bound does not exist. The essential reason is that the measure over which the function is significantly nonzero can decrease while the maximum of the function increases, as we change the functional form.[22] Thus, a natural generalization of the inelasticity parameter seems unlikely. This is not altogether surprising in view of the inherently complicated nature[23] of three-body unitarity.

## VI. RESONANCE APPROXIMATION

The most widely used approximation in elastic three-body computations is the resonance approximation, largely because it yields one-dimensional equations. The purpose of this section is to show that the inelastic problem, under either approach, also becomes one-dimensional under this approximation. The physical origin of the resonance approximation is the assumption that each two-body scattering amplitude is dominated by a single partial wave in which the amplitude has a resonance. At the resonance pole, the residue is known to be separable in the initial and final variables; this form is chosen to hold everywhere. Specifically, if there is a resonance in the $l$th partial wave, we assume that the off-shell multichannel two-body amplitude has the form[24]

$$\langle \mathbf{q}|t_{\mu\nu}(\sigma)|\mathbf{q}'\rangle=(2l+1)P_l(\hat{q}\cdot\hat{q}')\frac{g_\mu(q)g_\nu(q')}{D(\sigma)}, \quad (6.1)$$

where $\mathbf{q}$ and $\mathbf{q}'$ are the initial and final relative momenta in the two-body center-of-mass system, $\hat{q}$ is the unit vector in the direction of $\mathbf{q}$, $\sigma$ is the extended energy, $g_\mu$ is a form factor, and $D(\sigma)$ is a function which vanishes at the resonance position and insures unitarity and analyticity. One such choice of $D(\sigma)$ in the nonrelativistic case is

$$D(\sigma)=1-\frac{1}{\pi}\int_0^\infty \frac{dE}{E-\sigma}\sum_{\alpha=1}^{N}\rho_\alpha(E)[g_\alpha(E)]^2, \quad (6.2)$$

where $\rho_\alpha(E)$ is the phase-space factor for channel $\alpha$.

[22] If $\mathrm{Im}\langle M\omega_1\omega_2\omega_3|T_{11}{}^c|xyz\rangle$ is, as a function of $x$, $y$, and $z$, a function of bounded variation $\epsilon$ over the surface of integration, then $B\leqslant\epsilon+1/4A$, where $A$ is the area of the surface.

[23] For an expression of three-body unitary in its full glory in $s$-matrix form, see Ref. 2.

[24] For a discussion of the resonance approximation see, for example, Refs. 2, 4, or 6.

If we substitute Eq. (6.1) into Eq. (5.7), we find that the angular integration can be done, yielding a sum of terms, each of which is a product of a function of the $\omega_i$ (and $\mu$) times a function of the $\omega_i'$ (and $\nu$).[25] In a condensed notation, we have (where $\omega$ stands for $\omega_1$, $\omega_2$,$\omega_3$)

$$\langle M\omega|t_{\mu\nu}{}^i|M'\omega'\rangle =\delta(\omega_i-\omega_i')\sum_\lambda f_\mu{}^i(\lambda,M,\omega)h_\nu{}^i(\lambda,M',\omega'). \quad (6.3)$$

The sum over $\lambda$ arises from both the integration and the sum in Eq. (5.7). With the more complicated approximation of any number of contributions of the type of Eq. (6.1) to the total amplitude, one still obtains an expression of the form of Eq. (6.3). For the sake of clarity, we shall choose arbitrary (but fixed) values of $J$, $\Lambda$, $\nu$, $M'$, and $\omega'$ and write

$$T_\mu{}^i(M,\omega)=\langle JM\Lambda\omega|T_{\mu\nu}{}^i|JM'\Lambda\omega'\rangle \quad (6.4)$$

and similarly for $t^i$. The multichannel partial-wave Faddeev equations then become

$$T_\mu{}^i(M,\omega)=t_\mu{}^i(M,\omega)+\sum_{M''\alpha}\int d\omega''\langle M\omega|t_{\mu\alpha}{}^i|M''\omega''\rangle \times G_\alpha(\omega'')[T_\alpha{}^j(M'',\omega'')+T_\alpha{}^k(M'',\omega'')]. \quad (6.5)$$

Substitution of Eq. (6.3) into Eq. (6.5) yields

$$T_\mu{}^i(M,\omega)=\sum_\lambda f_\mu{}^i(\lambda,M,\omega)\phi^i(\lambda,\omega_i), \quad (6.6)$$

where

$$\phi^i(\lambda,\omega_i)=\delta(\omega_i-\omega_i)h_\nu{}^i(\lambda,M',\omega') +\sum_{M''\alpha}\int d\omega''\delta(\omega_i'-\omega_i'')h_\alpha{}^i(\lambda,M'',\omega'') \times G_\alpha(\omega'')\sum_{\lambda''}[f_\alpha{}^j(\lambda'',M'',\omega'')\phi^j(\lambda'',\omega_j'') +f_\alpha{}^k(\lambda'',M'',\omega'')\phi^k(\lambda'',\omega_k'')]. \quad (6.7)$$

This is easily recognized as a coupled set of one-dimensional equations. This is, of course, to be expected, since the multichannel kernel has exactly the same connectedness structure as the kernel for the single-channel elastic case.

Let us now turn our attention to the results of the complex-potential approach. In order to use the resonance approximation in the simple form of Eq. (6.1), we shall use the results of Sec. IV, which express the complex-potential approach in the case of $N$ three-body channels. By combining Eqs. (6.3) and (4.14), we may write the partial-wave projection of $\alpha^i$ in the form

$$\langle M\omega|\alpha^i|M''\omega''\rangle=\sum_\lambda f_1{}^i(\lambda,M,\omega)a^i(\omega_i,\lambda,M'',\omega''). \quad (6.8)$$

[25] For a general treatment see J. L. Basdevant, Phys. Rev. 138, B892 (1965).

By taking the partial-wave projection of Eq. (4.9) for $u^i$, we obtain

$$\langle M\omega|u^i|M''\omega''\rangle=\sum_\lambda f_1{}^i(\lambda,M,\omega)[\delta(\omega_i-\omega_i'')h_1{}^i(\lambda,M'',\omega'')+I^i(\omega_i,\lambda,M'',\omega'')], \quad (6.9)$$

where

$$I^i(\omega_i,\lambda,M'',\omega'')=a^i(\omega_i,\lambda,M'',\omega'')+\sum_{M'''\lambda'''}\int d\omega''' a^i(\omega_i,\lambda,M''',\omega''')G(\omega''')\times f_1{}^i(\lambda''',M''',\omega''')\{\delta(\omega_i'''-\omega_i'')h_1{}^i(\lambda''',M''',\omega'')+I^i(\omega_i''',\lambda''',M'',\omega'')\}. \quad (6.10)$$

Again, note that the double-primed variables appear only as parameters in the inhomogeneous term of Eq. (6.10). Applying the notation of Eq. (6.4) to Eqs. (5.5) and (6.9), we have

$$T^i(M,\omega)=\sum_\lambda f_1{}^i(\lambda,M,\omega)\phi^i(\lambda,\omega_i), \quad (6.11)$$

where

$$\phi^i(\lambda,\omega_i)=\delta(\omega_i-\omega_i')h_1{}^i(\lambda,M',\omega')+I^i(\omega_i,\lambda,M',\omega')+\sum_{M''}\int d\omega''\{\delta(\omega_i-\omega_i'')h_1{}^i(\lambda,M'',\omega'')+I^i(\omega_i,\lambda,M'',\omega'')\}\times G(\omega'')\sum_{\lambda''}[f_1{}^j(\lambda'',M'',\omega'')\phi^j(\lambda'',\omega_j'')+f_1{}^k(\lambda'',M'',\omega'')\phi^k(\lambda'',\omega_k'')]. \quad (6.12)$$

Equations (6.12) are clearly a coupled set of one-dimensional equations, and we expect this feature to persist in the more general case involving non-three-body channels, since an equation of the form of Eq. (6.8) should hold whenever a resonance approximation can be defined. Equations (6.12) differ from the purely elastic equations in two ways: First, the two-body amplitudes are modified due to the presence of inelastic states; second, there is now an additional term in both the inhomogeneous term and the kernel. However, in actual calculations, the experimental resonances are always used[26] to determine the two-body amplitudes, so the inelasticity actually only appears in the additional term. We may remark again that probably one of the most significant (and annoying, from a computational point of view) features of this term is its complete connectedness, which manifests itself in that fact that the term $I^i$ must not contain a $\delta$ function in $\omega_i$.

## VII. FURTHER APPROXIMATIONS

In this section we shall examine two cases for which $I$ takes on a simple form. The first case is when the interactions are such that if particles $j$ and $k$ annihilate into something else with $i$ as a spectator, the only way to come back to the elastic channel is to have something annihilate into $j$ and $k$, with $i$ again as a spectator. An example of such a situation is provided by 3-pion scattering with $\pi K\bar{K}$ and $\pi N\bar{N}$ as the inelastic channels. Equation (4.14) for $\alpha^i$ becomes

$$\alpha^i=\sum_{\beta,\gamma=2}^{N} t_{1\beta}{}^iG_\beta O_{\beta\gamma}{}^iG_\gamma\tau_{\gamma1}{}^i, \quad (7.1)$$

where $O_{\beta\gamma}{}^i$ is the part of $(S_{\beta\gamma}{}^j+S_{\beta\gamma}{}^k)$ which does not end in $\tau^i$ [Eq. (4.5) ensures us that $S^j+S^k$ does have such a decomposition]. Now, the equation for $\tau^i$ analogous to Eq. (6.3) for $t^i$ is

$$\langle M\omega|\tau_{\mu\nu}{}^i|M'\omega'\rangle=\delta(\omega_i-\omega_i')\sum_\lambda \tilde{f}_\mu{}^i(\lambda,M,\omega)h_\nu{}^i(\lambda,M',\omega'), \quad (7.2)$$

where the function $h$ is the same in Eqs. (6.3) and (7.2). Thus, from Eqs. (7.1) and (6.8) we may write

$$a^i(\omega_i,\lambda,M'',\omega'')=\sum_{\lambda''}\theta^i(\omega_i,\lambda,\omega_i'',\lambda'')h_1{}^i(\lambda'',M'',\omega''). \quad (7.3)$$

Substitution of this form into Eq. (6.10) yields

$$I^i(\omega_i,\lambda,M'',\omega'')=\sum_{\lambda''}J^i(\omega_i,\lambda,\omega_i'',\lambda'')h_1{}^i(\lambda'',M'',\omega''), \quad (7.4)$$

where

$$J^i(\omega_i,\lambda,\omega_i'',\lambda'')=\theta^i(\omega_i,\lambda,\omega_i'',\lambda'')\sum_{M'''}\int d\omega''' a^i(\omega_i,\lambda,M''',\omega''')G(\omega''')\times f_1{}^i(\lambda'',M''',\omega''')\delta(\omega_i'''-\omega_i'')+\sum_{M'''\lambda'''}\int d\omega'''\times a^i(\omega_i,\lambda,M''',\omega''')G(\omega''')f^i(\lambda''',M''',\omega''')\times J^i(\omega_i''',\lambda''',\omega_i'',\lambda''). \quad (7.5)$$

We may summarize by saying that for this case if we write the resonance approximation as

$$\langle M\omega|t^i|M'\omega'\rangle=\sum_{\lambda,\lambda'} f^i(\lambda,M,\omega)\times\{\delta_{\lambda\lambda'}\delta(\omega_i-\omega_i')\}h^i(\lambda',M',\omega') \quad (7.6)$$

then

$$\langle M\omega|u^i|M'\omega'\rangle=\sum_{\lambda,\lambda'} f^i(\lambda,M,\omega)\{\delta_{\lambda\lambda'}\delta(\omega_i-\omega_i')+J^i(\omega_i,\lambda,\omega_i',\lambda')\}h^i(\lambda',M',\omega'). \quad (7.7)$$

The form in Eq. (7.7) provides the simplest non-trivial fashion in which inelasticity can appear. At the same time, it includes an entire class of possible inelastic diagrams (see Fig. 4). We emphasize that, for

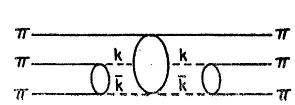


FIG. 4. Inelastic diagrams included by Eq. (7.7) for the case of the channels $\pi\pi\pi$ and $\pi K\bar{K}$.

[26] See, for example, Ref. 6.

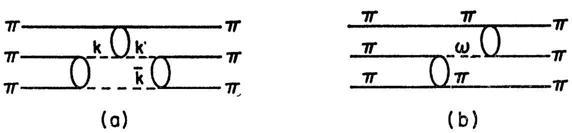


FIG. 5. (a) Lowest order inelastic contributions to $u$ for allowed two-body reactions of the type $\pi\pi \rightarrow K\bar{K}$. (b) Lowest order inelastic contributions to $u$ for allowed two-body reactions of the type $\pi\pi \rightarrow \pi\omega$.

certain models, such as the ones considered here, this class may in fact be the complete set. If we consider $J^i$ as a function of the over-all energy, we can see from the definition of $\theta^i$ that $J^i$ contains all the inelastic branch cuts, as well as the elastic cut.

In the general case, Eq. (7.2) leads to

$$a^i(\omega_i,\lambda,M'',\omega'') =\sum_{\lambda''} \sum_{l=1}^{3} \theta^{il}(\omega_i,\lambda,\omega_l'',\lambda'')h_1^l(\lambda'',M'',\omega''). \quad (7.8)$$

This gives

$$I^i(\omega_i,\lambda,M'',\omega'') =\sum_{\lambda''} \sum_{l=1}^{3} J^{il}(\omega_i,\lambda,\omega_l'',\lambda'')h_1^l(\lambda'',M'',\omega'') \quad (7.9)$$

and a correspondingly more complicated version of Eq. (7.7).

Another case in which $I$ takes on a simple form is the weak-coupling approximation, in which we approximate the contribution of the inelastic states by the lowest nonvanishing terms. We may expand $u^i$ to lowest order using Eqs. (4.9), (4.14), and (4.5) as

$$u^i=t_{11}{}^i+\sum_{\beta=2}^{N} t_{1\beta}{}^iG_\beta(\tau_{\beta 1}{}^j+\tau_{\beta 1}{}^k)(1+G_1t_{11}{}^i). \quad (7.10)$$

This first-order correction in fact vanishes for the model considered above, since the particle $i$ must be a spectator in the final interaction. Thus, the lowest nonvanishing correction is the second-order term in the expansion of $u^i$ [see Fig. 5(a)]. However, there are models for which the first-order correction does not vanish, for example $\pi\pi\pi$ scattering with the inelastic channels $\pi\pi\omega$, and $\pi\omega\omega$ [see Fig. 5(b)]. By using the Eqs. (6.3) and (7.2) for $t$ and $\tau$ in Eq. (7.10), one may obtain a representation for $I^i$ of the form of Eq. (7.9) as an explicit integral over known functions.

One of the physically most interesting applications of lowest-order equations such as Eq. (7.10) is the computation of the shift due to inelasticity of the mass of three-body resonances and bound states. Our equations are of the form [e.g., see Eq. (6.12)]

$$\phi(s)=f(s)+K(s)\phi(s). \quad (7.11)$$

There is a pole[2] at $s=M$ if there is a vector $v$ such that

$$K(M)v=v. \quad (7.12)$$

Because of time-reversal invariance, for our kernels this implies that there is also a vector $u$ such that

$$uK(M)=u. \quad (7.13)$$

Then

$$\lim_{s\to M}[(s-M)\phi(s)]=-v(uf)/uK'(M)v. \quad (7.14)$$

If we add a small term $\Delta(s)$ to the kernel, then the new pole position $M_I$ is given to lowest order as

$$M_I=M-u\Delta(M)v/uK'(M)v. \quad (7.15)$$

In our case, $\Delta(s)$ is just the Green's function times the inelastic contribution to Eq. (7.10); the other quantities in Eq. (7.15) can be computed from the elastic problem. Thus, the results of an elastic calculation allow one to estimate the effect of a small inelastic contribution relatively easily in the three-body problem, as in the two-body problem.

## VIII. DISCUSSION

In the preceding sections we have shown how one may incorporate the influence of inelastic scattering on elastic three-body amplitudes. Although we have restricted ourselves to two-body interactions in the elastic channel, the extension to three-body interactions can be accomplished without difficulty. In the same spirit as in the two-body problem, we have considered two distinctly different approaches, namely a multichannel approach and a formulation derived from a complex potential approach. Both these approaches were shown to lead to Faddeev-type equations, and were shown to be equivalent in their common domain of validity. The formulation based on the complex potential approach led to equations identical with the elastic equations, with the exception that the two-body amplitude was now replaced by the coupled two-body amplitude plus a completely connected term due entirely to inelastic effects, and which thus contained all the complicated cut structure associated with these processes. Both formulations reduced to one-dimensional equations, in a fashion completely analogous to the elastic case, under the resonance approximation. Further simplification of the completely connected term occurred in some special cases.

The relative merits and demerits of the two approaches in the context of practical calculation are essentially the same as for the corresponding approaches in the two-body problem. The multichannel approach may be more useful in a physical situation which can be described realistically by a few three-body channels. For a larger number of channels or for a more general situation the single-channel modified Faddeev equations of the complex approach may be more appropriate. Since the number of channels does increase quite rapidly with the number of production processes, the latter approach is probably necessary (due to computer limitations) in all except the simplest situations.

One interesting application of the inelastic equations, particularly in the multichannel form, is in the self-consistent dynamical calculations (bootstrap) of three-body resonances. For example, the $\omega$ meson should appear as a resonance in the coupled three-body amplitudes for the states $\pi\pi\pi$ and $\pi\pi\omega$, where we take as two-body input the coupled amplitudes for $\pi\pi$ and $\pi\omega$. Since the $\omega$ meson is involved both as an input and as an output, a self-consistent calculation is therefore possible. Similarly, one can look for the $N^*$ as a resonance in the coupled $\pi\pi N$ and $\pi\pi N^*$ channels and do a bootstrap calculation. This calculation should be of interest, since the three-body states are known[27] to contribute significantly to the nucleon states.

In the analysis of the three-body scattering at high energies, the most natural approach is through the function $I$ of Sec. VI and VII. It plays a role similar to that played by the inelasticity parameter in the two-body problem, but unfortunately the requirements on it are not nearly so simple. What is actually needed are realistic models for this function, embodying both the correct connectedness and analyticity structures, and depending on only a few parameters. Once this is achieved, one should have a powerful and fruitful approach for the analysis strong-interaction three-body problems.

Of course, in order to treat a realistic physical situation, one must take into account the necessary complications arising from relativistic considerations, spin, and statistics. However, these are independent considerations, and the inelastic equations are no more difficult to generalize than the purely elastic equations. Thus, we believe that the essential features of inelastic scattering, as outlined in the preceding sections, would be unaltered even in a fully realistic treatment.

## APPENDIX

In this Appendix, we should like to illustrate in slightly greater detail some of the points mentioned at the beginning of Sec. II. The requirement that no channel contain two identical labels is essential for a meaningful formulation of the problem, even in the single-channel case (symmetrization on labels of physically identical particles is a separate question). As an illustration, we shall consider the three-pion problem with $K\bar{K}$ as the two-body inelastic state. We label the three pions as 1, 2, and 3. We have $1+2\rightarrow 4+5$, where 4 is a $K$ and 5 is a $\bar{K}$. We are not allowed to write $2+3\rightarrow 4+5$, since then we would have three-body channels containing identical labels, namely 112 and 233. Thus, we write $2+3\rightarrow 6+7$ and similarly $1+3\rightarrow 8+9$. This gives us the channels 123, 345, 167, and 289. We remark that the physical amplitude for $\pi\pi\pi\rightarrow \pi K\bar{K}$ has contributions from all of the last three channels, which collapse after symmetrization on the pions.

[27] See, for example, P. Nath and K. V. Vasavada, Phys. Rev. **152**, 1259 (1966). See also Ref. 2.

In order to clarify our notation and procedures, let us consider a somewhat more complicated problem. In addition to the potentials for all elastic scatterings, denoted by $E$ subscripted with the appropriate particle labels, we allow the following production potentials:

$$1+3\leftrightarrow 3+5 \tag{A1a}$$
$$2+3\leftrightarrow 3+4 \tag{A1b}$$
$$1+2\leftrightarrow 1+4 \tag{A1c}$$
$$1+4\leftrightarrow 4+5. \tag{A1d}$$

This leads to the channels containing particles 123, 134, 235, and 345, which we label as channels 1, 2, 3, and 4, respectively. Our matrices $U^i$ now have the forms

$$U^1=\begin{bmatrix} E_{23} & b & 0 & 0\\ b & E_{34} & 0 & 0\\ 0&0&0&0\\0&0&0&0\end{bmatrix},\quad U^2=\begin{bmatrix} E_{13} & 0 & a & 0\\ 0&0&0&0\\ a & 0 & E_{35} & 0\\0&0&0&0\end{bmatrix},$$
$$U^3=\begin{bmatrix} E_{12} & c & 0 & 0\\ c & E_{14} & 0 & d\\ 0&0&E_{25}&0\\0&d&0&E_{45}\end{bmatrix},\quad U^4=\begin{bmatrix} 0&0&0&0\\ 0&E_{13}&0&a\\ 0&0&0&0\\0&a&0&E_{35}\end{bmatrix},$$
$$U^5=\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&E_{23}&b\\0&0&b&E_{34}\end{bmatrix}.$$

If we now relabel the channels according to our positional relabeling rule, they become, respectively, 123, 143, 523, and 543. The potentials $V^i$ are now obtained from Eq. (2.8) as

$$V^1=U^1+U^5,$$
$$V^2=U^2+U^4,$$
$$V^3=U^3.$$

The amplitudes defined by Eq. (2.9) and these $V^i$ have matrix elements which are clearly the multichannel two-body scattering amplitudes of all of our allowed processes.

An apparent difficulty may arise in this straightforward position labeling procedure when the reactions are such that a particle is forced to have different positions or two particles compete for the same position in some channel. For example, if we take the reactions

$$1+2\leftrightarrow 1+4,$$
$$1+2\leftrightarrow 2+4,$$
$$1+4\leftrightarrow 2+4,$$

the channel are 123, 143, and 423. It is not possible to relabel the channels so that every particle always occupies the same position, and thus it is not clear that one can even define the $V^i$. In fact, it is not possible to define them simply by using Eq. (2.8). However, they do exist, and the reason is that particle 4 actually plays two distinct roles in the reactions, and thus $U^4$

may be divided into two non-interacting parts. Specifically, we shall replace particle 4 by particles 5 and 6 and rewrite the reactions as

$$1+1 \leftrightarrow 1+5\,,$$
$$1+2 \leftrightarrow 2+6\,,$$
$$1+5 \leftrightarrow 2+6\,.$$

The channels are now 123, 153, and 623; and $V^i$ are given by Eq. (2.8). The physical content is unchanged. If one tries to add reactions which mix the roles played by particle 4, and thus prevent the separation into 5 and 6, there necessarily result three-body channels containing identical labels. For example, if we allow $1+3 \rightarrow 4+3$ we have the additional channel 443. In the general case, it seems that if the reactions are such as to give only channels not containing identical labels, then any confusion is only apparent and can be resolved in the manner illustrated above.

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# Nonlinear Density Effect in the Transmission of $K_2$ Mesons through Matter*

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We note that the attenuation of a monochromatic beam of $K_2$ mesons in an absorber is not simply describable in terms of a single "mean free path," even when the extent of the absorber is large in comparison with the $K_1$ mean decay length. The normal attenuation factor exp $(-N\sigma L)$ is to be corrected by a factor whose logarithm is nonlinear in the density $N$. The magnitude of this extra effect is discussed with the aid of a simple-minded model. The relation of the effect to the $K_1-K_2$ mass difference is briefly noted.

## INTRODUCTION

THE purpose of this paper is to point out that the transmission of a pure, monochromatic beam of $K_2$ mesons through a sample of material which is much longer than the $K_1$ decay length cannot be completely described in terms of a single "mean free path," but that the transmitted intensity contains an extra factor whose logarithm is nonlinear in the density of the material. The effect in question is *not* due to the interference of two exponential terms. The effect is latent in the usual formalism for the propagation of neutral kaons through matter, but has not (to our knowledge) been pointed out previously because it is obscured by certain standard approximations.

We also discuss very briefly the possibility of using this density effect to determine the sign of the $K_2-K_1$ mass difference.

## ASSUMPTIONS

The appropriate general equations for the propagation of neutral kaons through an absorber have been written down and solved exactly by Good.[1] The equations, and their solutions for the case of a pure $K_2$ incident beam, respectively, may be written

$$\frac{d}{dx}\begin{pmatrix}\alpha_1(x)\\ \alpha_2(x)\end{pmatrix}=i[1/\lambda\!\!\!^{-}+2\pi\lambda\!\!\!^{-}Nf_{22}(0)]\begin{pmatrix}\alpha_1(x)\\ \alpha_2(x)\end{pmatrix}+[2\pi\lambda\!\!\!^{-}Nf_{21}(0)]\begin{pmatrix}\alpha_2(x)\\ -\alpha_1(x)\end{pmatrix}-\frac{1}{\beta\gamma c}\begin{pmatrix}[i\omega_1+1/2\tau_1]\alpha_1(x)\\ [i\omega_2+1/2\tau_2]\alpha_2(x)\end{pmatrix};\quad(1)$$

$$\begin{pmatrix}\alpha_1(L)\\ \alpha_2(L)\end{pmatrix}=\frac{\alpha_2(0)}{1-R^2}\left\{\exp\left[i(2\pi N\lambda\!\!\!^{-2}f_{22}(0)+1)\frac{L}{\lambda\!\!\!^{-}}\right]\right\}\times\left\{\begin{pmatrix}R\\ 1\end{pmatrix}\exp\left[-\left(\frac{i\omega_2}{\beta\gamma c}+\frac{1}{2\Lambda_2}\right)L+\tfrac{1}{2}(\mu-i\delta)[(1-\epsilon^2)^{1/2}-1]\frac{L}{\Lambda_1}\right]-\begin{pmatrix}R\\ R^2\end{pmatrix}\exp\left[-\left(\frac{i\omega_1}{\beta\gamma c}+\frac{1}{2\Lambda_1}\right)L-\tfrac{1}{2}(\mu-i\delta)[(1-\epsilon^2)^{1/2}-1]\frac{L}{\Lambda_1}\right]\right\}.\quad(2)$$

The quantities in (1) and (2) are as follows: $\alpha_{1,2}(x)$ is the probability amplitude for $K_{1,2}$ mesons at a distance $x$ into the absorber. $\omega_{1,2}$ indicate the de Broglie frequencies

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[1] M. L. Good, Phys. Rev. **106**, 591 (1957). See also K. M. Case, *ibid.* **103**, 1449 (1956).