where $\bar{L}$ and $\bar{L}'$ are the minimum orbital angular momentum summed in the generalized MacDowell relations of Eq. (20), will have no impermissible kinematic singularities or zeros in the variable W. If any of the conditions of Eqs. (22) are satisfied, then $\bar{L}=L$ and $\bar{L}'=L'$, so that the variable s may be used and Eq. (26) simplifies to

$$\bar{A}_{L'S',LS}{}^J(s)=s^{\alpha_{S'S}}\{[s-(m+\mu)^2]\times[s-(m-\mu)^2]\}^{-(L+L')/2}A_{L'S',LS}{}^J(s). \quad (27)$$

If condition (22d) is satisfied, then the modified partial-wave amplitude of Eq. (27) must be divided by $s^{1/2}$ to remove that over-all factor. In Eqs. (26) and (27), the powers $\alpha_{S'S}$ are the asymptotic powers of the u- and t-channel backward amplitudes which are given in Regge-pole theory by $\alpha_{S'S}=\alpha_{S'S}(s=0)$ for the leading direct-channel Regge trajectory for spin $S\rightarrow S'$ scattering and $0>\alpha_{S'S}\geq-1$ if there is no direct-channel Regge pole.

Of course, in a practical calculation, one may choose not to work with the $\bar{A}_{L'S',LS}{}^J(W)$ amplitudes as given by Eq. (26) or (27). A variety of approximate treatments of the kinematic singularities is possible, but the forms given here seem to be the point at which these approximations should start.

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Extinct Ghosts in Potential Theory*

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To resolve the difficulties that arise if the P and/or P' Regge trajectories pass through $J=0$ at a negative value of the center-of-mass energy squared, Chew has conjectured that the determinant of the physical D matrix does indeed vanish, but that the N matrix is such as to lead to vanishing residues at the pole. We investigate whether this phenomenon of a simultaneous zero of the N and D functions can occur in potential theory. Standard arguments exclude this possibility for sufficiently well-behaved potentials. However, it is easy to explicitly construct amplitudes which do involve a coincident zero. Using the Gel'fand-Levitan-Marchenko equations, we derive a representation for the potential in terms of the Fredholm determinant of the integral operator that appears in the N/D equations. We show that if the s-wave amplitude has coincident zeros, the corresponding potential behaves like $1/r^2$ near the origin; conversely, such potentials give rise, in general, to coincident zeros. However, these zeros are unrelated to any Regge trajectory, so that (except perhaps for potentials which diverge more strongly than $1/r^2$ at the origin) the phenomenon hypothesized by Chew cannot occur in potential theory.

I. INTRODUCTION

THE assumption that the Pomeranchuk and P' Regge trajectories are approximately linear suggests that they cross $J=0$ at negative values of s, the square of the center-of-mass energy. However, a pole in a $J=0$ amplitude at a negative value of s would correspond to a particle of imaginary mass, as well as to a singularity in the physical region of the crossed reaction. Thus the residues of the P and P' poles in all physical processes would have to vanish at $J=0$. Gell-Mann has hypothesized[1] that this occurs because of a dynamical dominance of channels with spin for which $J=0$ is nonphysical. It is difficult to understand, however, why the P and P' should choose these "nonsense" channels when the force in the $\pi\pi$ s wave, for example, appears to be strongly attractive. Chew has suggested[2] that they choose "sense," meaning that the determinant of the physical $J=0$ D matrix does vanish, but that the physical N matrix is such as to lead to vanishing residues at the pole. We refer to this phenomenon as a *Regge ghost*.

Where there is only one physical channel, a Regge ghost is simply a simultaneous zero of N and D, say at $s=s_G$. This implies that D also vanishes on the unphysical sheet at s_G, so that there is a virtual Regge trajectory which at $J=0$ passes directly beneath the trajectory on the physical sheet. When there are N coupled two-body channels, one can show by writing the S-matrix element in terms of the determinant of D that the latter vanishes on all 2^N sheets at $s=s_G$, so that there are 2^N trajectories coinciding at $J=0$. If for some value of J one of these virtual trajectories comes close to the physical region of the physical sheet, it could have experimentally observable consequences.

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[1] M. Gell-Mann, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 539.

[2] G. Chew, Phys. Rev. Letters **16**, 60 (1966).

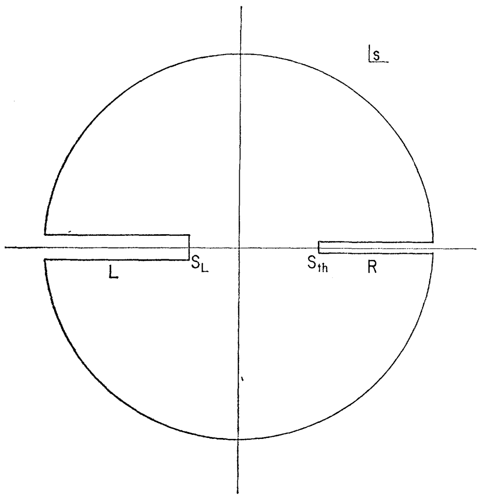

FIG. 1. The contour C, composed of L, R, and a circle at infinity.

[In what follows we shall refer to a simultaneous zero of N and D for some specific value of the angular momentum as an *extinct bound state*,[3] or *extinct ghost* (EG), without necessarily implying that any Regge trajectory is involved. The term *Regge ghost* will be reserved for an extinct ghost which does lie along a Regge trajectory.]

Atkinson and Halpern[3] have shown that a coincident zero of N and D implies that the *homogeneous* integral equations for N and D can be solved. This can be understood by noting that when $N(s_G)=D(s_G)=0$, one may divide out the zero; the quantities

$$n(s)=\frac{N(s)}{s-s_G}, \tag{1.1a}$$

$$d(s)=\frac{D(s)}{s-s_G}, \tag{1.1b}$$

will obviously satisfy a homogeneous system of N/D equations, since $d\to 0$ at ∞. It follows that the Fredholm determinants of the Fredholm integral equations for N and for D vanish. Furthermore,[3] since the inhomogeneous equations satisfied by N and D are also satisfied by

$$N'(s)=N(s)+\lambda n(s)=[s-(s_G-\lambda)]n(s), \tag{1.2a}$$

$$D'(s)=D(s)+\lambda d(s)=[s-(s_G-\lambda)]d(s), \tag{1.2b}$$

where λ is arbitrary, the position of the EG is not determined by the N/D equations for one channel and one partial wave.

In this paper we investigate whether extinct bound states can occur in ordinary potential theory, and, if so, what dynamical conditions will produce them. For sufficiently well-behaved potentials, standard arguments exclude the possibility of extinct ghosts.[4,5] If

$$\int_0^\infty r|V(r)|dr<\infty, \tag{1.3a}$$

and

$$\int_a^\infty e^{\mu r}|V(r)|dr<\infty \tag{1.3b}$$

for some positive μ and a, then the usual s-wave Jost functions $f(\pm k)$ can be defined and are analytic in the strip $|\mathrm{Im}k|<\mu/2$, and it can be shown that

$$D(k)=f(-k), \tag{1.4a}$$

$$N(k^2)=\frac{f(k)-f(-k)}{2ik}, \tag{1.4b}$$

with D defined so that $D(k^2)\to 1$ as $k^2\to\infty$. Here k is the center-of-mass momentum; from now on we shall use s to denote k^2. [If $V(r)$ has the representation $V(r)=\int_{\mu_0}^\infty d\mu\,\rho(\mu)e^{-\mu r}/r$, (1.3) implies that $\mu_0>0$ and that $\int_{\mu_0}^\infty d\mu\,\rho(\mu)/\mu<\infty$.]

In terms of the Jost wave functions $\psi(\pm k,r)$, which are solutions of the s-wave Schrödinger equation approaching $e^{\mp ikr}$ asymptotically, the Jost functions $f(\pm k)$ are given by

$$f(\pm k)=\psi(\pm k,0). \tag{1.5}$$

If there were an EG, we would have

$$f(k)=f(-k)=0 \tag{1.6}$$

for some point k on the imaginary axis. Then the Wronskian $W[\psi(k,r),\psi(-k,r)]$ would vanish at $r=0$, which is impossible since W is r-independent and equal to $2ik$ at $r\to\infty$. [If $f(k)$ vanishes at $k=0$, the zero is simple,[4] so again no EG is possible.]

On the other hand, it is easy to construct s-wave S matrices which exhibit extinct ghosts (Sec. II). These S matrices are found to correspond to potentials which go like $1/r^2$ near the origin (Sec. III). Specific examples are discussed (Sec. IV). Conversely, potentials with $1/r^2$ behavior for small r can be shown to lead in general to simultaneous zeros in N and D for some values of the angular momentum l (Sec. V). However (Sec. VI), the extinct ghosts which we have found do not lie on Regge trajectories, and hence do not correspond to the phenomenon conjectured by Chew.

II. CONSTRUCTION OF AMPLITUDES CONTAINING EXTINCT GHOSTS

In order to construct examples of partial-wave amplitudes containing extinct ghosts, we shall utilize Levinson's theorem, which relates the variation of the

[3] D. Atkinson and M. B. Halpern, Phys. Rev. **149**, 1133 (1966).

[4] M. Goldberger and K. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 279.

[5] That no EG will be produced by such potentials has been argued by Professor R. Sawyer, whom we wish to thank for several private communications.

phase shift from threshold to infinite energy to the number of zeros of the D function:

$$(1/\pi)[\delta_l(\infty)-\delta_l(\text{th})]=-N_B-N_G$$
$$\text{(no Castillejo-Dalitz-Dyson poles).} \quad (2.1)$$

To prove this theorem we merely require that the partial-wave S matrix approach unity as $s\rightarrow\infty$ (faster than $1/\ln s$). A more useful relation is obtained by evaluating the contour integral

$$\frac{1}{2\pi i}\oint\left(\frac{d\ln S}{ds}\right)ds=N_Z-N_P, \quad (2.2)$$

where the contour C is illustrated in Fig. 1 and $N_Z(N_P)$ is the number of zeros (poles) of S inside the contour C. The contribution from the circle at infinity vanishes, since $S\rightarrow 1$, $|s|\rightarrow\infty$, and the integral across the right-hand cut gives

$$\frac{1}{2\pi i}\int_R\frac{d\ln S}{ds}ds=\frac{2}{\pi}[\delta(\infty)-\delta(\text{th})]=-2(N_B+N_G). \quad (2.3)$$

On the physical sheet, a bound state appears as a pole of S, a virtual state as a zero, and a resonance as a pair of zeros at complex conjugate positions. Thus

$$N_P-N_Z=N_B-N_V-2N_R. \quad (2.4)$$

Therefore, using (2.3) and (2.4) in (2.2) we derive

$$\frac{1}{2\pi i}\oint_L\frac{d\ln S}{ds}ds-N_B-N_V-2N_R=2N_G. \quad (2.5)$$

In particular, if the left-hand cut of S consists solely of N_L poles, then

$$\frac{1}{2\pi i}\oint_L\frac{d\ln S}{ds}ds=N_L,$$

and

$$N_L-N_B-N_V-2N_R=2N_G. \quad (2.6)$$

In this case we can explicitly exhibit the S matrix in terms of the positions of the poles. For example, the s wave nonrelativistic S matrix whose only singularities in the $k=\sqrt{s}$ plane are poles is just

$$S(k)=\prod_{i=1}^{N_L}\left(\frac{k+ip_i}{k-ip_i}\right)\prod_{j=1}^{N_B}\left(\frac{k+ib_j}{k-ib_j}\right)\times\prod_{k=1}^{N_V}\left(\frac{k-iv_k}{k+iv_k}\right)\prod_{m=1}^{N_R}\frac{(k-i\Gamma_m)^2-\alpha_m{}^2}{(k+i\Gamma_m)^2-\alpha_m{}^2}, \quad (2.7)$$

where p_i, b_j, v_k, Γ_m, α_m are positive real numbers. Note that the left-hand side of (2.6) must be even or we cannot satisfy the condition $S\rightarrow 1$ both as $k\rightarrow 0$ and as $k\rightarrow\infty$. Equation (2.6) is satisfied even if there are an infinite number of left-hand cut poles, as in the case of the exponential potential or the Hulthèn potential in the s wave. These potentials give rise to an infinite number of virtual states. For example the s-wave D function for the Hulthèn potential,

$$V(r)=-\frac{\lambda}{2ma^2}\left(\frac{1}{e^{r/a}-1}\right),$$

is

$$D(k)=\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n(n-2ika)}\right),$$

which vanishes on the unphysical sheet at $k=-(i/2a)\times(n-\lambda/n)$, $n=1,\cdots,\infty$, $n>\sqrt{\lambda}$.

If the right-hand side of (2.6) is nonzero we have extinct ghosts. The simplest example is that of 2 left-hand poles and nothing else, so that

$$S(k)=\frac{k+ip_1}{k-ip_1}\frac{k+ip_2}{k-ip_2}, \quad (2.8)$$

and there is one extinct ghost ($\delta(\infty)-\delta(\text{th.})=-\pi$). (In the S matrix it is, of course, impossible to distinguish a bound-state pole from a "force" pole; however, the corresponding potential does depend on how the poles are interpreted.)

III. POTENTIALS CORRESPONDING TO S MATRIX WITH A GHOST

(a) Gel'fand-Levitan-Marchenko Equations

In the previous section we have constructed examples of amplitudes containing extinct ghosts. However, as was pointed out in the Introduction, such extinct ghosts cannot exist in potential theory for sufficiently well-behaved potentials. In order to determine what type of potential *can* produce extinct ghosts, we shall use the Gel'fand-Levitan equations, which enable one to construct a local potential that reproduces a given partial-wave amplitude.

Given an s-wave, nonrelativistic S matrix that satisfies

(a) $|S(k)|=S(0)=S(\infty)=1$,
(b) $S(-k)^{-1}=S(k)=S^*(k)^{-1}$,
(c) $S(k)-1$ has an absolutely integrable Fourier transform, one can always construct a potential that reproduces this S matrix in the s wave by using the procedure given by Marchenko[6,7]:

$$V(r)=-2\frac{d}{dr}A(r,r), \quad (3.1)$$

where $A(x,y)$ is the solution of the equation[8]

$$A(x,y)+F(x+y)+\int_x^\infty dt\,A(x,t)F(t+y)=0, \quad (3.2)$$

[6] V. Marchenko, Dokl. Akad. Nauk. SSSR **104**, 433 (1955).

[7] L. Faddeyev, Uspekhi Matem. Nauk. **14**, 57 (1959) [English transl.: B. Seckler, J. Math. Phys. **4**, 72 (1963)].

[8] The scale is determined by choosing the reduced mass $\mu=\frac{1}{2}$, so that Schrödinger's equation is

$$\frac{d^2}{dx^2}\psi(x)=[V(x)-k^2]\psi(x).$$

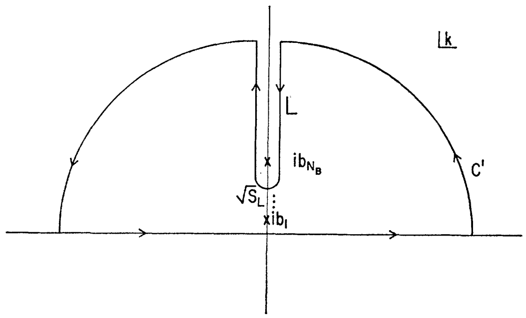

FIG. 2. The contour C'.

and $F(r)$ is given by

$$F(r)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\, e^{ikr}[1-S(k)]+i\sum_{m=1}^{N_B}\operatorname{Res}S(k)\bigg|_{k=ib_m}e^{-b_m r}. \quad (3.3)$$

Here ib_m, $m=1,\cdots,N_B$, are the positions of the bound-state poles in the k plane.

The potential which produces a given s-wave amplitude is unique only if there are no bound states. If there are N_B bound states, there will exist an N_B parameter family of "phase-equivalent" potentials, all of which reproduce the given amplitude. However, they will all have the same behavior at the origin.[7] Also, the potential determined by (3.1)–(3.3) is still unique.

$F(r)$ can be expressed in terms of the left-hand cut of the amplitude

$$A(k=\sqrt{s})=\frac{S(k)-1}{2ik}.$$

Considering

$$\frac{1}{2\pi}\oint_{C'}[1-S(k)]e^{ikr}dk$$

(C' is the contour pictured in Fig. 2) for $r>0$, and noting that the last term in (3.3) cancels the bound-state pole contributions, we derive

$$F(r)=-\int_L f(s)e^{-r\sqrt{(-s)}}ds. \quad (3.4)$$

Here

$$f(s)=\frac{\operatorname{Im}A(s)}{\pi},\quad -\infty\leqq s\leqq s_L<0$$

(excluding contributions from any bound-state poles located in this region). If we now represent $A(x,y)$ in the form

$$A(x,y)=\int_L a(s,x)e^{-y\sqrt{(-s)}}ds, \quad (3.5)$$

we can derive from (3.2) a Fredholm integral equation for $a(x,s)$:

$$a(s,x)-\int_L ds'\frac{f(s)e^{-x[\sqrt{(-s)}+\sqrt{(-s')}]}}{\sqrt{(-s)}+\sqrt{(-s')}}a(s',x)=f(s)e^{-x\sqrt{(-s)}}. \quad (3.6)$$

Or, if we define the integral operator $\mathcal{K}$:

$$\mathcal{K}(s,s';x)=I-K(s,s';x)=\delta(s-s')-\frac{f(s)e^{-x[\sqrt{(-s)}+\sqrt{(-s')}]}}{\sqrt{(-s)}+\sqrt{(-s')}}, \quad (3.7)$$

we can write (3.6) as

$$\mathcal{K}(s,s';x)\otimes a(s',x)=f(s)e^{-x\sqrt{(-s)}}. \quad (3.8)$$

We note that the kernel K is L^2 for all $x\geqq 0$, since $s^{1/2}f(s)\to 0$, $s\to-\infty$ and $s_L<0$.

(b) Relation to the N/D Equations

If we write, as usual, $A=ND^{-1}$, such that D is real analytic with zeros at the bound-state poles of the amplitude and N has only the left-hand cut, then N and D satisfy the coupled singular integral equations

$$N(s)=\int_L\frac{f(s')D(s')ds'}{s'-s}, \quad (3.9a)$$

$$D(s)=1-\frac{1}{\pi}\int_0^\infty\frac{(\sqrt{s'})N(s')ds'}{s'-s}. \quad (3.9b)$$

We have normalized D to approach unity as $s\to\infty$. (Such a D function always exists under the assumptions we have made about the S matrix.)

If we substitute (3.9a) into (3.9b) we derive a Fredholm equation for $D(s)$:

$$D(s)-\int_L ds'\frac{f(s')}{\sqrt{(-s)}+\sqrt{(-s')}}D(s')=1. \quad (3.10)$$

In terms of the operator $\mathcal{K}(s,s';x)$ defined by (3.7) this can be rewritten as

$$\mathcal{K}^\dagger(s,s';x=0)\otimes D(s')=1, \quad (3.11)$$

where

$$\mathcal{K}^\dagger(s,s';x)=\mathcal{K}(s',s;x).$$

Also interesting is the fact that $n(s)\equiv\operatorname{Im}N(s)/\pi=f(s)D(s)$ satisfies the equation

$$\mathcal{K}(s,s';x=0)\otimes n(s')=f(s). \quad (3.12)$$

Therefore if the operator $\mathcal{K}(x)$ is not singular at $x=0$ it is clear that $a(s,\,x=0)=n(s)$. If the potential

$$V(x)=-2\frac{d}{dx}\int_L ds\, a(s,x)e^{-x\sqrt{(-s)}}$$

is integrable, and $s_L<0$ (no infinite range forces), we can derive an amusing relation between $V(x)$ and $N(s)$:

$$\frac{1}{2}\int_0^\infty V(x)dx=\frac{1}{\pi}\int_L \mathrm{Im}N(s)ds. \qquad (3.13)$$

(c) Potential near the Origin

If there exist extinct ghosts then the Fredholm determinant of $\mathcal{K}$ is equal to zero when $x=0$, and therefore $a(s,x)$ and correspondingly $V(x)$ will develop singularities at $x=0$. To investigate the nature of these singularities we will derive an expression for the potential in terms of the Fredholm determinant of $\mathcal{K}$.

The resolvent of the operator $\mathcal{K}$ is defined by

$$\mathcal{K}^{-1}(s,s';x)=\delta(s-s')+\frac{R(s,s';x)}{\Delta(x)},$$

where $\Delta(x)=\det\mathcal{K}(s,s';\,x)$. Therefore the solution of (3.8) is

$$a(s,x)=f(s)e^{-x\sqrt{(-s)}}+\frac{1}{\Delta(x)}\times\int_L R(s,s';x)f(s')e^{-x\sqrt{(-s')}}ds'.$$

Let us now evaluate

$$\begin{aligned}\Delta(x)A(x,x)&=\Delta(x)\int_L ds\, a(s,x)e^{-x\sqrt{(-s)}}\\&=\int_L ds\,\Delta(x)f(s)e^{-2x\sqrt{(-s)}}+\int_L ds\int_L ds'\\&\quad\times R(s,s';x)f(s')e^{-x[\sqrt{(-s)}+\sqrt{(-s')}]}. \qquad (3.14)\end{aligned}$$

We shall prove that

$$\Delta(x)\,A(x,x)=\frac{d\Delta(x)}{dx}. \qquad (3.15)$$

$\Delta(x)$ is given by the Fredholm series

$$\Delta(x)=1+\sum_{n=1}^{\infty}\Delta_n(x);$$

$$\begin{aligned}\Delta_n(x)&=\frac{(-)^n}{n!}\int_L ds_1\cdots ds_n\begin{vmatrix}K(s_1s_1)\cdots K(s_1s_n)\\ \vdots \qquad\qquad \vdots\\ K(s_ns_1)\cdots K(s_ns_n)\end{vmatrix}\\&\equiv\frac{(-)^n}{n!}\int ds_1\cdots ds_nK\begin{pmatrix}s_1\cdots s_n\\ s_1\cdots s_n\end{pmatrix}.\end{aligned}$$

In the derivative of $\Delta_n(x)$ the n terms involving derivatives of the diagonal elements have the form

$$\begin{aligned}&\frac{(-)^n}{n!}\int_L ds_1\cdots ds_n[-f(s_i)e^{-2x\sqrt{(-s_i)}}]\\&\qquad\times K\begin{pmatrix}s_1\cdots s_{i-1}s_{i+1}\cdots s_n\\ s_1\cdots s_{i-1}s_{i+1}\cdots s_n\end{pmatrix}\\&\qquad=\frac{1}{n}\int ds\, f(s)e^{-2x\sqrt{(-s)}}\Delta_{n-1}(x),\quad n\geqq 1.\end{aligned}$$

The remaining $n(n-1)$ terms have the form $(n\geqq 2)$

$$\begin{aligned}&\frac{(-)^n}{n!}\int_L ds_1\cdots ds_n[-f(s_i)e^{-x[\sqrt{(-s_i)}+\sqrt{(-s_j)}]}]\\&\qquad\times K\begin{pmatrix}s_1\cdots s_{i-1}s_{i+1}\cdots s_n\\ s_1\cdots s_{j-1}s_{j+1}\cdots s_n\end{pmatrix}(-)^{i+j}.\end{aligned}$$

Using the Fredholm expansion of $R(s,s';x)$,

$$R(s,s';x)=\sum_{n=0}^{\infty}R_n(s,s';x);$$

$$R_n(s,s';x)=\frac{(-)^n}{n!}\int ds_1\cdots ds_nK\begin{pmatrix}s\,s_1\cdots s_n\\ s's_1\cdots s_n\end{pmatrix},$$

this is

$$\frac{1}{n(n-1)}\int dsds'\,R_{n-2}(s,s';x)f(s')e^{-x[\sqrt{(-s)}+\sqrt{(-s')}]}.$$

The addition of all these terms yields

$$\begin{aligned}\Delta_n'(x)&=\int_L ds\,\Delta_{n-1}(x)f(s)e^{-2x\sqrt{(-s)}}\\&\quad+\int_L\int dsds'\,R_{n-2}(s,s';x)f(s')e^{-x[\sqrt{(-s)}+\sqrt{(-s')}]}.\end{aligned}$$

Summing over n, we recover the right-hand side of (3.14). Therefore

$$\begin{aligned}V(x)&=-2\frac{d}{dx}\left[\frac{1}{\Delta(x)}\frac{d}{dx}\Delta(x)\right]=-2\frac{d^2}{dx^2}\ln\Delta(x)\\&=2\frac{[\Delta'(x)]^2-\Delta(x)\Delta''(x)}{[\Delta(x)]^2}\end{aligned} \qquad (3.16)$$

expresses the potential in terms of the Fredholm determinant of $\mathcal{K}(s,s';x)$. Although the integral equation (3.2) may look rather intractable, we now see that if an S matrix model is being constructed with some guess about the left cut as input, it is always practical to find out what kind of nonrelativistic potential corresponds to this left cut by calculating $\Delta(x)$ and applying (3.16).

The procedure is particularly simple if the left cut is a finite sum of poles, but even in the most general case, one has only to calculate a Fredholm determinant and two of its derivatives.

Some general properties of $V(x)$ follow immediately from the fact that the operator $\mathcal{K}$ is an entire function of x. Fredholm theory then tells us that $\Delta(x)$ is an entire function of x in any domain of the x plane where $\mathcal{K}$ is square integrable. Therefore the potential $V(x)$ is a meromorphic function of x in the right-half x plane ($\mathrm{Re}x>0$), the only singularities being double poles at points x_0 where

$$\Delta(x)\approx c(x-x_0)^n\,,\quad \mathrm{Re}x>0\,,\quad x\approx x_0$$

$$V(x)\approx\frac{2n}{(x-x_0)^2}\,,\quad x\approx x_0\,.$$

Furthermore if the left-hand cut does not extend to infinity (in particular if it consists of a finite number of poles), then $\mathcal{K}(s,s';x)$ is an L^2 kernel for *all* finite x, and thus $V(x)$ is a meromorphic function of x for all finite x, its only singularities being double poles when $\Delta(x)=0$.

In this case, if there is an extinct ghost, so that $\Delta(0)=0$, then $\Delta(x)\approx cx^n$, $x\approx 0$, $n=$integer, and thus

$$V(x)\approx\frac{2n}{x^2}\,,\quad x\sim 0\,;$$

extinct ghost, finite left-hand cut, (3.17a)

whereas

$V(0)=$const.; no extinct ghost, finite left-hand cut. (3.17b)

If the left-hand cut extends to infinity, then $\Delta(x)$ is not, in general, analytic at $x=0$. [For example, for a Yukawa potential $\Delta(x)_{x\to 0}\sim x^x$.] However we can still show that an extinct ghost gives rise to a potential that behaves like $1/x^2$ near the origin. An extinct ghost, as we have seen, corresponds to the vanishing of the Fredholm determinant of $\mathcal{K}(s,s';x=0)$; i.e., to the existence of a solution of the homogeneous integral equation

$$\mathcal{K}(s,s';x=0)\otimes n'(s')=0\,.$$

However, the inhomogeneous equation (3.12) also has a solution,[3] since if $d'(s)$ is a solution of the adjoint homogeneous equation

$$\mathcal{K}^\dagger(s,s';x=0)\otimes d'(s')=0\,,$$

then $d'(s)$ is orthogonal to $f(s)$:

$$\int_L d'(s)f(s)ds=\int_L n'(s)ds=\int_L \mathrm{Im}N'(s)ds=0\,,$$

since $sN'(s)\to 0$, $s\to\infty$. We have shown that

$$A(x,x)=\int_L ds\, e^{-x\surd(-s)}a(s,x)$$

$$=\int_L ds\, e^{-x\surd(-s)}\frac{1}{\Delta(x)}\times\int_L \mathrm{adj}\mathcal{K}(s,s';x)f(s')e^{-x\surd(-s')}ds'$$

$$=\frac{\Delta'(x)}{\Delta(x)}\,.$$

Therefore

$$\Delta'(x)=\int_L ds\times e^{-x\surd(-s)}\left[\int_L ds'\,\mathrm{adj}\mathcal{K}(s,s';x)f(s')e^{-x\surd(-s')}\right],$$

where

$$\int_L \mathrm{adj}\mathcal{K}(s,s';x)\mathcal{K}(s',s'';x)ds'=\Delta(x)\delta(s-s'')\,.$$

Since there exists a function $n(s)\not\equiv 0$ such that

$$\int_L \mathcal{K}(s,s';x=0)n(s')ds'=f(s)\,,$$

even when $\Delta(0)=0$, it follows that

$$\int_L \mathrm{adj}\mathcal{K}(s,s';x=0)f(s')ds'=\Delta(0)n(s)=0\,.$$

Therefore, if

$$\Delta(0)=0\,,$$

then

$$\Delta'(0)=\int_L ds\left[\int_L ds'\,\mathrm{adj}\mathcal{K}(s,s';0)f(s')\right]=0\,.$$

An extinct ghost thus is equivalent to $\Delta(0)=\Delta'(0)=0$, and therefore to

$$\Delta(x)\sim x^c f(x)\,,\quad x\sim 0\,,$$

where

$$\frac{\ln f(x)}{\ln x}\to 0\,,\quad x\to 0\,,$$

and $c\geqq 1$. Hence the leading singularity at the origin is[9]

$$V(x)\approx 2c/x^2\,,\quad c\geqq 1\,. \tag{3.18}$$

[9] We are assuming that $\Delta(x)$ does not have an essential singularity at $x=0$ which could lead to singular potentials that behave worse than $1/r^2$ at the origin. Such potentials give rise to unbounded phase shifts at infinite energy and thus to essential singularities of the partial-wave amplitudes.

We have thus shown that if the S matrix has an extinct ghost the potential (or potentials) that generates it has a repulsive $1/x^2$ singularity at the origin.

IV. EXAMPLES

When the left-hand cut of the S matrix consists solely of n poles, the integral equation (3.6) reduces to an $n\times n$ matrix equation. If n is small, the potential (3.16) can be found without too much effort. As an illustration, the two-pole S matrix (2.8), which has one ghost, leads to

$$\Delta(x)=1-\frac{p_2+p_1}{p_2-p_1}(e^{-2p_1x}-e^{-2p_2x})-e^{-2(p_1+p_2)x}. \quad (4.1)$$

The corresponding $V(x)$ was plotted for the case $p_2=2p_1$ and found to be a monotonic decreasing repulsive potential with an exponential tail. Near the origin, independently of p_1,p_2,

$$V(x)\cong 6/x^2, \quad (4.2)$$

in conformity with (3.17a) and (3.17b). Note that it is not necessary for the potential to include any attraction in order to produce an EG. As another example, consider the s wave amplitude with three left-hand cut poles and one bound state, which according to (2.6) also has one ghost. In the notation of (2.7),

$$S(k)=\frac{k+ip_1}{k-ip_1}\frac{k+ip_2}{k-ip_2}\frac{k+ip_3}{k-ip_3}\frac{k+ib}{k-ib}, \quad (4.3)$$

and we suppose that $b<p_1<p_2<p_3$. The analytical expression for the potential given by the Gel'fand-Levitan equations is somewhat involved, so we present a plot of $V(x)$ for the case $p_1=1$, $p_2=2$, $p_3=3$, $b=\frac{1}{2}$ (Fig. 3). As in the previous example, V has an exponential tail and approaches $6/x^2$ at small distances. Of course it has an attractive well, since it generates a bound state.

It is interesting to see what happens if one tries $b=\frac{3}{2}$, since $p_1<b<p_2<p_3$ corresponds to a bound state whose residue has the "wrong" sign. In this case the solution of (3.6) is found to lead to the potential shown in Fig. 4, which is purely repulsive and has a double pole at a finite x. This pathological potential is clearly incapable of supporting any true bound states.

As Auberson and Wanders[10] discovered for the case of a two-pole left cut, the N/D solution for the amplitude is very unstable if the input parameters are near the values which would produce an EG. Obviously, instability of the amplitude near a ghost is a completely general feature arising from the near vanishing

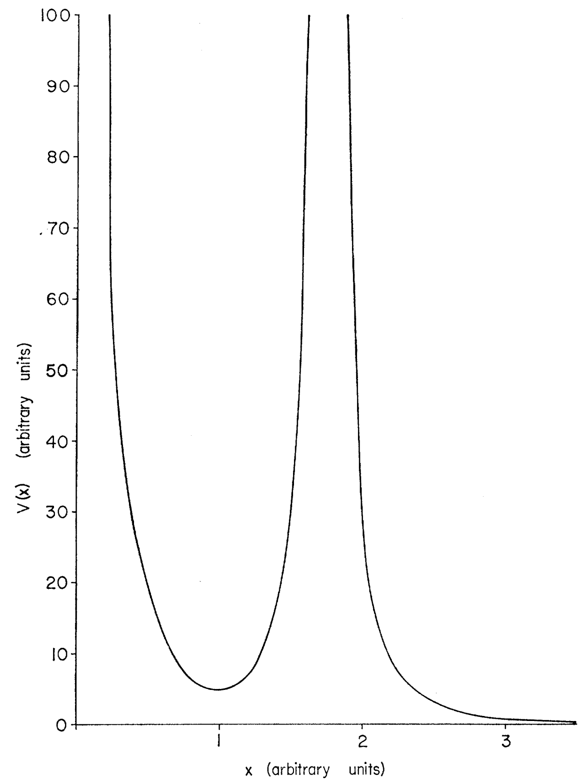

FIG. 4. $V(x)$ for $p_1=1$, $p_2=2$, $p_3=3$, $b=\frac{3}{2}$.

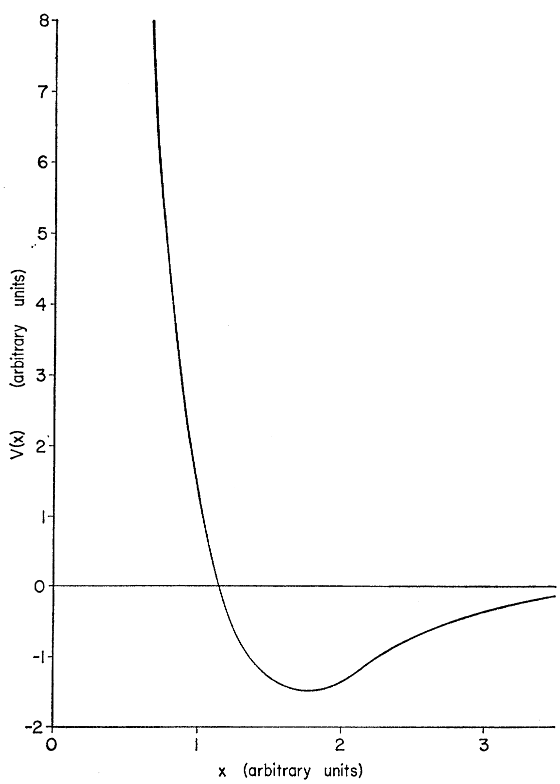

FIG. 3. $V(x)$ for $p_1=1$, $p_2=2$, $p_3=3$, $b=\frac{1}{2}$.

[10] G. Auberson and G. Wanders (unpublished report).

of the Fredholm determinant of $\mathcal{K}(s,s';0)$. However, this behavior can hardly be construed as a shortcoming of the N/D method, since the appearance of a ghost corresponds to a rather violent change in the potential; namely, the introduction of a $1/r^2$ singularity near the origin. Naturally the amplitude can behave wildly in such circumstances, no matter what method one uses to calculate it. However, if one constructs a model based on a left cut as input and discovers that one is near a ghost, extreme caution should be used in drawing conclusions from the calculation.

V. PHASE SHIFTS AND EG PRODUCED BY $\sim 1/r^2$ POTENTIALS

Will a potential which has repulsive $1/r^2$ behavior at small distances always lead to extinct ghosts? To investigate this question we first find out what phase shifts are produced by such potentials. We suppose that $V(r)$ is exponentially small at large distances, nonsingular at finite r, and that as $r \to 0$

$$V(r) \cong \lambda(\lambda+1)/r^2, \tag{5.1}$$

where λ is a positive constant. To settle ambiguities of π in the phase shift, we imagine that $V(r)$ is gradually turned on, and that the displacement of the wave function from the free wave is measured.

Consider first the s wave, and suppose that the energy $s=k^2$ is so large that except for the $1/r^2$ peak, $V(r) \ll s$. Then, except near the origin, the wavelength of the wave function changes slowly as a function of r. If $p(r)$ is the wave number, then

$$\delta p = p(r) - p(\infty) = -(1/v)V(r), \tag{5.2}$$

(where v is the velocity of the particles), since δ(kinetic energy)$= v\delta p = -V(r)$. Then since $d\phi(r)/dr = p(r)$, where $\phi(r)$ is the phase of the wave function, the phase shift produced by the part of the potential at $r > r_0$, r_0 being outside the $1/r^2$ peak, is just

$$\delta = -\frac{1}{v}\int_{r_0}^{\infty} V(r)dr, \tag{5.3}$$

which goes to zero at infinite energy.

Thus the phase shift at $k \to \infty$ comes only from the $\lambda(\lambda+1)/r^2$ singularity at the origin. This would also be true if the potential were exactly $\lambda(\lambda+1)/r^2$ at *all* r, but for the latter potential $\delta(\infty) = \delta$(any energy) $= -\frac{1}{2}\lambda\pi$. Therefore for the actual $V(r)$

$$\delta(\infty) = -\tfrac{1}{2}\lambda\pi. \tag{5.4}$$

To find $\delta(k=0)$, we note that the low-energy behavior and the bound states cannot be affected by cutting off the $1/r^2$ peak at, say, 10^{137} BeV. The low-energy wave functions are, after all, strongly excluded from this peak. Now the amplitude generated by the cut-off $V(r)$ will have the standard analytic and asymptotic properties which allow one to construct the N/D representation and prove the orthodox Levinson theorem $1/\pi[\delta(0) - \delta(\infty)] = N_B$. With our normalization of phase shifts, $\delta(\infty) = 0$ for this potential, so

$$\delta(0) = \pi N_B. \tag{5.5}$$

Turning now to arbitrary angular momentum, we recall that the amplitudes generated by $V(r)$ can be continued in l.[11] Writing $V(r) = \lambda(\lambda+1)/r^2 + v(r)$, one may group the $1/r^2$ term with the centrifugal potential $l(l+1)/r^2$; the remaining potential $V(r)$ is finite at $r \to 0$, and has a $1/r^2$ tail. Thus the amplitude will be analytic in the "effective angular momentum" L defined by

$$L(L+1) = l(l+1) + \lambda(\lambda+1), \tag{5.6}$$

hence in the actual angular momentum l. Owing to the square-root singularity in the map connecting l to L, (5.6), the amplitude will have a branch cut connecting the points $l = -\frac{1}{2} \pm i[\lambda(\lambda+1)]^{1/2}$. This is a trivial complication; we restrict attention to $\mathrm{Re}\, l > -\frac{1}{2}$.

For l real and non-negative the effective potential $V_{\mathrm{eff}} = l(l+1)/r^2 + V(r)$ has a real, repulsive $1/r^2$ singularity at the origin. By the same argument as for s waves, we conclude that $\delta_l(0) = \pi N_B^l$. But now the $1/r^2$ peak leads to

$$\delta_l(\infty) = -\tfrac{1}{2}(L-l)\pi. \tag{5.7a}$$

It is convenient at this point to redefine the phase shift by setting $\delta_l(0) = 0$, whereupon (5.7a) becomes

$$\delta_l(\infty) = -\pi[N_B^l + \tfrac{1}{2}(L-l)]. \tag{5.7b}$$

When there are N_B^l bound states and N_G^l extinct ghosts, the D function may be represented in the form $[\delta_l(0) \equiv 0]$

$$D_l(s) = \exp\left[-\frac{s-s_0}{\pi}\int_0^{\infty} ds' \frac{\delta_l(s')}{(s'-s)(s'-s_0)}\right] \times \prod_{i=1}^{N_B^l} \frac{s - s_{B_i}^l}{s_0 - s_{B_i}^l} \prod_{j=1}^{N_G^l} \frac{s - s_{G_j}^l}{s_0 - s_{G_j}^l}, \tag{5.8}$$

where s_0 is the subtraction point, and $s_{B_i}^l$ and $s_{G_i}^l$ are the positions of the bound states and extinct ghosts.

It follows from (5.8) that for large s, apart from possible logarithmic factors,

$$D_l(s) \propto s^{\delta_l(\infty)/\pi + N_B^l + N_G^l}. \tag{5.9}$$

Or, from (5.7b),

$$D_l(s) \propto s^{-\frac{1}{2}(L-l) + N_G^l}. \tag{5.10}$$

Although the partial-wave S matrix generated by $V(r)$ approaches a constant as $k \to \infty$, this constant is not, in general, unity, because $\frac{1}{2}(L-l)$ is not, in general, an integer. Correspondingly $D_l(s)$ does not approach a constant. Now in defining an EG as a simultaneous zero

[11] H. Bethe and T. Kinoshito, Phys. Rev. **128**, 1418 (1962).

of "N" and "D," one must obviously specify the asymptotic behavior of D to within one power of s, to preclude the possibility of adding or deleting simultaneous zeros at will by multiplying or dividing both N and D by factors of the form $(s-s_G)$. In order to fix precisely what asymptotic behavior is to be allowed, we note that if N and D are to obey dispersion relations containing no arbitrary subtraction constants apart from normalization, D must not diverge as fast as $\sqrt{s}$. On the other hand, if D converges more rapidly than $1/\sqrt{s}$, it is possible to multiply N and D by a factor of the form $(s-s_G)$, thereby obtaining an N and D which contain an additional simultaneous zero but which still obey dispersion relations involving no subtraction parameters. Consequently, from (5.10) we have the condition

$$-\tfrac{1}{2}\leqq-\tfrac{1}{2}(L-l)+N_G{}^l<\tfrac{1}{2}. \tag{5.11}$$

Thus for given real l (in the range of interest), the number of extinct ghosts is determined by the condition

$$\tfrac{1}{2}(L-l)-\tfrac{1}{2}\leqq N_G{}^l<\tfrac{1}{2}(L-l)+\tfrac{1}{2}. \tag{5.12}$$

Since $L>l$, $N_G{}^l$ will be non-negative, as it should be.

Taking $l=0$ as an illustration, it follows from (5.6) and (5.12) that $V(r)$ will produce no s-wave EG if $\lambda\leqq 1$. There will be one EG if $1<\lambda\leqq 3$, two if $3<\lambda\leqq 5$, etc. In the examples of Sec. IV, the S matrix was constructed to approach unity asymptotically, so that $D_0(s)\to 1$, $s\to\infty$. In this case the existence of one ghost corresponds to $\tfrac{1}{2}(L-l)=1$, so that $\lambda=2$, which is why the coefficient of the $1/r^2$ peak always came out to be 6.

VI. REGGE TRAJECTORIES

Do the extinct ghosts which $\sim 1/r^2$ (small r) potentials produce lie on Regge trajectories? Disregarding our relativistic motivation, there is no reason to expect that they do, since an EG in some specific partial wave is *not* a pole of the S matrix. In fact, since $\tfrac{1}{2}(L-l)$ is a continuous function of l, it is clear that if there is an EG for one l there will also be one for a continuous range of l surrounding this value. Thus these EG are not "Regge ghosts" of the type proposed by Chew where a Regge trajectory is extinguished at only one value of l. If they lie on Regge trajectories at all, they correspond to a situation in which a virtual trajectory lies directly beneath a bound-state trajectory in the s plane for a continuous range of l. Such trajectories must, of course, remain "parallel" for all l, since they are analytic functions of l. One may verify, however, that $d[\tfrac{1}{2}(L-l)]/dl<0$, with $(L-l)\to 0$ [cf. (5.6)] as $l\to+\infty$. Thus, from (5.12), $N_G{}^l$ must be zero for sufficiently large l, so that there cannot be an EG for all l. Consequently the EG which we have found in potential theory do not lie on Regge trajectories.[12]

VII. CONCLUSION

Whereas for well-behaved potentials extinct ghosts do not exist, it is easy to construct partial-wave amplitudes, relativistic or nonrelativistic, which do exhibit these ghosts. We have shown that such nonrelativistic amplitudes are generated by potentials with repulsive $1/r^2$ singularities at the origin. Conversely, a potential with a $\lambda(\lambda+1)/r^2$ singularity at the origin ($\lambda>1$) will always produce an extinct ghost in the s wave. However, if an extinct ghost exists for some value of the angular momentum l_0 then one will appear for all $l\leqq l_0$, although for sufficiently large l there will be no extinct ghosts. Therefore these potential-theory extinct ghosts do *not* lie on Regge trajectories and thus, except perhaps for singular potentials (worse than $1/r^2$ at the origin), the phenomenon conjectured by Chew cannot take place in potential theory. Even though most of our intuition about Regge poles arises from potential theory this result does not necessarily make Chew's mechanism unlikely—since only in a crossing symmetric relativistic theory does the physical requirement for such a mechanism arise.

ACKNOWLEDGMENTS

We are indebted to Jerome Finkelstein for a direct numerical verification that two of the potentials discussed in Sec. IV reproduce the S matrices from which they were derived. Discussions with Dr. A. Goldhaber and especially with Professor C. Schwartz were instrumental in determining the relation between the potential near the origin and the phase shift at infinite energy.

[12] One may easily argue that any potential which, near $r=0$, behaves like $1/r^2$ plus some less singular part will also have phase shifts obeying (5.7b). Hence (5.12) again gives the number of EG, and thus even if the amplitudes generated by this potential are analytic in l, the EG will not lie on Regge trajectories.